Unique Solutions of 2×2 Conservation Laws with Large Data

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ABSTRACT. For a 2×2 hyperbolic system of conservation laws, we first consider a Riemann problem with arbitrarily large data.

A stability assumption is introduced, which yields the existence of a Lipschitz semigroup of solutions, defined on a domain containing all suitably small BV perturbations of the Riemann data.

We then establish a uniqueness result for large BV solutions, valid within the same class of functions where a local existence theorem can be proved.

1. Introduction. This paper is concerned with the local existence and uniqueness of weak solutions to the Cauchy problem for a 2×2 hyperbolic system of conservation laws:

$$(1.1) \quad u_t + [F(u)]_x = 0,$$

$$(1.2) \quad u(0, x) = \bar{u}(x),$$

assuming that the total variation of $\bar{u}$ is bounded but possibly large.

In order to study the continuous dependence of solutions of (1.1) on the initial data, a new approach was recently introduced in [2], [3], [4]. Its main steps are:

- (i) Within a suitable class of functions with bounded variation, construct a Lipschitz continuous flow compatible with the self-similar solutions to the Riemann problems.
- (ii) Prove that such a flow is necessarily unique.
- (iii) Characterize the trajectories of this flow as solutions of (1.1) in a suitable *viscosity* sense.

In [4], the authors constructed a globally Lipschitz continuous semigroup of solutions of (1.1), defined on a set of functions with sufficiently small total variation. In the first part of the present paper, we prove the existence of a Lipschitz semigroup defined on a more general domain $\mathcal{D}$, containing all functions which are sufficiently close (in the norm of total variation) to a given Riemann data

$$(1.3) \qquad u(0, x) = \begin{cases} u^b & \text{if } x < 0, \\ u^\# & \text{if } x > 0. \end{cases}$$

A linearized stability condition on the Riemann problem (1.1)–(1.3) will be assumed.

The trajectories of the semigroup are obtained as limits of piecewise constant approximations, using the wave-front tracking algorithm developed in [4]. As in [4], the uniform Lipschitz continuity is proved by introducing a Riemann-type metric, equivalent to the usual $\mathbf{L}^1$ distance, which is non-increasing w.r.t. the flow generated by (1.1). The coefficients of this metric, however, must be chosen in a quite different way, since the domain $\mathcal{D}$ of our semigroup now contains functions with large total variation. As soon as a Lipschitz semigroup has been constructed, its uniqueness and the characterization of its trajectories as *viscosity solutions* of (1.1) can be proved exactly as in [3].

Relying on the finite propagation speed of the system, we then establish a local existence and uniqueness theorem for general initial data $\bar{u} \in \text{BV}$, provided that at every large jump the corresponding Riemann problem satisfies our linearized stability conditions. We remark that, in the standard literature, the existence of solutions is proved in the space of BV functions [8], [12], [14], while uniqueness is established only within a smaller class of functions satisfying additional regularity conditions [6], [7], [9], [11]. A major feature of the present paper is that the existence and the uniqueness of solutions are both proved within the same class of BV functions.

The local existence of solutions for large BV initial data was established in [1] for the equations of isentropic gas dynamics and in [13] for general $n \times n$ systems. In the case of a 2×2 Riemann problem solved in terms of two shocks, our stability condition (2.12) is somewhat stronger than the corresponding one in [13]. In Section 8 we show that the assumption (2.12) is sharp: if the opposite inequality holds, local solutions to (1.1) may still exist in BV, but they do not

depend Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm. This section is also concerned with the equations of isentropic gas dynamics in Lagrange coordinates [12], [14]:

$$(1.4) \quad (u_1)_t - (u_2)_x = 0, \quad (u_2)_t + [p(u_1)]_x = 0,$$

assuming $p' < 0$, $p'' > 0$. We prove that, if a given Riemann problem is solved without the appearance of the vacuum state, then the stability condition (2.12) holds.

2. Notations and Main Results. In the following, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the euclidean norm and inner product on $\mathbb{R}^2$, respectively. Let $\Omega \subset \mathbb{R}^2$ be an open convex set, let $F : \Omega \mapsto \mathbb{R}^2$ be a smooth vector field and call $A(u) \doteq DF(u)$ the Jacobian matrix of F at u . Throughout this paper, we assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. For $u, u' \in \Omega$, define the averaged matrix

$$(2.1) \quad A(u, u') \doteq \int_0^1 A(\vartheta u + (1 - \vartheta)u') d\vartheta.$$

For $i = 1, 2$, let $\lambda_i(u, u')$, $r_i(u, u')$ and $l_i(u, u')$ be respectively the i^{th} eigenvalue and a right and left i^{th} eigenvectors of $A(u, u')$, satisfying

$$(2.2) \quad \langle l_i, r_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

When $u' = u$, we of course have $A(u) = A(u, u)$, and write $\lambda_i(u)$, $r_i(u)$, $l_i(u)$ for the eigenvalues and eigenvectors of $A(u)$. For simplicity, we shall assume that

$$(2.3) \quad -\lambda^* \leq \lambda_1(u, u') \leq -\lambda_* < 0 < \lambda_* \leq \lambda_2(u, u') \leq \lambda^* \quad \forall u, u' \in \Omega,$$

for some positive constants λ_* , λ^* .

By the Rankine-Hugoniot equations, two states $u^b, u^h \in \Omega$ are joined by a shock of the first characteristic family iff they satisfy the scalar equation

$$(2.4) \quad \Phi_2(u^b, u^h) \doteq \langle l_2(u^b, u^h), u^h - u^b \rangle = 0.$$

Similarly, the states $u^h, u^\sharp$ are connected by a 2-shock iff

$$(2.5) \quad \Phi_1(u^h, u^\sharp) \doteq \langle l_1(u^h, u^\sharp), u^\sharp - u^h \rangle = 0.$$

The differentials of Φ_i w.r.t. its first and second argument will be written as $D_1\Phi_i$, $D_2\Phi_i$, respectively. For example,

$$D_1\Phi_2(u^b, u^\sharp) \cdot \mathbf{v} \doteq \lim_{\varepsilon \rightarrow 0} \frac{\Phi_2(u^b + \varepsilon \mathbf{v}, u^\sharp) - \Phi_2(u^b, u^\sharp)}{\varepsilon}.$$

We say that the 1-shock joining u^b , $u^\sharp$ is *stable* provided that

$$(2.6) \quad D_2\Phi_2(u^b, u^\sharp) \cdot r_2(u^\sharp) \neq 0.$$

Similarly, we say that a 2-shock joining $u^\sharp$, $u^\flat$ is *stable* if

$$(2.7) \quad D_1\Phi_1(u^\sharp, u^\flat) \cdot r_1(u^\flat) \neq 0.$$

Now assume that the Riemann problem (1.1), (1.3) admits a solution with two shocks, having $u^\sharp$ as middle state. Consider an infinitesimally small wave-front bouncing back and forth between the shocks (Fig. 1a). At a time t_1 where the front interacts with the 1-shock, the ratio between the size of the outgoing 2-wave and the incoming 1-wave is given by

$$(2.8) \quad \kappa_1 \doteq - \frac{D_2\Phi_2(u^b, u^\sharp) \cdot r_1(u^\sharp)}{D_2\Phi_2(u^b, u^\sharp) \cdot r_2(u^\sharp)}$$

