The Range of the Complex Monge-Ampère Operator, II

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ABSTRACT. If ν is a complex Monge-Ampère mass of a bounded plurisubharmonic function then so is every non-negative Borel measure $\mu \leq \nu$.

0. Introduction. In this paper we continue our investigation ([KO1], [KO2]) of the Dirichlet problem for the complex Monge-Ampère equation in a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Given $\varphi \in C(\partial\Omega)$ and a nonnegative Borel measure $d\mu$ we look for a plurisubharmonic (psh) function u satisfying

$$\begin{aligned}
 &u \in \text{PSH} \cap L^\infty(\Omega), \\
 (*) \quad &(dd^c u)^n = d\mu, \\
 &\lim_{z \rightarrow x} u(z) = \varphi(x) \text{ for } x \in \partial\Omega.
 \end{aligned}$$

We refer to [B], [K], [KO1], [KO2] for a more detailed historical account on the subject. Here we only record for later reference the following two basic results.

Theorem 0.1 ([BT1]). *If $d\mu = f d\lambda$ (λ denotes the Lebesgue measure), $f \in C(\bar{\Omega})$, then (*) has a unique continuous solution.*

Theorem 0.2 ([CKNS]). *If $d\mu = f d\lambda$, $f \in C^\infty(\bar{\Omega})$, $f > 0$ and the boundary of Ω is C^∞ smooth, then (*) has a unique solution $u \in C^\infty(\bar{\Omega})$.*

A bounded psh function v is called a *subsolution* for (*) if $(dd^c v)^n \geq d\mu$ and $\lim_{z \rightarrow x} v(z) = \varphi(x)$, $x \in \partial\Omega$. The main result of this paper says

Theorem A. *If there exists a subsolution for the Dirichlet problem (*), then the problem is solvable.*

In other words, if $(*)$ is solvable for a measure ν and $\mu \leq \nu$, then $(*)$ is solvable for μ as well.

In [KO1, Thm. 2] a weaker version of Theorem A was proved. There we needed one extra assumption, namely that μ is locally dominated by capacity (see [KO1] for the definition). The strategy of the proofs here and in [KO1] hinges upon careful investigation of the solutions of $(*)$ for some special regularizing sequence for μ . If the sequence of solutions is uniformly bounded and μ itself is locally dominated by capacity, then $(*)$ is solvable ([KO1, Thm. 1]). It would be interesting to know if the first hypothesis alone is sufficient since one can quite accurately describe, in terms of the relative capacity of Bedford and Taylor, the set of measures for which this approximating sequence of solutions is uniformly bounded (see [KO2]).

The regularizing sequence mentioned above for μ must be judiciously chosen since arbitrary weakly convergent sequences $\mu_j \rightarrow \mu$ may lead to divergent sequences of solutions (see [CK, Remark 2]). Moreover, even having “good” μ_j ’s, we still do not know if the approximating sequence of solutions is uniformly bounded, although, due to the Domination Principle, the boundedness of the alleged solution is guaranteed by the hypothesis. This makes the proof quite technical.

Theorem A settles one of the open problems in Bedford’s survey [B].

The referee remarks helped very much to clarify the exposition of the proof.

1. Preliminaries. For the convenience of the reader we collect in this section definitions and results of pluripotential theory which are used in the sequel. The Lebesgue measure will be denoted throughout by λ and the Kähler form in $\mathbb{C}^n$ by $\beta := dd^c|z|^2 = \sum (i/2)dz_j \wedge d\bar{z}_j$. For locally bounded psh functions u_k , the wedge product

$$dd^c u_1 \wedge \cdots \wedge dd^c u_N$$

is a well defined positive current (see [BT1]). In particular, it is continuous with respect to monotone sequences of functions appearing in it.

Theorem 1.1 (Convergence Theorem, [BT2]). *Let $\{u_k^j\}_{j=1}^\infty$ be increasing (resp. decreasing) sequences of psh functions in Ω for $k = 1, 2, \dots, N$; and let $u_k^j \rightarrow u_k \in \text{PSH} \cap L_{\text{loc}}^\infty(\Omega)$ almost everywhere as $j \rightarrow \infty$ for $k = 1, 2, \dots, N$. Then*

$$dd^c u_1^j \wedge \cdots \wedge dd^c u_N^j \rightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_N$$

in the weak topology of currents.

The easier part of the theorem above (for decreasing sequences) follows almost immediately from Chern-Levine-Nirenberg inequality.

Theorem 1.2 ([CLN]). *If $\Omega' \Subset \Omega$, then for a constant $C = C(\Omega', \Omega)$ the following inequality holds*

$$\int_{\Omega'} dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq C \|u_1\|_{\Omega} \cdots \|u_n\|_{\Omega},$$

for any set of $u_k \in \text{PSH} \cap L^{\infty}(\Omega)$, where $\|\cdot\|$ denotes the sup norm of a function.

If $(dd^c u)^n = 0$ in an open set Ω , we call $u \in \text{PSH}(\Omega)$ a *maximal function* in Ω . This is justified by the following crucial result.

Theorem 1.3 (Comparison Principle, [BT2]). *If $u, v \in \text{PSH} \cap L^{\infty}(\Omega)$ and $\liminf_{z \rightarrow \partial\Omega} (u(z) - v(z)) \geq 0$, then*

$$\int_{\{u < v\}} (dd^c v)^n \leq \int_{\{u < v\}} (dd^c u)^n.$$

Following [BT2] one may define for a compact set $K \subset \Omega$ the *relative extremal function* and the *relative capacity*

$$u_K(z) := \sup\{v(z) : v \in \text{PSH}(\Omega), -1 \leq v < 0, v \leq -1 \text{ on } K\},$$

$$\text{cap}(K, \Omega) := \sup \left\{ \int_K (dd^c u)^n : u \in \text{PSH}(\Omega), -1 < u < 0 \right\}.$$

Let v^* denote the upper semicontinuous regularization of v . Then the function u_K^* is psh and cap extends as an inner capacity to open sets. The capacity dominates the Lebesgue measure in the sense that given $\Omega' \Subset \Omega$ there exists a constant $C = C(\Omega', \Omega)$ such that

$$(1.1) \quad \lambda(K) \leq C \text{cap}(K, \Omega) \quad \text{for } K \subset \Omega',$$

(see [BT2]).

A sequence u_j of functions defined in Ω is said to *converge with respect to capacity* to u if for any $\delta > 0$ and $K \Subset \Omega$,

$$\lim_{j \rightarrow \infty} \text{cap}(K \cap \{|u - u_j| > \delta\}, \Omega) = 0.$$

This convergence is good enough to ensure the convergence of currents.

Theorem 1.4 ([X]). *If u_j is a uniformly bounded sequence of psh functions in Ω converging with respect to capacity to $u \in \text{PSH}(\Omega)$, then*

$$(dd^c u_j)^n \rightarrow (dd^c u)^n$$

in the sense of currents.

Plurisubharmonic functions are *quasicontinuous*.

Theorem 1.5 ([BT2]). *If $u \in \text{PSH}(\Omega)$, then given $\varepsilon > 0$ one can find an open set U with $\text{cap}(U, \Omega) < \varepsilon$ such that $u|_{\Omega \setminus U}$ is continuous.*

Locally “discontinuity” of a psh function is controlled by means of the following lemma.

Lemma 1.6 ([KO1]). *Given $z_0 \in \Omega$ and two numbers $M > 1$, $R_0 > 0$ such that*

$$B_M := \{|z - z_0| \leq e^M R_0\} \Subset \Omega,$$

and given $v \in \text{PSH} \cap C(\Omega)$, $-1 < v < 0$, denote by $E = E(\delta)$ the set

$$\{z \in B_0 : (1 - \delta)v(z) \leq \sup_{B_0} v\},$$

where $\delta \in (0, 1)$ and $B_0 = \{|z - z_0| < R_0\}$. Then

$$\text{cap}(E, B_1) < C_0 \frac{1}{M\delta},$$

where $B_1 = \{|z - z_0| < eR_0\}$ and C_0 is an independent constant.

For later reference we record here the following consequence of the lemma above and (1.1).

Corollary 1.7. *If $\delta > 0$ and $\Omega' \Subset \Omega$, then*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{I, v} \left[\frac{\lambda(I \cap E(v, I, \delta))}{\lambda(I)} \right] = 0,$$

where $E(v, I, \delta) := \{z \in I : v(z) < \sup_I v - \delta\}$ and the sup is taken over all cubes $I \subset \Omega'$ of diameter $< \varepsilon$ and all $v \in \text{PSH} \cap C(\Omega)$, $-1 < v < 0$.

Proof. Fix $a > 0$. Let B_0, B_1, B_M be defined as in Lemma 1.6 for some z_0, R_0, M . Then the constant C in (1.1) corresponding to (B_0, B_1) in place of (Ω', Ω) does not depend on z_0 and R_0 provided that we normalize the measure replacing $\lambda(K)$ by $\lambda(K)/\lambda(B_0)$ in this formula.

For $\varepsilon > 0$ small enough there exists $M > 0$ so large that

$$\frac{C_0 C}{M\delta} < a,$$

where C_0 comes from the statement of Lemma 1.6, and, moreover, for any cube $I \subset \Omega'$ of diameter $< \varepsilon$ one can find z_0, R_0 such that $\text{diam } I = 2R_0$ and $I \subset B_0 \subset B_M \subset \Omega$. Then, by (1.1) and Lemma 1.6 we get

$$\begin{aligned} \frac{\lambda(I \cap E(v, I, \delta))}{\lambda(I)} &\leq C \frac{\lambda(B_0)}{\lambda(I)} \text{cap}(I \cap E(v, I, \delta), B_1) \\ &\leq \frac{CC_0 \lambda(B_0)}{M\delta \lambda(I)} \leq a \frac{\lambda(B_0)}{\lambda(I)}. \end{aligned}$$

By the choice of B_0 , the quotient $\lambda(B_0)/\lambda(I)$ depends only on the dimension of the space and thus the result follows. $\square$

2. Proof of Theorem A. It is enough to consider only measures μ which have compact support. Then, given a non-compactly supported measure μ , one can find solutions corresponding to $\chi_j \mu$, where χ_j is a non-decreasing sequence of cut-off functions $\chi_j \uparrow 1$ on Ω . The solutions will be bounded from below by the given subsolution (Thm. 1.3) and they will decrease to the solution for μ . When μ has compact support, a subsolution v can be modified so that $\lim_{z \rightarrow x} v(z) = 0$ for any $x \in \partial\Omega$, which is accomplished by adding a suitable psh function. Then, by the balayage procedure, one can make the support of $d\nu := (dd^c v)^n$ compact. Without loss of generality we also assume that $\|v\| < 1$, $\nu(\Omega) < 1$, the boundary values φ are non-positive and that Ω is a subset of the unit cube $I_0 := [0, 1]^{2n} \subset \mathbb{C}^n$.

We shall consider a sequence $\mathcal{B}_s$ of subdivisions of I_0 into 3^{2sn} congruent open cubes of equal size which are pairwise disjoint but their closures cover I_0 . It is no restriction to assume that for each s we have $\nu(\bigcup_{I \in \mathcal{B}_s} \partial I) = 0$.