(for a detailed analysis of first order perturbations for piecewise Lipschitz solutions of (1.1), see [5]). Similarly, at a time t_2 where the front hits the 2-shock, the ratio between the outgoing 1-wave and the incoming 2-wave is computed by

$$(2.9) \quad \kappa_2 \doteq - \frac{D_1\Phi_1(u^\sharp, u^\flat) \cdot r_2(u^\sharp)}{D_1\Phi_1(u^\sharp, u^\flat) \cdot r_1(u^\sharp)}.$$

Stability of the Cauchy problem w.r.t. small perturbations of the Riemann data requires

$$(2.10) \quad |\kappa_1 \kappa_2| < 1.$$

Otherwise, small wave-fronts bouncing back and forth several times between the two large shocks could increase their size indefinitely. A stronger assumption, however, will be needed to ensure the Lipschitz continuity of solutions w.r.t. the initial data. Indeed, consider a small wave-front located between the two shocks. If σ_0 denotes its strength at an initial time t_0 , after two reflections its strength will be $\sigma = |\kappa_1 \kappa_2| \cdot \sigma_0$ (neglecting higher order terms). If the initial location of the front is shifted in the x -direction by a small quantity Δx_0 , an elementary

Figure 1a

Figure 1b

computation shows that, after two reflections (Fig. 1b), the position of the wave-front will be shifted by the amount $\Delta x = |\Theta_1 \Theta_2| \cdot \Delta x_0$, with

$$(2.11) \quad \Theta_1 = \frac{\Lambda_1 - \lambda_2(u^\sharp)}{\Lambda_1 - \lambda_1(u^\sharp)}, \quad \Theta_2 = \frac{\Lambda_2 - \lambda_1(u^\sharp)}{\Lambda_2 - \lambda_2(u^\sharp)}.$$

Here $\Lambda_1 = \lambda_1(u^b, u^\sharp)$ and $\Lambda_2 = \lambda_2(u^\sharp, u^\sharp)$ are the speeds of the two large shocks. Since the number of reflections may be arbitrarily large, a natural condition to ensure stability in the $\mathbf{L}^1$ norm is

$$(2.12) \quad |\kappa_1 \Theta_1| \cdot |\kappa_2 \Theta_2| < 1.$$

Observe that (2.12) implies (2.10), because $\Theta_1, \Theta_2 < -1$. We also remark that, by genuine nonlinearity

$$(2.13) \quad \lambda_1(u^b, u^\sharp) - \lambda_1(u^\sharp) \geq c|u^\sharp - u^b|, \quad \lambda_2(u^\sharp) - \lambda_2(u^\sharp, u^\sharp) \geq c|u^\sharp - u^\sharp|$$

for some constant $c > 0$. This, together with the estimates

$$|\kappa_1| \leq C|u^\sharp - u^b|^2, \quad |\kappa_2| \leq C|u^\sharp - u^\sharp|^2,$$

implies that the condition (2.12) certainly holds if at least one of the quantities $|u^\sharp - u^b|$, $|u^\sharp - u^\sharp|$ is sufficiently small. In addition to the Rankine-Hugoniot

equations, we assume that genuinely nonlinear shocks satisfy the usual admissibility conditions

$$(2.14) \quad \lambda_1(u^b) > \lambda_1(u^b, u^{\natural}) > \lambda_1(u^{\natural}), \quad \lambda_2(u^{\natural}) > \lambda_2(u^{\natural}, u^{\sharp}) > \lambda_2(u^{\sharp}).$$

Our first result states the existence of a semigroup of entropic solutions for (1.1), whose domain contains all suitably small perturbations of a Riemann data. More precisely, let $\tilde{u} = \tilde{u}(t, x)$ be the self-similar solution to the standard Riemann problem (1.1)–(1.3) and consider the family $\mathcal{F}$ of all smooth increasing diffeomorphisms $\varphi : \mathbb{R} \mapsto \mathbb{R}$. For a given $\delta > 0$, define the domain

$$(2.15) \quad \tilde{\mathcal{D}}_{\rho} \doteq \left\{ u : \mathbb{R} \mapsto \mathbb{R}^2; \text{ there exists } \varphi \in \mathcal{F} \text{ such that} \right. \\ \left. \|u(\cdot) - \tilde{u}(1, \varphi(\cdot))\|_{\mathbf{L}^1} < \infty, \text{ Tot.Var. } \{u(\cdot) - \tilde{u}(1, \varphi(\cdot))\} \leq \rho \right\}.$$

Theorem 1. *Let $F : \Omega \mapsto \mathbb{R}^2$ be as above, and let the Riemann problem (1.1)–(1.3) admit a self-similar solution $\tilde{u} = \tilde{u}(t, x)$. Assume that*

- (A1) *If $\tilde{u}$ contains a 1-shock or a 2-shock, the corresponding stability conditions (2.6)–(2.7) hold, together with (2.14).*
- (A2) *If two shocks are present, the “non-resonance” condition (2.12) holds.*

Then there exist constants $L, \rho > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}_{\text{loc}}^1$ and a continuous semigroup $S : [0, \infty[\times \mathcal{D} \mapsto \mathcal{D}$ such that, for all $\bar{u}, \bar{v} \in \mathcal{D}$ and $s, t \geq 0$, one has

- (i) $\tilde{\mathcal{D}}_{\rho} \subset \mathcal{D}$,
- (ii) $S_0 \bar{u} = \bar{u}$, $S_t(S_s \bar{u}) = S_{t+s} \bar{u}$,
- (iii) $\|S_t \bar{u} - S_s \bar{w}\|_{\mathbf{L}^1} \leq L(|t - s| + \|\bar{u} - \bar{w}\|_{\mathbf{L}^1})$,
- (iv) *each trajectory $t \mapsto S_t \bar{u}$ is a weak, entropy-admissible solution of (1.1),*
- (v) *if $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t, \cdot) = S_t \bar{u}$ coincides with the solution of (1.1)–(1.2) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.*

To characterize the trajectories of the semigroup S constructed in Theorem 1, we recall here the definition of *viscosity solution* of (1.1), introduced in [2].

Let $u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}^n$ be a locally integrable function with $u(t, \cdot) \in \text{BV}$ for each t , and fix any point (τ, ξ) in the domain of u . Call $\omega = \omega(t, x)$ the self-similar solution of the Riemann problem

$$(2.16) \quad \omega_t + [F(\omega)]_x = 0, \quad \omega(0, x) = \begin{cases} u(\tau, \xi -) & \text{if } x < 0, \\ u(\tau, \xi +) & \text{if } x > 0. \end{cases}$$

Let λ^* be an upper bound for all characteristic speeds, as in (2.3). For $t > \tau$, define

$$U_{(u;\tau,\xi)}^\#(t,x) \doteq \begin{cases} \omega(t-\tau, x-\xi) & \text{if } |x-\xi| \leq \lambda^*(t-\tau), \\ u(\tau, x) & \text{if } |x-\xi| > \lambda^*(t-\tau). \end{cases}$$

The function $t \mapsto U_{(u;\tau,\xi)}^\#(t, \cdot)$ is then Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, and approaches $u(\tau, \cdot)$ as $t \rightarrow \tau+$.

Next, call $\hat{A} \doteq DF(u(\tau, \xi))$ the Jacobian matrix of F computed at the point $u(\tau, \xi)$. For $t > \tau$, define $U_{(u;\tau,\xi)}^b(t, x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + \hat{A}w_x = 0, \quad w(\tau, x) = u(\tau, x).$$

In the following, $\text{Tot.Var.}\{u(\tau); I\}$ denotes the total variation of the function $u(\tau, \cdot)$ over the set I .