Let us denote by χ_{I_0} the characteristic function of I_0 and fix a sequence χ_s of C^∞ functions on $\bar{I}_0$ satisfying the following conditions:

- (a) $\chi_s \rightarrow \chi_{I_0}$ uniformly on compact subsets of I_0 ,
- (b) $|\chi_s| < 1 + 1/s$,
- (c) $\int_{I_0} \chi_s d\lambda = 1$,
- (d) χ_s vanishes to infinite order on ∂I_0 .

Next we define two sequences f_s, g_s of smooth functions on I_0 requiring that on $I \in \mathcal{B}_s$,

$$f_s = \mu(I)(\chi_s \circ T_I), \quad g_s = \nu(I)(\chi_s \circ T_I),$$

where $T_I : I \rightarrow I_0$ is the unique affine isomorphism between I and I_0 which is a superposition of a translation and a homothety. Both f_s and g_s extend to C^∞ functions if we put $f_s = g_s = 0$ on ∂I , $I \in \mathcal{B}_s$.

Note that from the property (c) of χ_s and the construction of f_s, g_s we have

$$(2.0) \quad \begin{aligned} f_s &\leq g_s, \\ \int_I f_s d\lambda &= \mu(I), \\ \int_I g_s d\lambda &= \nu(I). \end{aligned}$$

Moreover, $f_j d\lambda \rightarrow d\mu, g_j d\lambda \rightarrow d\nu$ in the weak* topology.

Using Theorem 0.1 we are able to solve the following Dirichlet problems:

$$\begin{cases} u_s \in \text{PSH}(\Omega) \cap C(\bar{\Omega}), \\ (dd^c u_s)^n = f_s d\lambda, \\ u_s(x) = \varphi(x) \text{ for } x \in \partial\Omega. \end{cases}$$

$$\begin{cases} w_s \in \text{PSH}(\Omega) \cap C(\bar{\Omega}), \\ (dd^c w_s)^n = g_s d\lambda, \\ w_s = 0 \text{ on } \partial\Omega. \end{cases}$$

From the sequence u_s obtained this way, we pick a subsequence converging in L^1_{loc} (see [H]). Renumbering, we shall still denote this converging sequence by u_s . (In what follows we shall always stick to the original notation whenever passing to a subsequence.) Set $u := (\limsup u_s)^*$; this is our candidate for the solution of (*). Note that u remains unchanged if we use a subsequence of u_s in its definition.

First we prove that u solves (*), provided that for any $a > 0$ and any compact $K \subset \Omega$ we have

$$(2.1) \quad \lim_{s \rightarrow \infty} \int_{E_s(a) \cap K} (dd^c u_s)^n = 0, \quad \text{where } E_s(a) := \{u - u_s \geq a\}.$$

Indeed, if (2.1) holds, then for any j one can find $s(j)$ such that

$$\int_{E_s(1/j) \cap K} (dd^c u_s)^n < \frac{1}{j}, \quad s \geq s(j).$$

Set $\rho_j := \max(u_{s(j)}, u - 1/j)$. Then $(dd^c \rho_j)^n = (dd^c u_{s(j)})^n$ on the interior of $K \setminus E_{s(j)}(1/j)$ and so the inequality above implies that any accumulation point of $\{(dd^c \rho_j)^n\}$ is $\geq d\mu$ on $\text{int } K$. On the other hand, by the definition of ρ_j and a version of Hartog's lemma given in [H], $\rho_j \rightarrow u$ uniformly on any compact E such that $u|_E$ is continuous. So it follows from Theorem 1.5 that ρ_j converge to u

with respect to capacity. Therefore, applying Theorem 1.4 we obtain $(dd^c \rho_j)^n \rightarrow (dd^c u)^n$ and further, $(dd^c u)^n \geq d\mu$. Then it easily follows that $(dd^c u)^n = d\mu$ since the equality $\rho_j = u_{s(j)}$ on a neighbourhood of $\partial\Omega$ implies, via the Stokes theorem, that $\int_{\Omega} (dd^c \rho_j)^n = \int_{\Omega} (dd^c u_{s(j)})^n \rightarrow \int_{\Omega} d\mu$ and, therefore, $\int_{\Omega} (dd^c u)^n \leq \int_{\Omega} d\mu$.

From now on we shall work under the hypothesis that (2.1) is false. So, we assume that for some $a > 0$, $A > 0$ and $K \Subset \Omega$,

$$(2.2) \quad \int_{E_s(a) \cap K} (dd^c u_s)^n \geq A,$$

where we again pass to a subsequence if necessary. Observe that (2.2) remains valid if u_s are replaced by w_s since from (2.0) it follows that $(dd^c u_s)^n \leq (dd^c w_s)^n$.

The proof is organized as follows. Its crucial point is Lemma 2.7, where we “almost” show that (2.2) leads to the statement

$$\int_{E_s(a')} (dd^c w_k)^n \geq A',$$

for k large enough, $s > s(k)$, and some positive a' , A' .

Since $(dd^c w_k)^n$ is equicontinuous with the Lebesgue measure, this inequality, with k fixed and $s \rightarrow \infty$, would contradict the fact that $u_s \rightarrow u$ in L^1 .

The trouble is that the inductive proof of Lemma 2.7 works only for a modified sequence $v_k = \max(w_k, \psi)$, $\psi \in \text{PSH} \cap L^\infty(\Omega)$, as we do not know if w_k is uniformly bounded. But the v_k are no longer smooth, so to save the argument from the preceeding paragraph, we need to show, in Lemma 2.6, that

$$\lim_{k \rightarrow \infty} \int_{\{w_k \leq \psi\}} (dd^c v_k)^n = 0.$$

In the proof of Lemma 2.7 we also require $(\limsup u_s)^*$ to be bounded. This follows from Theorem 2.1 and the Comparison Principle.

Theorem 2.1. $v = w := (\limsup w_s)^*$.

Proof. We begin with showing the inequality $v \leq w$.

Lemma 2.2. For any $t > 1$,

$$\lim_{k \rightarrow \infty} \int_{V(k,t)} (dd^c w_k)^n = 0,$$

where $V(k,t) := \{w_k < tv\}$.

Proof. If the statement were not true, then passing to a subsequence we could find $C > 0$ and $t_0 > 1$ such that

$$(2.3) \quad \int_{V(k,t)} (dd^c w_k)^n > C, \quad 1 < t < t_0.$$

Since v is quasicontinuous (Thm. 1.5) one can find an open set U satisfying the inequality

$$(2.4) \quad \int_U (dd^c v)^n \leq \text{cap}(U, \Omega) \leq \frac{C}{8} \left(\frac{(t_0+1)^n}{2^n} - 1 \right)$$

and the condition that $v|_{\Omega \setminus U}$ is continuous. Set $G(k, t) := \bigcup \{I \in \mathcal{B}_k : I \subset V(k, t) \cup U\}$.

Since $\int_{\{(t_0+1)/2\}v \leq w_k < tv} (dd^c v)^n$ is an increasing function of t , it is possible to choose t and t' satisfying $(t_0+1)/2 < t' < t < t_0$ and (again extracting a subsequence of $\{w_k\}$)

$$(2.5) \quad \int_{\{tv \leq w_k < t'v\}} (dd^c v)^n \leq \frac{C}{8} (t^n - 1), \quad k = 1, 2, \dots$$

By the construction of w_k (see (2.0)), we have

$$\begin{aligned} \int_I (dd^c w_k)^n &= \int_I (dd^c v)^n \\ &= \int_{I \cap V(k,t)} (dd^c v)^n + \int_{I \setminus V(k,t)} (dd^c v)^n, \quad I \in \mathcal{B}_k. \end{aligned}$$

Hence

$$\int_{I \cap V(k,t)} [(dd^c w_k)^n - (dd^c v)^n] \leq \int_{I \setminus V(k,t)} (dd^c v)^n.$$

Summing up these inequalities over $I \subset G(k, t') \setminus G(k, t)$, we get, in view of the inclusion $G(k, t') \setminus V(k, t) \subset [V(k, t') \setminus V(k, t)] \cup U$, (2.4) and (2.5), the following estimate

$$\begin{aligned} (2.6) \quad & \int_{(G(k, t') \setminus G(k, t)) \cap V(k, t)} [(dd^c w_k)^n - (dd^c v)^n] \\ & \leq \int_{\{tv \leq w_k < t'v\} \cup U} (dd^c v)^n \leq \frac{C}{4} (t^n - 1). \end{aligned}$$

Let F be a compact subset of Ω such that $\text{supp } \nu \Subset \text{int } F$. Then $|v| > \eta > 0$ on F for some η . Since $v|_{F \setminus U}$ is continuous, for k_1 large enough we have

$$|v(x) - v(y)| < \varepsilon := \frac{\eta(t - t')}{2t}, \quad x, y \in (I \setminus U) \cap F, \quad I \in \mathcal{B}_k, \quad k > k_1.$$

Hence, fixing $I \in \mathcal{B}_k$, $I \subset F$ and $z_0 \in (I \setminus U) \setminus V(k, t')$, we get for any $z \in (I \setminus U) \cap V(k, t)$:

$$w_k(z) < tv(z) < t(v(z_0) + \varepsilon) < t'v(z_0) - \frac{\eta(t-t')}{2} \leq w_k(z_0) - \frac{\eta(t-t')}{2}.$$

Thus $(I \setminus U) \cap V(k, t) \subset I \cap E(\delta, k)$ if $I \subset F \setminus G(k, t')$, where

$$E(\delta, k) := \bigcup_{I \in \mathcal{B}_k} \{z \in I : w_k(z) \leq \sup_I w_k - \delta\}, \quad \delta = (\eta(t-t'))/2.$$

Therefore, from Corollary 1.7 and the construction of $(dd^c w_k)^n$, one infers that

$$\left[\sup_{I \subset F \setminus G(k, t')} \left(\frac{\int_{(I \setminus U) \cap V(k, t)} (dd^c w_k)^n}{\int_I (dd^c w_k)^n} \right) \right] \leq \frac{C(t^n - 1)}{8}, \quad k > k_2 > k_1,$$

for some k_2 . Since $\int_{\Omega \setminus F} (dd^c w_k)^n \rightarrow 0$ as $k \rightarrow \infty$, and $\int_U (dd^c w_k)^n$ is controlled by (2.4), one obtains from the inequality above, upon increasing k_2 , that

$$(2.7) \quad \int_{(\Omega \setminus G(k, t')) \cap V(k, t)} (dd^c w_k)^n \leq \frac{C(t^n - 1)}{4}, \quad k > k_2.$$

Now, we apply the Comparison Principle (Thm. 1.3), (2.6) and (2.7) to the effect

$$\begin{aligned} (2.8) \quad t^n \int_{V(k, t)} (dd^c v)^n &\leq \int_{V(k, t)} (dd^c w_k)^n \\ &\leq \int_{G(k, t)} + \int_{(G(k, t') \setminus G(k, t)) \cap V(k, t)} + \int_{(\Omega \setminus G(k, t')) \cap V(k, t)} \\ &\leq \int_{G(k, t)} (dd^c w_k)^n + \int_{(G(k, t') \setminus G(k, t)) \cap V(k, t)} (dd^c v)^n \\ &\quad + \frac{C}{2}(t^n - 1), \quad k > k_2. \end{aligned}$$