Definition 1. Let $u : [0, T] \mapsto \text{BV}$ be continuous w.r.t. the topology of $\mathbf{L}_{\text{loc}}^1$. We say that u is a *viscosity solution* of the system of conservation laws if there exists a constant $C > 0$ such that, at each point $(\tau, \xi) \in [0, T] \times \mathbb{R}$, for all $\rho, \varepsilon > 0$ sufficiently small one has,

$$(2.17) \quad \frac{1}{\varepsilon} \int_{\xi-\rho+\varepsilon\lambda^*}^{\xi+\rho-\varepsilon\lambda^*} |u(\tau+\varepsilon, x) - U_{(u;\tau,\xi)}^\#(\tau+\varepsilon, x)| dx$$

$$\leq C \cdot \text{Tot.Var.}\{u(\tau);]\xi-\rho, \xi[\cup]\xi, \xi+\rho[\},$$

$$(2.18) \quad \frac{1}{\varepsilon} \int_{\xi-\rho+\varepsilon\lambda^*}^{\xi+\rho-\varepsilon\lambda^*} |u(\tau+\varepsilon, x) - U_{(u;\tau,\xi)}^b(\tau+\varepsilon, x)| dx$$

$$\leq C \cdot (\text{Tot.Var.}\{u(\tau);]\xi-\rho, \xi+\rho[\})^2.$$

If also $u(0, \cdot) = \bar{u}$, then we say that u is a viscosity solution of the Cauchy problem (1.1)–(1.2).

If $\mathcal{D}$ is the domain of the semigroup S constructed in Theorem 1, the same arguments used in [2] now show that a continuous map $u : [0, T] \mapsto \mathcal{D}$ is a viscosity solution of (1.1)–(1.2) if and only if

$$(2.19) \quad u(t, \cdot) = S_t \bar{u} \quad \forall t \geq 0.$$

Thanks to the finite propagation speed, from the previous results we shall obtain a local existence and uniqueness theorem for viscosity solutions with large BV data. At this stage, however, a technical difficulty should be pointed out. Indeed, the characterization (2.19) for a viscosity solution u was proved under

the crucial assumption that all values $u(t, \cdot)$ lie within the domain $\mathcal{D}$ of a given semigroup. On the other hand, functions with large total variation may not be contained in any such domain. We thus need to replace the assumption $u(t, \cdot) \in \mathcal{D}$ with a weaker condition. Roughly speaking, we shall require that, in a forward neighborhood of each point $(\bar{t}, \bar{x})$, the restriction of $u(t, \cdot)$ lies within the domain of some semigroup, possibly depending on $\bar{t}$, $\bar{x}$. A more precise condition is formulated below.

Let $u : \mathbb{R} \mapsto \mathbb{R}^2$ have bounded variation. Then $\mu \doteq u_x$ is a vector measure, which can be decomposed into a continuous and an atomic part: $\mu = \mu^c + \mu^a$. For $i = 1, 2$ we define the signed measure $\mu_i = \mu_i^c + \mu_i^a$ as follows. The continuous part of μ_i is the Radon measure such that

$$\int \varphi d\mu_i^c \doteq \int l_i(u) \cdot \varphi d\mu^c$$

for every scalar continuous function φ with compact support. The atomic part of μ_i is the measure concentrated on the countable set $\{x_\alpha; \alpha = 1, 2, \dots\}$ where u has a jump, and $\mu_i^a(\{x_\alpha\})$ is the strength of the i^{th} wave in the solution of the Riemann problem with data $u(x_\alpha+)$, $u(x_\alpha-)$. Of course, we assume here that a basis of left eigenvectors $l_i(u)$ has already been selected, continuously depending on u , and that all of the Riemann problems above are uniquely solvable.

Call μ_i^+ , μ_i^- the positive and negative parts of the signed measure μ_i , so that

$$|\mu_i| = \mu_i^+ + \mu_i^-, \quad \mu_i = \mu_i^+ - \mu_i^-.$$

We then define the *interaction potential* among the waves in u as

$$\begin{aligned} (2.20) \quad Q(u) \doteq & (|\mu_2| \times |\mu_1|)(\{(x, y) \in \mathbb{R}^2; x < y\}) \\ & + (\mu_1^- \times |\mu_1|)(\{(x, y) \in \mathbb{R}^2; x \neq y\}) \\ & + (\mu_2^- \times |\mu_2|)(\{(x, y) \in \mathbb{R}^2; x \neq y\}). \end{aligned}$$

Observe that, when u is piecewise constant, the definition above coincides with the standard one. For convenience, a wave of negative size is here considered to be approaching every other wave of the same family, even in the linearly degenerate case.

Now fix any point $(\bar{t}, \bar{x})$ and consider the forward triangular neighborhood

$$(2.21) \quad \Delta \doteq \{(t, x); t \geq \bar{t}, x \in [\bar{x} - \rho + \lambda^*(t - \bar{t}), \bar{x} + \rho - \lambda^*(t - \bar{t})]\},$$

for $\rho > 0$ small. Given any function $u = u(t, x)$ defined on a domain containing Δ , we set

$$(2.22) \quad u_\Delta(t, x) \doteq \begin{cases} u(\bar{t}, \bar{x}-) & \text{if } (t, x) \notin \Delta \text{ and } x < \bar{x}, \\ u(t, x) & \text{if } (t, x) \in \Delta, \\ u(\bar{t}, \bar{x}+) & \text{if } (t, x) \notin \Delta \text{ and } x \geq \bar{x}. \end{cases}$$

We now introduce an assumption, stating that the interaction potential of the truncated functions u_Δ is small, in a forward neighborhood of each point. More precisely:

- (A3) At every point $(\bar{t}, \bar{x})$, for every $\varepsilon > 0$, there exist $\rho > 0$, $\rho' \in]0, \rho/\lambda^*]$ small enough such that the corresponding function u_Δ in (2.22) satisfies

$$Q(u_\Delta(t, \cdot)) < \varepsilon \quad \forall t \in [\bar{t}, \bar{t} + \rho'].$$

The condition that “ u locally lies within the domain of some semigroup” will be derived as a consequence of (A3). Using the condition above, an existence and a uniqueness theorem for solutions of the system (1.1) with large data can now be stated.

Theorem 2. *Let $F : \Omega \mapsto \mathbb{R}^2$ be as above. Let $\bar{u} : \mathbb{R} \mapsto \mathbb{R}^2$ have bounded variation and take values within a compact subset $K \subset \Omega$. Assume that at each point x where $\bar{u}$ has a jump, the corresponding Riemann problem admits a unique self-similar solution, satisfying the conditions (2.6), (2.7), (2.14) and (2.12) in case of shocks. Then the Cauchy problem (1.1)–(1.2) admits a viscosity solution satisfying (A3), defined on $[0, T] \times \mathbb{R}$, for some $T > 0$ sufficiently small.*

Theorem 3. *Let $u : [0, T] \mapsto \text{BV}$ be a viscosity solution of (1.1)–(1.2), satisfying (A3). Assume that, at each point (t, x) where u has a jump, the Riemann problem with data $u(t, x+)$, $u(t, x-)$ is stable and non-resonant, so that (A1) and (A2) hold. If v is a second solution with all of the properties above, then $u = v$ on $[0, T] \times \mathbb{R}$.*

3. Outline of the proof of Theorem 1. To prove Theorem 1, three different cases will be considered, depending on whether the Riemann problem (1.1)–(1.3) with large data admits a self-similar solution consisting of

- (RR) two rarefaction waves,
- (SS) two shocks,
- (SR) a shock (say, of the first family), and a rarefaction wave (say, of the second family).