Since $G(k, t) \subset V(k, t) \cup U$ and $\int_{G(k, t)} (dd^c w_k)^n = \int_{G(k, t)} (dd^c v)^n$ (see (2.0)), the formulae (2.4) and (2.8) lead to

$$\begin{aligned} t^n \int_{G(k, t)} (dd^c w_k)^n &= t^n \int_{G(k, t)} (dd^c v)^n \\ &\leq t^n \int_{V(k, t)} (dd^c v)^n - t^n \int_{(G(k, t') \setminus G(k, t)) \cap V(k, t)} (dd^c v)^n \\ &\quad + t^n \int_U (dd^c v)^n \end{aligned}$$

$$\leq \int_{G(k,t)} (dd^c w_k)^n + (t^n - 1) \left(\frac{t^n C}{8} + \frac{C}{2} \right).$$

Hence

$$\int_{G(k,t)} (dd^c w_k)^n \leq \frac{t^n C}{8} + \frac{C}{2}, \quad k > k_3 \text{ (} k_3 \text{ large enough)}$$

which contradicts (2.3), if t is sufficiently close to 1, and thus proves the lemma. $\square$

Corollary 2.3. $v \leq w$.

Proof. Suppose it were not true. Then for some $z_0 \in \Omega$, $\varepsilon > 0$, we would have $w(z_0) < v(z_0) - 3\varepsilon$. By the Hartogs lemma and the upper semicontinuity of w we have $w_s < w(z_0) + \varepsilon$ on a ball $B = B(z_0, r)$ for $s > s_0$. Apply the Comparison Principle to obtain

$$\int_{G_s} \left(dd^c \left(v + \varepsilon \left(\frac{|z|^2}{n} - 1 \right) \right) \right)^n \leq \int_{G_s} (dd^c w_s)^n,$$

where $G_s = \{w_s < v + \varepsilon(|z|^2/n - 1)\}$. Since $|z|^2/n - 1$ is negative on Ω the inclusion $G_s \Subset \Omega$ holds. If r is so small that $v \leq v(z_0) + \varepsilon$ on B , then by the mean value inequality for subharmonic functions $v \geq v(z_0) - \varepsilon$ on a subset F of B such that $2\lambda(F) \geq \lambda(B)$. Since $F \subset G_s$, the left hand side integral is not smaller than $(\varepsilon/n)^n \int_F \beta^n > 0$. But by Lemma 2.2 the right hand side tends to 0 as $s \rightarrow \infty$, which leads to a contradiction. $\square$

To show the inequality $w \leq v$ we need

Lemma 2.4. $\lim_{k \rightarrow \infty} \int_{V(k,a)} (dd^c v)^n = 0$, $V(k,a) := \{tv < w_k - a\}$, for any $t < 1$ and $a > 0$.

Proof. Suppose the lemma is not true, and there exist $t < 1$, $C > 0$, $a_0 > 0$ such that the inequality

$$C < \int_{V(k,a_0)} (dd^c v)^n$$

holds for any k (after passing to a subsequence). Fix ε , a , b such that $0 < \varepsilon < C(1 - t^n)/8$, $0 < a < b < a_0$ and

$$(2.9) \quad \int_{V(k,a) \setminus V(k,b)} (dd^c v)^n \leq \varepsilon,$$

for all k after yet another renumbering. Using Theorem 1.5 we choose an open set U such that $v|_{\Omega \setminus U}$ is continuous and

$$(2.10) \quad \int_U (dd^c v)^n \leq \text{cap}(U, \Omega) \leq \varepsilon.$$

Then for k_1 large enough and any $x, y \in I \setminus U$, $I \in \mathcal{B}_k$, $k > k_1$, we have

$$(2.11) \quad |v(x) - v(y)| < \delta := \frac{b-a}{4t}.$$

Increasing k_1 if necessary we can also assume, according to Corollary 1.7, that

$$(2.12) \quad \int_{I \cap E(\delta, k)} (dd^c w_k)^n < \varepsilon \int_I (dd^c w_k)^n,$$

where $E(\delta, k)$ is defined as in the previous lemma and $k > k_1$.

Consider $I \in \mathcal{B}_k$, $k > k_1$ such that $(I \setminus U) \cap V(k, b) \neq \emptyset$. Then for any $z \in I \setminus [E(\delta, k) \cup U]$, we have $v(z) \leq v(z_0) + \delta$ (see (2.11)) and $w_k(z) \geq w_k(z_0) - \delta$, where $z_0 \in (I \setminus U) \cap V(k, b)$. Therefore,

$$w_k(z) - tv(z) \geq w_k(z_0) - tv(z_0) - \delta(t+1) \geq b - \frac{(b-a)(t+1)}{4t} \geq \frac{a+b}{2}.$$

Thus $I \setminus [E(\delta, k) \cup U] \subset V(k, a)$. Hence, by (2.0) and (2.12) if $(I \setminus U) \cap V(k, b) \neq \emptyset$, then

$$\begin{aligned} \int_{I \cap V(k, a)} (dd^c w_k)^n &\geq (1-\varepsilon) \int_I (dd^c w_k)^n - \int_{I \cap U} (dd^c w_k)^n \\ &\geq (1-\varepsilon) \int_I (dd^c v)^n - \int_{I \cap U} (dd^c w_k)^n, \quad k > k_1. \end{aligned}$$

Let $\mathcal{B}'_k := \{I \in \mathcal{B}_k : (I \setminus U) \cap V(k, b) \neq \emptyset\}$. Summing up the inequality above over $I \in \mathcal{B}'_k$, $k > k_1$, and applying (2.9) and (2.10) we get

$$\begin{aligned} \int_{V(k, a)} (dd^c w_k)^n &\geq (1-\varepsilon) \sum_{\mathcal{B}'_k} \int_I (dd^c v)^n - \varepsilon \\ &\geq (1-\varepsilon) \left(\int_{V(k, a)} (dd^c v)^n - \int_{(V(k, a) \setminus V(k, b)) \cup U} (dd^c v)^n \right) - \varepsilon \\ &\geq \int_{V(k, a)} (dd^c v)^n - 4\varepsilon. \end{aligned}$$

This combined with the Comparison Principle gives

$$\int_{V(k,a)} (dd^c v)^n - 4\varepsilon \leq \int_{V(k,a)} (dd^c w_k)^n \leq t^n \int_{V(k,a)} (dd^c v)^n,$$

and finally, by our hypothesis,

$$C(1 - t^n) \leq (1 - t^n) \int_{V(k,a)} (dd^c v)^n < 4\varepsilon < \frac{C(1 - t^n)}{2},$$

which is a contradiction. $\square$

Corollary 2.5. $w \leq v$.