Let $u^\sharp$ be the intermediate state occurring in the solution of (1.1)–(1.3). For a fixed $\varepsilon > 0$, a family of piecewise constant approximate solutions will be constructed by the wave-front tracking algorithm introduced in [4], which is now briefly described.

As a first step, we construct local Riemann coordinates $v = (v_1, v_2)$, so that a basis $\{r_1, r_2\}$ of right eigenvectors for the Jacobian matrix $A(u) = DF(u)$ in these coordinates takes the form $r_1 \equiv (1, 0)$, $r_2 \equiv (0, 1)$. More precisely, in Case (RR), we construct a single set of Riemann coordinates, defined on an open

Figure 2a

connected neighborhood of the two rarefaction curves joining u^b with $u^\sharp$ and $u^\sharp$ with $u^\sharp$.
In Case (SS), we choose three sets of Riemann coordinates, defined on disjoint neighborhoods of the points u^b , $u^\sharp$, $u^\sharp$. In Case (SR), we choose two sets of Riemann coordinates, defined on a neighborhood of the point u^b and on a neighborhood of the rarefaction curve joining $u^\sharp$ with $u^\sharp$, respectively (see Fig. 2a, b, c).

Figure 2b

Figure 2c

In a given set of Riemann coordinates, the i -rarefaction curve φ_i^+ and the i -shock curve φ_i^- through the point v can now be parametrized as

$$\begin{aligned} \varphi_1^+(v, \sigma) &= (v_1 + \sigma, v_2), & \varphi_1^-(v, \sigma) &= (v_1 + \sigma, v_2 + \hat{\varphi}_2(v, \sigma)\sigma^3), \\ \varphi_2^+(v, \sigma) &= (v_1, v_2 + \sigma), & \varphi_2^-(v, \sigma) &= (v_1 + \hat{\varphi}_1(v, \sigma)\sigma^3, v_2 + \sigma), \end{aligned} \tag{3.1}$$

for suitable functions $\hat{\varphi}_1, \hat{\varphi}_2$. Choose any C^∞ function $\varphi: \mathbb{R} \mapsto \mathbb{R}$ such that

$$\begin{cases} \varphi(s) = 1 & \text{if } s \leq -2, \\ \varphi'(s) \in [-2, 0] & \text{if } s \in [-2, -1], \\ \varphi(s) = 0 & \text{if } s \geq -1, \end{cases}$$

and, for a fixed $\varepsilon > 0$, define the interpolation between the i -shock and the i -rarefaction curve,

$$(3.2) \quad \psi_i^\varepsilon(v, \sigma) \doteq \varphi\left(\frac{\sigma}{\sqrt{\varepsilon}}\right) \cdot \varphi_i^-(v, \sigma) + \left(1 - \varphi\left(\frac{\sigma}{\sqrt{\varepsilon}}\right)\right) \cdot \varphi_i^+(v, \sigma), \quad i = 1, 2.$$

Given a right and a left state u^ℓ, u^r , assuming that they both belong to the domain of the same chart and have Riemann coordinates $v^r = (v_1^r, v_2^r)$, $v^\ell = (v_1^\ell, v_2^\ell)$, an approximate solution to the Riemann problem with initial data

$$\bar{u}(x) = \begin{cases} u^\ell & \text{if } x < 0, \\ u^r & \text{if } x > 0, \end{cases}$$

is constructed as follows.

First, using the implicit function theorem, we determine unique values σ_1 and σ_2 and a middle state v^m such that

$$(3.3) \quad v^r = \psi_2^\varepsilon(v^m, \sigma_2), \quad v^m = \psi_1^\varepsilon(v^\ell, \sigma_1).$$

If $\sigma_1 \geq 0$, then the states v^ℓ, v^m are connected by a rarefaction wave. Let the integers h, k be such that

$$h\varepsilon \leq v_1^\ell < (h+1)\varepsilon \quad k\varepsilon \leq v_1^m < (k+1)\varepsilon.$$

Introducing the states

$$\omega_1^j \doteq (j\varepsilon, v_2^\ell), \quad \hat{\omega}_1^j \doteq \left(\frac{2j+1}{2}\varepsilon, v_2^\ell\right), \quad j = h, \dots, k,$$

we construct the ε -approximate solution on the quadrant where $x \leq 0$ as a rarefaction fan:

$$(3.4) \quad v^\varepsilon(t, x) = \begin{cases} v^\ell & \text{if } x < \lambda_1(\hat{\omega}_1^h)t, \\ \omega_1^j & \text{if } \lambda_1(\hat{\omega}_1^{j-1})t < x < \lambda_1(\hat{\omega}_1^j)t, \\ v^m & \text{if } \lambda(\hat{\omega}_1^k) < x \leq 0. \end{cases} \quad j = h+1, \dots, k,$$

On the other hand, if $\sigma_1 < 0$, the states v^ℓ and v^m are connected by a single shock:

$$(3.5) \quad v^\varepsilon(t, x) \doteq \begin{cases} v^\ell & \text{if } x < \lambda_1^\varphi(v^\ell, \sigma_1)t, \\ v^m & \text{if } \lambda_1^\varphi(v^\ell, \sigma_1)t < x \leq 0. \end{cases}$$

The shock speed λ_1^φ is here defined as

$$\lambda_1^\varphi(v^\ell,\sigma_1) \doteq \varphi\left(\frac{\sigma_1}{\sqrt{\varepsilon}}\right) \cdot \lambda_1^s(v^\ell,\sigma_1) + \left(1 - \varphi\left(\frac{\sigma_1}{\sqrt{\varepsilon}}\right)\right) \cdot \lambda_1^r(v^\ell,\sigma_1),$$

with

$$\begin{aligned} \lambda_1^s(v^\ell,\sigma_1) &\doteq \lambda_1(v^\ell,\varphi_1^-(v^\ell,\sigma_1)), \\ \lambda_1^r(v^\ell,\sigma_1) &\doteq \sum_j \frac{\text{meas}([j\varepsilon,(j+1)\varepsilon] \cap [v_1^m,v_1^l])}{|\sigma_1|} \lambda_1(\hat{\omega}_1^j). \end{aligned}$$

Observe that the jump in (3.5) provides an exact solution to the Rankine-Hugoniot equations as soon as $\sigma_1 \leq -2\sqrt{\varepsilon}$. The construction of the ε -approximate solution on the quadrant where $x \geq 0$ is entirely similar, repeating the above construction with waves of the second family. See [4] for details.

On the other hand, if the data u^ℓ and u^r are not contained within the domain of a single chart, the Riemann problem is solved with the introduction of a large shock, satisfying the Rankine-Hugoniot equations and the entropy-admissibility conditions as well. For example, in Case (SS), if u^l lies in the neighborhood of $u^\natural$ and u^r lies in the neighborhood of $u^\sharp$, we choose a unique middle state u^m such that, in the corresponding coordinates v , one has

$$v^m = \psi_1^\varepsilon(v^l,\sigma_1)$$

for some σ_1 , while u^m and u^r are connected by a true shock of the second family. On the quadrant where $x < 0$ the piecewise constant ε -approximate solution is then constructed as in (3.4) or (3.5), according to the sign of σ_1 .