Proof. The proof is a copy of the proof of Corollary 2.3 with the roles of w_s and v interchanged and obvious modifications. We only have to choose z_0 satisfying $v(z_0) < w(z_0) - 3\varepsilon$ in such a manner that $w(z_0) = \limsup w_s(z_0)$. This is possible since the set of points where the equality is not valid is pluripolar (see [BT2]). Obviously in place of Lemma 2.2 we now apply Lemma 2.4.

Thus the proof of Theorem 2.1 has been completed. $\square$

To proceed further we need yet to modify the functions u_s and w_s since we do not know if the sequences are uniformly bounded. First, let us note that by the Comparison Principle $\|u_s\| \leq \|w_s\| + \|\varphi\|$. Therefore, since $w = v$ is bounded, so is u . It is no harm to assume that $\|u\| < 1$, that the compact K from the formula (2.1) contains the support of ν and that u_K is continuous. Set

$$\begin{aligned} v_s &:= \max(u_K, w_s), \\ h_s &:= \max(u_K, u_s), \\ h &:= (\limsup h_s)^*. \end{aligned}$$

It follows from (2.0) and (2.2) that

$$(2.13) \quad \int_{K \cap E_s(a)} (dd^c w_s)^n > A.$$

Due to our previous assumptions (v is maximal away from K , $\|v\| < 1$) we have $tv \geq u_K$ for some $t > 1$. Thus $v = (\limsup v_s)^*$ (this fact will be used to obtain (2.18) below) and $\{w_s \leq u_K\} \subset \{w_s < tv\}$. The last inclusion and Lemma 2.2 give

$$(2.14) \quad \lim_{s \rightarrow \infty} \int_{\{w_s \leq u_K\}} (dd^c w_s)^n = 0.$$

Combining (2.13) and (2.14), we get for s large enough,

$$(2.15) \quad \int_{K \cap E_s(a)} (dd^c v_s)^n \geq \int_{K \cap E_s(a) \cap \{w_s > u_K\}} (dd^c v_s)^n \\ = \int_{K \cap E_s(a) \cap \{w_s > u_K\}} (dd^c w_s)^n \geq \frac{A}{2}.$$

Further, if only a is small enough ($a < 1 - \|u\|$), then

$$(2.16) \quad K \cap E_s(a) = K \cap \{h - h_s \geq a\}.$$

Indeed, $h = u$ on K and obviously $u - u_s \geq h - h_s$. In the other direction, if $x \in K$ with $(u - u_s)(x) \geq a$ and $h_s(x) > u_s(x)$, then $h_s(x) = -1$ and so $(h - h_s)(x) \geq u(x) + 1 > a$.

The following property is the reason for introducing v_s and h_s . Let ρ be a psh defining function for Ω such that $\rho < u_K$ in Ω . Then all the functions h , v_s , h_s can be extended, by ρ , as psh functions to the domain of definition of ρ . Hence Theorem 1.2 implies that for some $C > 0$ and any set of positive integers j, k, s we have

$$(2.17) \quad \int_{\Omega} (dd^c h)^{\alpha} \wedge (dd^c h_j)^{\beta} \wedge (dd^c v_k)^{\gamma} \wedge (dd^c v_s)^{n-\alpha-\beta-\gamma} < C,$$

where $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma \leq n$.

Next we generalize Lemma 2.2, showing the following result.

Lemma 2.6. *If g_1 and g_2 are bounded negative psh functions in Ω , then*

$$\lim_{k \rightarrow \infty} \int_{V(k,t) \cap \Omega'} (dd^c g_1)^{\alpha} \wedge (dd^c g_2)^{\beta} \wedge (dd^c v_k)^{n-\alpha-\beta} = 0,$$

for $\alpha, \beta \geq 0$, $\alpha + \beta \leq n$, $1 < t < t_1$, $\Omega' \Subset \Omega$ and $V(k, t) = \{w_k < tv\}$.

Proof. Apply the Comparison Principle to the pair of functions w_k and $tv + \varepsilon(g_1 + g_2 + v_k)$, $\varepsilon > 0$, on the set $G := \{w_k < tv + \varepsilon(g_1 + g_2 + v_k)\}$, to obtain

$$\int_G [dd^c (tv + \varepsilon(g_1 + g_2 + v_k))]^n \leq \int_G (dd^c w_k)^n.$$

Since $G \subset V(k, t)$, one infers from Lemma 2.2 that the right hand side of this inequality tends to zero as $k \rightarrow \infty$. So the same must be true about all the terms of the Newton expansion of the integrand on the left hand side; a multiple of $\int_G \varepsilon^n (dd^c g_1)^{\alpha} \wedge (dd^c g_2)^{\beta} \wedge (dd^c v_k)^{n-\alpha-\beta}$ being among them. To get the statement, it is enough to observe that $\Omega' \cap V(k, t') \subset G$ if ε is so small that

$$\varepsilon(g_1 + g_2 + v_k) > \frac{t-1}{2}v \quad \text{on } \Omega',$$

and $t' = (t+1)/2$. □

Here is the crucial step in our argument.