Let now $\bar{u}$ be a piecewise constant initial condition. An ε -approximate solution to the Cauchy problem (1.1)–(1.2), within the class of piecewise constant functions, is constructed as follows. At the initial time $\tau_0 = 0$ we solve the Riemann problems determined by the jumps of $\bar{u}$ applying the algorithm previously described. This yields a piecewise constant approximate solution $u = u^\varepsilon(t,x)$ defined up to the time $\tau_1 > 0$ where the first set of wave-front interactions takes place. We then solve these new Riemann problems by applying again the above algorithm. The solution is prolonged up to the time τ_2 where the second set of interactions takes place, etc....

We will show that, if the initial condition $\bar{u}$ lies in a suitable domain $\mathcal{D}_\delta^\varepsilon$, then the corresponding ε -approximate solution $u^\varepsilon : [0,\infty[\times\mathbb{R} \mapsto \mathbb{R}^2$ is well defined and satisfies $u^\varepsilon(t,\cdot) \in \mathcal{D}_\delta^\varepsilon$ for all $t \geq 0$. In particular, on any bounded time interval $[0,T]$ the number of wave-front interactions is finite, and each of the new Riemann problems determined by the interactions can be uniquely solved by the algorithm. The definition of the domain $\mathcal{D}_\delta^\varepsilon$ is given below, in the three

distinct cases. We shall always be concerned with piecewise constant functions $u = u(x)$ of the form

$$(3.6) \quad u = u^b \chi_{]-\infty, x_1]} + \sum_{\alpha=1}^{n-1} u^\alpha \chi_{]x_\alpha, x_{\alpha+1}]} + u^\sharp \chi_{]x_n, +\infty[}.$$

Whenever $u^{\alpha-1}$, u^α belong to the same chart and have coordinates $v^{\alpha-1}$, v^α , we shall assume that the Riemann problem determined by the jump at x_α can be uniquely solved by the algorithm above, in terms of waves with sizes $\sigma_{1,\alpha}$, $\sigma_{2,\alpha}$. Recalling (3.1)–(3.2), this means

$$(3.7) \quad v^\alpha = \psi_2^\varepsilon(\psi_1^\varepsilon(v^{\alpha-1}, \sigma_{1,\alpha}), \sigma_{2,\alpha}).$$

Case (RR). In this case, there exists a single set of Riemann coordinates covering a neighborhood of the rarefaction curve Γ_1 joining u^b with $u^\sharp$ and a neighborhood of the rarefaction curve Γ_2 joining $u^\sharp$ with $u^\sharp$. In particular, if b , q , d are the coordinates corresponding to the left, intermediate and right state in the solution of the Riemann problem (1.1)–(1.3), we can assume that the domain of the chart contains the set

$$(3.8) \quad \mathcal{U} \doteq \{(v_1, v_2); v_1 \in [b_1 - \delta_0, q_1 + \delta_0], |v_2 - b_2| \leq \delta_0\} \\ \cup \{(v_1, v_2); |v_1 - d_1| \leq \delta_0, v_2 \in [q_2 - \delta_0, d_2 + \delta_0]\}$$

for some $\delta_0 > 0$. Observe that $b_2 = q_2$ and $q_1 = d_1$.

A positively invariant set $\mathcal{D}_\delta^\varepsilon$ of piecewise constant functions is now constructed as follows. Define

$$(3.9) \quad \Upsilon(u) \doteq V(u) + Q(u), \\ V(u) \doteq \sum_{\alpha=1}^n \sum_{i=1}^2 |\sigma_{i,\alpha}|, \\ Q(u) \doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}} |\sigma_{i,\alpha} \sigma_{j,\beta}|,$$

where, as usual, $\mathcal{A}$ denotes the set of all couples of approaching waves. Here and in the following, in connection with a fixed system of Riemann coordinates, we can regard Υ , V , Q alternatively as functions of u or as functions of the coordinates $v = (v_1(u), v_2(u))$. The wave sizes $\sigma_{i,\alpha}$ will be always defined by (3.7), in terms of the coordinates v . In connection with (3.8)–(3.9), define

$$(3.10) \quad \mathcal{D}_\delta^\varepsilon \doteq \left\{ u \text{ as in (3.6); } v^\alpha \in \mathcal{U} \text{ for each } \alpha, \Upsilon(u) \leq \tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta, Q(u) \leq \frac{\delta}{2} \right\},$$

where $\tilde{\sigma}_1 = q_1 - b_1$ and $\tilde{\sigma}_2 = d_2 - q_2$ are the sizes of the first and second rarefaction waves in the solution of (1.1)–(1.3). Observe that, for fixed states $v^{\alpha-1}$, v^α , the wave sizes $\sigma_{i,\alpha}$ in (3.7) actually depend on ε . In turn, the functions Υ , V , Q in (3.9) depend on ε through these wave strengths, and so does the set $\mathcal{D}_\delta^\varepsilon$.

Case (SS). We begin by introducing three sets of Riemann coordinates on disjoint neighborhoods of u^b , $u^\natural$ and $u^\sharp$, respectively. We can assume that, for some $\delta_0 > 0$, the domains of these charts contain the disjoint sets

$$(3.11) \quad \begin{aligned} \mathcal{U}^b &= \{(v_1, v_2); |v_i - v_i^b| \leq \delta_0, i = 1, 2\}, \\ \mathcal{U}^\natural &= \{(v_1, v_2); |v_i - v_i^\natural| \leq \delta_0, i = 1, 2\}, \\ \mathcal{U}^\sharp &= \{(v_1, v_2); |v_i - v_i^\sharp| \leq \delta_0, i = 1, 2\}. \end{aligned}$$

Let u be as in (3.6) and assume that the corresponding coordinates v^α satisfy

$$(3.12) \quad v^0, \dots, v^{\alpha^b-1} \in \mathcal{U}^b, \quad v^{\alpha^b}, \dots, v^{\alpha^\natural-1} \in \mathcal{U}^\natural, \quad v^{\alpha^\natural}, \dots, v^{\alpha^\sharp} \in \mathcal{U}^\sharp.$$

The piecewise constant function u thus contains several small jumps, together with two large jumps connecting u^{α^b-1} with u^{α^b} and $u^{\alpha^\natural-1}$ with $u^{\alpha^\sharp}$. We assume that:

- (i) For every $\alpha \neq \alpha^b, \alpha^\natural$, the Riemann problem at x_α is solved according to (3.7).
- (ii) The Riemann problem at x_{α^b} is solved in terms of a middle state u_* such that the following holds. The states u^{α^b-1} and u_* are connected by an admissible 1-shock satisfying the Rankine-Hugoniot conditions, while u_* and u^{α^b} are connected by a 2-wave of size σ_{2,α^b} . The corresponding coordinates $v_*, v^{\alpha^b} \in \mathcal{U}^\natural$ satisfy

$$v^{\alpha^b} = \psi_2^\varepsilon(v_*, \sigma_{2,\alpha^b}).$$

- (iii) The Riemann problem at $x_{\alpha^\natural}$ is solved in terms of a middle state u^* such that the following holds. The states u^* and $u^{\alpha^\natural}$ are connected by an admissible 2-shock satisfying the Rankine-Hugoniot conditions, while $u^{\alpha^\natural-1}$ and u^* are connected by a 1-wave of size $\sigma_{1,\alpha^\natural}$. The corresponding coordinates $v^{\alpha^\natural-1}, v^* \in \mathcal{U}^\sharp$ satisfy

$$v^* = \psi_2^\varepsilon(v^{\alpha^\natural-1}, \sigma_{1,\alpha^\natural}).$$

We shall consider separately the strength of waves and the interaction potential in the regions on the left, in the middle and on the right of the two large jumps. More precisely, we define

$$\begin{aligned}
 (3.13) \quad V^b(u) &\doteq \sum_{\alpha=1}^{\alpha^b-1} \sum_{i=1}^2 |\sigma_{i,\alpha}|, & Q^b(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^b} |\sigma_{i,\alpha} \sigma_{j,\beta}| \\
 V^{\natural}(u) &\doteq \sum_{\alpha=\alpha^b}^{\alpha^\sharp} \sum_{i=1}^2 C_i |\sigma_{i,\alpha}|, & Q^{\natural}(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^{\natural}} C_i C_j |\sigma_{i,\alpha} \sigma_{j,\beta}| \\
 V^{\sharp}(u) &\doteq \sum_{\alpha=\alpha^\sharp+1}^n \sum_{i=1}^2 |\sigma_{i,\alpha}|, & Q^{\sharp}(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^{\sharp}} |\sigma_{i,\alpha} \sigma_{j,\beta}| \\
 \Upsilon^b(u) &\doteq V^b(u) + Q^b(u), & \Upsilon^{\natural}(u) &\doteq V^{\natural}(u) + Q^{\natural}(u), & \Upsilon^{\sharp}(u) &\doteq V^{\sharp}(u) + Q^{\sharp}(u)
 \end{aligned}$$

To simplify the notation, the convention $\sigma_{1,\alpha^b} = \sigma_{2,\alpha^\sharp} = 0$ is here used. C_1 and C_2 positive constants whose precise value will be determined later. The set $\mathcal{A}^b$ indicates all couples of approaching waves with $\alpha, \beta < \alpha^b$. The set $\mathcal{A}^{\natural}$ denotes all couples of approaching waves with $\alpha^b \leq \alpha, \beta \leq \alpha^\sharp$, while $\mathcal{A}^{\sharp}$ indicates couples of approaching waves with $\alpha, \beta > \alpha^\sharp$.

For suitable constants $C_3, C_7 > 1$, we then define

$$(3.14) \quad \Upsilon(v) \doteq C_3 \Upsilon^b(v) + \Upsilon^{\natural}(v) + C_3 \Upsilon^{\sharp}(v) + \frac{1}{C_7} \|v^{\alpha^b} - q\|.$$

and set

$$(3.15) \quad \mathcal{D}_\delta^\varepsilon \doteq \{u \text{ as in (3.6), (3.12); } \Upsilon(u) < \delta\}.$$

Case (SR). In this case there exist two disjoint sets of Riemann coordinates, covering a neighborhood of u^b and a neighborhood of the rarefaction curve Γ_2 joining $u^{\natural}$ with $u^{\sharp}$. We can assume that the domains of these charts contain the sets

$$\begin{aligned}
 (3.16) \quad \mathcal{U}^b &= \{(v_1, v_2); |v_i - v_i^b| \leq \delta_0, i = 1, 2\}, \\
 \mathcal{U}^{\natural} &= \{(v_1, v_2); |v_1 - v_1^{\natural}| \leq \delta_0, v_2 \in [v_2^{\natural} - \delta_0, v_2^{\sharp} + \delta_0]\}.
 \end{aligned}$$

Call $\tilde{\sigma}_2 = v_2^{\natural} - v_2^b$ the size of the rarefaction wave of the second family, in the solution of (1.1)–(1.3). Let u be as in (3.6) and assume that the corresponding coordinates v^α satisfy

$$(3.17) \quad v^0, \dots, v^{\alpha^b-1} \in \mathcal{U}^b, \quad v^{\alpha^b}, \dots, v^n \in \mathcal{U}^{\natural}.$$

The piecewise constant function u thus contains several small jumps, together with a single large jump connecting u^{α^b-1} with u^{α^b} . We assume that:

- (i) For every $\alpha \neq \alpha^b$, the Riemann problem at x_α is solved according to (3.7).
- (ii) The Riemann problem at x_{α^b} is solved in terms of a middle state u_* such that the following holds. The states u^{α^b-1} and u_* are connected by an admissible 1-shock satisfying the Rankine-Hugoniot conditions, while u_* and u^{α^b} are connected by a 2-wave of size σ_{2,α^b} . The corresponding coordinates v_* , $v^{\alpha^b} \in \mathcal{U}^\sharp$ satisfy

$$v^{\alpha^b} = \psi_2^\varepsilon(v_*, \sigma_{2,\alpha^b}).$$

For suitable weights C_1, C_2, C_3 , we then define

$$\begin{aligned} \Upsilon^b(u) &\doteq V^b(u) + Q^b(u), & \Upsilon^\natural(u) &\doteq V^\natural(u) + Q^\natural(u), \\ (3.18) \quad V^b(u) &\doteq \sum_{\alpha=1}^{\alpha^b-1} \sum_{i=1}^2 |\sigma_{i,\alpha}|, & V^\natural(u) &\doteq \sum_{\alpha=\alpha^b}^n \sum_{i=1}^2 C_i |\sigma_{i,\alpha}|, \\ Q^b(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^b} |\sigma_{i,\alpha} \sigma_{j,\beta}|, & Q^\natural(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^\natural} C_i C_j |\sigma_{i,\alpha} \sigma_{j,\beta}|, \end{aligned}$$

$$(3.19) \quad \Upsilon(v) \doteq C_3 \Upsilon^b(v) + \Upsilon^\natural(v) + \|v^{\alpha^b} - q\|.$$

As before, we use here the convention $\sigma_{1,\alpha^b} = 0$. The set $\mathcal{A}^b$ indicates all couples of approaching waves with $\alpha, \beta < \alpha^b$, while the set $\mathcal{A}^\natural$ indicates all couples of approaching waves with $\alpha, \beta \geq \alpha^b$. Calling $\tilde{\sigma}_2 = v_2^\sharp - v_2^\natural$ the size of the large rarefaction wave in the solution of the Riemann problem (1.1)–(1.3), we define

$$(3.20) \quad \mathcal{D}_\delta^\varepsilon \doteq \{u \text{ as in (3.6), (3.17)}; \Upsilon(u) < \tilde{\sigma}_2 + \delta\}.$$

With the definitions above, in all Cases (RR), (SS), (SR), one has:

Proposition 1. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions (A1)–(A2) in Theorem 1. Then there exists $\delta > 0$, and constants $C_1, C_2, C_3 > 0$ in the (SS) or (SR) cases, independent of ε such that, for any $\bar{u} \in \mathcal{D}_\delta^\varepsilon$, the wave-front tracking algorithm constructs a unique approximate solution $u^\varepsilon : [0, +\infty[\times \mathbb{R} \mapsto \mathbb{R}^2$ of (1.1)–(1.2), with the following properties:*

- (i) $u^\varepsilon(t, \cdot) \in \mathcal{D}_\delta^\varepsilon$ for all $t \geq 0$,
- (ii) the function $t \mapsto \Upsilon(u^\varepsilon(t, \cdot))$ is non-increasing,
- (iii) Any strip of the form $[0, T] \times \mathbb{R}$ contains finitely many interaction points of u^ε ,
- (iv) $\text{Tot.Var.}(u^\varepsilon(t, \cdot))$ is uniformly bounded.

In all the three cases, to denote this unique, globally defined, ε -approximate solution, we use the semigroup notation

$$(3.21) \quad u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u}.$$

As in [4], the next section of the proof works toward an estimate of the Lipschitz constant for the semigroup S^ε , in the $\mathbf{L}^1$ norm, independent of ε . The basic technique is to shift the locations of the jumps in the initial condition $\bar{u}$ at constant rates, and estimates the rates at which the jumps in the corresponding solution $u^\varepsilon(t, \cdot)$ are shifted, for any fixed $t > 0$.