Lemma 2.7. *Under the hypothesis (2.2), for any $0 \leq m \leq n$ there exist $a_m > 0$, $A_m > 0$, $k_1 > 0$ such that*

$$\int_{E'_s(a_m)} (dd^c v_s)^{n-m} \wedge (dd^c v_k)^m > A_m, \quad k > k_1, s > s(k),$$

where $E'_s(a_m) := \{h - h_s \geq a_m\}$ and a_m , A_m depend on a , A , and C .

Proof. We shall proceed by induction over m . For $m = 0$ the statement is true by (2.15) and (2.16). Let us assume that it holds for fixed $m < n$; we shall prove it for $m + 1$.

Denote by $T = T(s, k, m)$ the current $(dd^c v_s)^{n-m-1} \wedge (dd^c v_k)^m$ and set $v_{sk} := \max(v_s, v_k - 2\varepsilon)$ for some fixed $\varepsilon > 0$. By Hartog's Lemma, Theorem 2.1, Theorem 1.5 and the equality $v = (\limsup v_s)^*$, we have

$$(2.18) \quad \begin{aligned} \max(v_s, v_k) &\leq v + \varepsilon, \\ h_s &\leq h + \varepsilon, \end{aligned}$$

on $\Omega \setminus U$ for $s > s_0$, $k > k_0$ with s_0 , k_0 large enough and $\text{cap}(U, \Omega) < \varepsilon/3^n$. Throughout the proof we tacitly assume that $s > s_0$, $k > k_0$.

Integrating by parts twice, then using the inequalities $v_k - v_{sk} \leq 2\varepsilon$, $|v_k| \leq 1$ and (2.17), one obtains

$$\begin{aligned} \int_{\Omega} (h - h_s) dd^c(v_{sk} - v_k) \wedge T &= \int_{\Omega} (v_{sk} - v_k) dd^c(h - h_s) \wedge T \\ &= \int_{\Omega} (v_{sk} - v_k) dd^c h \wedge T + \int_{\Omega} (v_k - v_{sk}) dd^c h_s \wedge T \\ &\leq \int_{\Omega} (v_{sk} - v_k) dd^c h \wedge T + 2\varepsilon \int_{\Omega} dd^c h_s \wedge T \\ &\leq \int_{\{v_k < v_s - 2\varepsilon\}} \|v_k\| dd^c h \wedge T + 2\varepsilon \int_{\Omega} dd^c(h + h_s) \wedge T \\ &\leq \int_{\{v_k < v_s - 2\varepsilon\} \setminus U} dd^c h \wedge T + \int_U dd^c h \wedge T + 4\varepsilon C. \end{aligned}$$

To the first term on the right one can apply Lemma 2.6, since if $\rho \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ is chosen so that $\rho \leq v_k$, $k = 1, 2, \dots$, and $\rho = 0$ on the boundary of Ω , then $\{v_k < v_s - 2\varepsilon\} \subset \{\rho < -2\varepsilon\} \Subset \Omega$. Furthermore, for $t < \varepsilon + 1$ we have by (2.18), $v_s - 2\varepsilon < v - \varepsilon < tv$ away from U .

The second term is bounded from above by

$$\int_U (dd^c(h + v_s + v_k))^n \leq 3^n \text{cap}(U, \Omega) < \varepsilon.$$

Therefore, the right hand side does not exceed $4\varepsilon(C+1)$ for $k > k_1 > k_0$ provided k_1 is sufficiently large. We write this in the following form

$$(2.19) \quad \sup_{s>s_0} \left[\int_{\Omega} (h - h_s) dd^c v_{sk} \wedge T - \int_{\Omega} (h - h_s) dd^c v_k \wedge T \right] \leq 4\varepsilon(C+1), \quad k > k_1,$$

and proceed to estimate the first integral of (2.19) from below. Since $h_s < h + \varepsilon$ away from U (see (2.18)), we get

$$\begin{aligned} (2.20) \quad & \int_{\Omega} (h - h_s) dd^c v_{sk} \wedge T \\ & \geq a_m \int_{E'_s(a_m)} dd^c v_{sk} \wedge T - \varepsilon \int_{\Omega \setminus U} dd^c v_{sk} \wedge T - \int_U dd^c v_{sk} \wedge T \\ & \geq a_m \int_{E'_s(a_m)} dd^c v_{sk} \wedge T - \varepsilon(C+1) \end{aligned}$$

(because if we replace one of the $(1,1)$ currents from the integrand of (2.17) by $dd^c v_{sk}$ the inequality remains valid). Furthermore,

$$\begin{aligned} (2.21) \quad & \int_{E'_s(a_m)} dd^c v_{sk} \wedge T = \int_{E'_s(a_m) \cap \{v_s > v_k - 2\varepsilon\}} dd^c v_s \wedge T \\ & \quad + \int_{E'_s(a_m) \cap \{v_s \leq v_k - 2\varepsilon\}} dd^c v_{sk} \wedge T \\ & \geq \int_{E'_s(a_m)} dd^c v_s \wedge T - \int_U dd^c v_s \wedge T \\ & \quad - \int_{(E'_s(a_m) \cap \{v_s \leq v_k - 2\varepsilon\}) \setminus U} dd^c v_s \wedge T. \end{aligned}$$

To the last integral we again apply Lemma 2.6 with the index s in place of k . It is possible since if $t < \varepsilon + 1$, then (see (2.18)) $\{v_s \leq v_k - 2\varepsilon\} \setminus U \subset \{v_s < v - \varepsilon\} \subset \{v_s < tv\}$. Moreover, as explained above, the sets $\{v_s \leq v_k - 2\varepsilon\}$ are contained in a compact subset of Ω . Thus, given k one can find $s(k)$ such that for $s > s(k)$,

$$\int_{(E'_s(a_m) \cap \{v_s \leq v_k - 2\varepsilon\}) \setminus U} dd^c v_s \wedge T \leq \varepsilon.$$