Definition 2. Let $]a, b[$ be an open interval. An *elementary path* is a map $\gamma :]a, b[\mapsto \mathbf{L}_{\text{loc}}^1$ of the form

$$(3.22) \quad \gamma(\vartheta) = \sum_{\alpha=1}^N u^\alpha \cdot \chi_{]x_{\alpha-1}^\vartheta, x_\alpha^\vartheta[}, \quad x_\alpha^\vartheta = \bar{x}_\alpha + \xi_\alpha \vartheta,$$

with $x_{\alpha-1}^\vartheta < x_\alpha^\vartheta$ for all $\vartheta \in]a, b[$ and $\alpha = 1, \dots, N$.

Definition 3. A continuous map $\gamma : [a, b] \mapsto \mathbf{L}_{\text{loc}}^1$ is a *pseudopolygonal* if there exist countably many disjoint open intervals $J_h \subset [a, b]$ such that:

- (i) The restriction of γ to each J_h is an elementary path.
- (ii) The set $[a, b] \setminus \bigcup_{h \geq 1} J_h$ is countable.

Exactly as in [4], one can prove:

Proposition 2. Let $\gamma_0 : \vartheta \mapsto \bar{u}^\vartheta \in \mathcal{D}_\delta^\varepsilon$ be a pseudopolygonal. Then, for all $\tau > 0$, the path $\gamma_\tau : \vartheta \mapsto u^\vartheta(\tau, \cdot) = S_\tau^\varepsilon \bar{u}^\vartheta$ is also a pseudopolygonal. Indeed, there exist countably many open intervals J_h such that $[a, b] \setminus \bigcup J_h$ is countable and the wave-front configuration of the solution u^ϑ on $[0, \tau] \times \mathbb{R}$ remains the same as ϑ ranges on each J_h .

For a fixed $\varepsilon > 0$, we now introduce the *weighted length* of the elementary path γ in (3.22), given by

$$(3.23) \quad \|\gamma\| \doteq (b-a) \Upsilon_\varepsilon(u),$$

where the functional Υ_ε (also depending on ε) will be defined below, in the various cases.

Definition 4. The *weighted length* of a pseudopolygonal is the sum of the weighted lengths of its elementary paths. For any two piecewise constant functions $u, w \in \mathcal{D}^\varepsilon_\delta$, their *weighted distance* is

(3.24)
$$d_\varepsilon(u, w) \doteq \inf\{\|\gamma\|; \gamma : [0, 1] \mapsto \mathcal{D}^\varepsilon_\delta$$

is a pseudopolygonal joining u with w \}.

By carefully defining the functionals Υ_ξ , we will show that the distance (3.24) is uniformly (w.r.t. ε) equivalent to the $\mathbf{L}^1$ metric, and that the the function

$$t \mapsto d_\varepsilon(S^\varepsilon_t \bar{u}, S^\varepsilon_t \bar{w})$$

is non-increasing, for all $\bar{u}, \bar{w} \in \mathcal{D}^\varepsilon_\delta$. This will imply that the semigroup S^ε is uniformly Lipschitz continuous w.r.t. the usual $\mathbf{L}^1$ distance. Observe that, by taking

$$\Upsilon_\xi(u) = \sum_\alpha |u^\alpha - u^{\alpha-1}| |\xi_\alpha|,$$

the expression (3.23) would give precisely the length of γ in the usual $\mathbf{L}^1$ metric. This length, however, may well increase along the flow of the semigroup S^ε . For this reason, we shall consider a weighted length of the form

$$\Upsilon_\xi = \sum_{i,\alpha} |\sigma_{i,\alpha}| |\xi_\alpha| W_{i,\alpha},$$

where the weights $W_{i,\alpha}$ are defined below, in the various cases. To motivate the following definitions, we remark that, as in [4], the weights should be essentially of the form

$$W_{i,\alpha} = 1 + [\text{total strength of waves approaching } \sigma_{i,\alpha}]$$

$$+ [\text{global interaction potential}].$$

Observe that, if $\sigma_{i,\alpha} < 0$ is a shock of the i^{th} family, the amount of i -waves which can interact in the future with $\sigma_{i,\alpha}$ can be roughly bounded by

$$-[[\sigma_{i,\alpha}]]_- + 2 \sum_{\beta \neq \alpha} [[\sigma_{i,\beta}]]_-.$$

Indeed, a negative i -wave cannot interact with positive i -waves in an amount greater than its own size, otherwise it would be completely annihilated.

Case (RR). Let $u \in \mathcal{D}_\delta^\varepsilon$ be as in (3.6). With the notations introduced in (3.9)–(3.10), we then define

$$(3.25) \quad S_{i,\alpha}(u) \doteq 2 \left(\sum_{\beta=1}^n \sum_{j=1}^2 \llbracket \sigma_{j,\beta} \rrbracket_- \right) - \llbracket \sigma_{i,\alpha} \rrbracket_-,$$

$$R_\alpha(u) \doteq \sum_{\beta=1}^{\alpha-1} |\sigma_{2,\beta}| + \sum_{\beta=\alpha+1}^n |\sigma_{1,\beta}|,$$

$$(3.26) \quad \Upsilon_\xi(u) \doteq \sum_{\alpha=1}^n \sum_{i=1}^2 |\sigma_{i,\alpha} \xi_\alpha| (1 + K_1 S_{i,\alpha}(u)) e^{K_2 R_\alpha(u) + Q(u)},$$

where $\llbracket s \rrbracket_- \doteq (|s| - s)/2$ denotes the negative part of the real number s . A precise value for the constants K_1, K_2 will be selected later.

Case (SS). Let u be as in (3.6) and let (3.12) hold. We then set

$$(3.27) \quad p_{i,\alpha} \doteq 1 + \varepsilon_0 \cdot \text{sign}(\sigma_{i,\alpha})$$

for some $\varepsilon_0 > 0$ suitably small. In connection with the quantities defined at (3.13), we now introduce certain functionals V_ξ, Q_ξ , which also take the shift rates ξ_α into account:

$$(3.28) \quad \begin{aligned} V_\xi^b &\doteq \sum_{\alpha=1}^{\alpha^b-1} \sum_{i=1}^2 p_{i,\alpha} |\sigma_{i,\alpha} \xi_\alpha| \\ Q_\xi^b &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^b} |\sigma_{i,\alpha} \sigma_{j,\beta}| (p_{i,\alpha} |\xi_\alpha| + p_{j,\beta} |\xi_\beta|) \\ V_\xi^\natural &\doteq \sum_{\alpha=\alpha^b}^{\alpha^\sharp} \sum_{i=1}^2 C_i p_{i,\alpha} |\sigma_{i,\alpha} \xi_\alpha| \\ Q_\xi^\natural &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^\natural} C_i C_j |\sigma_{i,\alpha} \sigma_{j,\beta}| (p_{i,\alpha} |\xi_\alpha| + p_{j,\beta} |\xi_\beta|) \\ V_\xi^\sharp &\doteq \sum_{\alpha=\alpha^\sharp+1}^n \sum_{i=1}^2 p_{i,\alpha} |\sigma_{i,\alpha} \xi_\alpha| \\ Q_\xi^\sharp &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^\sharp} |\sigma_{i,\alpha} \sigma_{j,\beta}| (p_{i,\alpha} |\xi_\alpha| + p_{j,\beta} |\xi_\beta|) \end{aligned}$$

Again with reference to (3.13), we define

$$\begin{aligned} \Upsilon_\xi^b &\doteq V_\xi^b(1+Q^b)+K_3Q_\xi^b, \\ \Upsilon_\xi^{\natural} &\doteq V_\xi^{\natural}(1+Q^{\natural})+K_3Q_\xi^{\natural}, \\ \Upsilon_\xi^{\sharp} &\doteq V_\xi^{\sharp}(1+Q^{\sharp})+K_3Q_\xi^{\sharp}. \end{aligned} \tag{3.29}$$

Denoting by $\hat{\xi}_1, \hat{\xi}_2$ the shift rates of the two large shocks and recalling (3.14), we then set

$$\Upsilon_\xi \doteq (\Upsilon_\xi^{\natural}+K_8(\Upsilon_\xi^b+\Upsilon_\xi^{\sharp})+K_4(|\hat{\xi}_1|+|\hat{\xi}_2|))e^{K_9\Upsilon}, \tag{3.30}$$

where $K_3, \dots, K_9$ are suitable constants whose precise value will be selected later.