This inequality together with (2.21), $\int_U dd^c v_s \wedge T < \varepsilon$, and the induction hypothesis yield

$$\int_{E'_s(a_m)} dd^c v_{sk} \wedge T \geq A_m - 2\varepsilon, \quad s > s(k),$$

and further, via (2.20),

$$\int_{\Omega} (h - h_s) dd^c v_{sk} \wedge T \geq a_m(A_m - 2\varepsilon) - \varepsilon(C + 1).$$

Returning now to (2.19), we obtain, using the inequality above for ε small enough,

$$\int_{\Omega} (h - h_s) dd^c v_k \wedge T \geq \frac{a_m A_m}{2}, \quad k > k_1, s > s(k).$$

On the other hand, by (2.17) one gets for $d > 0$,

$$\begin{aligned} \int_{\Omega} (h - h_s) dd^c v_k \wedge T &\leq \int_{\{h_s < h-d\}} dd^c v_k \wedge T + d \int_{\Omega} dd^c v_k \wedge T \\ &\leq \int_{\{h_s < h-d\}} dd^c v_k \wedge T + dC. \end{aligned}$$

If we set $d = a_m A_m / 4C =: a_{m+1}$, then the last two formulae yield

$$\int_{E'_s(a_{m+1})} dd^c v_k \wedge T \geq \frac{a_m A_m}{4} := A_{m+1}, \quad k > k_1, s > s(k),$$

which concludes the proof of the inductive step and that of the lemma. $\square$

End of the proof of Theorem A. By Lemma 2.7, for fixed $k > k_1$ we have

$$\int_{E'_s(a_n)} (dd^c v_k)^n > A_n \quad \text{if } s > s(k).$$

If we choose k large enough then by Lemma 2.6 and the inclusion $\{w_k \leq u_K\} \subset \{w_k \leq tv\}$, $t > 1$ the following inequality is also satisfied:

$$\int_{\{w_k \leq u_K\}} (dd^c v_k)^n < \frac{A_n}{2}.$$

The two inequalities above lead to

$$\begin{aligned} \int_{\{w_k > u_K\} \cap E'_s(a_n)} (dd^c w_k)^n &= \int_{\{w_k > u_K\} \cap E'_s(a_n)} (dd^c v_k)^n \\ &= \int_{E'_s(a_n)} (dd^c v_k)^n - \int_{\{w_k \leq u_K\} \cap E'_s(a_n)} (dd^c v_k)^n \\ &\geq A_n/2, \quad s > s(k). \end{aligned}$$

Since $(dd^c w_k)^n$ is smooth, then for some $M > 0$ (independent of s) the following inequality holds $(dd^c w_k)^n \leq M\beta^n$. Therefore,

$$\lambda(E'_s(a_n)) = \frac{1}{n!} \int_{E'_s(a_n)} \beta^n > \frac{A_n}{2Mn!},$$

which contradicts the fact that $h_s \rightarrow h$ in L^1_{loc} . The proof of Theorem A is completed. $\square$

Remark. The argument above remains valid for any pseudoconvex Ω .

Proof. Let us first note that its statement holds true for upper semicontinuous boundary data as well if one replaces $\lim$ by $\limsup$ in the boundary condition of (*). Indeed, if $\varphi_j \downarrow \varphi$ is a sequence of continuous functions on $\partial\Omega$, then the corresponding solutions u_j decrease to a psh function u , which by Theorem 1.1 solves (*).

Let us now consider an exhaustion sequence of smooth strictly pseudoconvex sets $\Omega_j \uparrow \Omega$. In each Ω_j we solve (*) with boundary data equal to $v|_{\partial\Omega_j}$ and denote the resulting function by u_j . Then, by the Comparison Principle $u_j \geq v$ on $\partial\Omega_{j-1}$. So, again applying the Comparison Principle, $u_j \geq u_{j-1}$ on Ω_{j-1} . It now follows from Theorem 1.1 that $u = (\sup u_j)^*$ is the solution of (*) in Ω . $\square$

Corollary 2.8. If Ω is strictly pseudoconvex and $v_1, \dots, v_n \in \text{PSH} \cap L^\infty(\Omega)$, then there exists $u \in \text{PSH} \cap L^\infty(\Omega)$ satisfying any prescribed continuous boundary data and such that $(dd^c u)^n = dd^c v_1 \wedge \dots \wedge dd^c v_n$.

Proof. Since $v_1 + \dots + v_n$ becomes a subsolution for the measure in the statement after adding a suitable psh function to meet the boundary condition one can apply Theorem A. $\square$

The Monge-Ampère operator can be defined in a reasonable way for some classes of unbounded functions as well (see, e.g., [B]). In particular, $(dd^c v)^n$ is well defined if v is locally bounded away from a compact subset of Ω (see [D], [P]).

Corollary 2.9. If $v \in \text{PSH}(\Omega)$; $v \in L^\infty_{\text{loc}}(\Omega \setminus E)$, $E \Subset \Omega$ and if a Borel measure μ satisfies $\mu(E) = 0$, $d\mu \leq (dd^c v)^n$, then there exists a solution $u \geq v$ of (*) (except that u is not necessarily bounded).

Proof. Denote by U_s the set $\{v < -s\}$, $s = 1, 2, \dots$, and solve the Dirichlet problem (*) for measures $\mu_s := \mu|_{\Omega \setminus U_s}$. The solutions denote by u_s . Then by the Comparison Principle $u_s \geq \max(v, -s)$ and so $u = \lim \downarrow u_s$ is the desired solution. $\square$

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