Case (SR). Let (3.6) and (3.17) hold. We then define

$$\begin{aligned} V_\xi^b(u) &\doteq \sum_{\alpha=1}^{\alpha^b-1} \sum_{i=1}^2 p_{i,\alpha} |\sigma_{i,\alpha} \xi_\alpha|, \\ Q_\xi^b(u) &\doteq \sum_{((i,\alpha),(j,\beta)) \in \mathcal{A}^b} |\sigma_{i,\alpha} \sigma_{j,\beta}| (p_{i,\alpha} |\xi_\alpha| + p_{j,\beta} |\xi_\beta|), \end{aligned} \tag{3.31}$$

$$S_{i,\alpha}(u) \doteq 2 \Big(\sum_{\beta=\alpha^b}^n \sum_{j=1}^2 \llbracket \sigma_{j,\beta} \rrbracket - \rrbracket \sigma_{i,\alpha} \rrbracket \Big), \tag{3.32}$$

$$R_\alpha(u) \doteq \sum_{\beta=\alpha^b}^{\alpha-1} |\sigma_{2,\beta}| + \sum_{\beta=\alpha+1}^n |\sigma_{1,\beta}|,$$

with $p_{i,\alpha}$ as in (3.27). Here $\mathcal{A}^{\natural}$ denotes the set of couples of approaching waves with $\alpha, \beta > \alpha^b$. Moreover, we set

$$\begin{aligned} \Upsilon_\xi^b &\doteq V_\xi^b(1+Q^b)+K_3Q_\xi^b(u), \\ \Upsilon_\xi^{\natural} &\doteq \sum_{\alpha=\alpha^b}^n \sum_{i=1}^2 K_i |\sigma_{i,\alpha} \xi_\alpha| (1+K_4S_{i,\alpha}) e^{K_5R_\alpha}, \end{aligned} \tag{3.33}$$

and finally,

$$\Upsilon_\xi(u) \doteq (K_8\Upsilon_\xi^b+\Upsilon_\xi^{\natural}(u)+K_6|\hat{\xi}_1|)e^{K_9\Upsilon}, \tag{3.34}$$

where $K_1, \dots, K_9$ are suitable constants.

Using the definitions of Υ_ξ above in the formula (3.23) for the weighted length of a path, in each of the three cases (RR), (SS), (SR), one has:

Proposition 3. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions (A1)–(A2) as in Theorem 1. Then there exists $\delta > 0$ and positive constants K_i in the definitions of Υ_ε above, independent of ε , such that the following holds. If $\gamma_0 : \vartheta \mapsto \bar{u}^\vartheta \in \mathcal{D}_\delta^\varepsilon$ is a pseudopolygonal, then the weighted length $\|\gamma_\tau\|$ of the pseudopolygonal $\gamma_\tau : \vartheta \mapsto u^\vartheta(\tau, \cdot) = S_\tau^\varepsilon \bar{u}^\vartheta$ is a non-increasing function of time.*

Proposition 4. *Given $\delta > 0$, there exists some $\delta' \in]0, \delta]$ such that any two functions u, u' in $\mathcal{D}_{\delta'}^\varepsilon$, can be joined by a pseudopolygonal entirely contained in $\mathcal{D}_\delta^\varepsilon$. Moreover, the weighted length of this pseudopolygonal is uniformly equivalent to the usual distance $\|u - u'\|_{\mathbf{L}^1}$.*

Proposition 5. *Let the Riemann problem (1.1)–(1.3) satisfy the stability assumptions (A1)–(A2) as in Theorem 1. Then there exists $\delta' \in]0, \delta]$, independent of ε , such that the semigroup*

$$S^\varepsilon : [0, +\infty[\times \mathcal{D}_{\delta'}^\varepsilon \mapsto \mathcal{D}_{\delta'}^\varepsilon,$$

defined by (3.21) is uniformly Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, with a constant independent of ε .

As in [4], to complete the proof of Theorem 1 we now consider a sequence of semigroups S^{ε_n} with $\varepsilon_n \rightarrow 0$, and construct the limit semigroup as $n \rightarrow \infty$. More precisely, we fix $\delta' > 0$ according to Proposition 5 and define the closed domain

$$(3.35) \quad \mathcal{D} \doteq \{\bar{u}; \exists \bar{u}_n \rightarrow \bar{u}, \bar{u}_n \in \mathcal{D}_{\delta'}^\varepsilon, \forall n\}.$$

For $\bar{u} \in \mathcal{D}$ and $t \geq 0$, we then define

$$(3.36) \quad S_t \bar{u} \doteq \lim_{n \rightarrow \infty} S_t^{\varepsilon_n} \bar{u}_n,$$

where $\bar{u}_n \in \mathcal{D}_{\delta'}^{\varepsilon_n}$ is any sequence approaching $\bar{u}$ in $\mathbf{L}^1$. One concludes by proving:

Proposition 6. *The closed domain $\mathcal{D}$ in (3.35) and the semigroup S in (3.36) are well defined and satisfy all conditions (i)–(v) in Theorem 1, for suitable constants $L, \rho > 0$.*

4. Bounds on the total variation. Throughout this and the following section, the value of ε will be kept fixed. Hence, to simplify the notation, the dependence on ε of the various functionals will not be explicitly written. Since we are eventually interested in the limit of ε -approximate solutions as $\varepsilon \rightarrow 0$, it is understood that all of the estimates given below remain valid uniformly w.r.t. ε , as ε ranges in a suitably small interval $]0, \varepsilon_*]$. In all three cases (RR), (SS), (SR), the same argument used in [4] shows that, as long as the total variation remains uniformly bounded, the set of interaction points of an ε -approximate solution can have no limit point in the t - x plane. In particular, the interaction times will satisfy $\lim_{n \rightarrow \infty} \tau_n = \infty$. To prove the global existence of piecewise constant approximate solutions, stated in Proposition 1, it thus suffices to derive a-priori bounds on the quantities $Q(u)$, $\Upsilon(u)$, valid for all $t > 0$.

With the Landau symbol $O(1)$ we denote a function whose absolute value remains uniformly bounded. The positive and negative part of a real number s are written

$$[s]_+ \doteq \frac{|s| + s}{2}, \quad [s]_- \doteq \frac{|s| - s}{2}.$$

For convenience, we use the sup-norm on the space $\mathbb{R}^2$ of the Riemann coordinates, so that

$$(4.1) \quad \|v' - v\| \doteq \max\{|v'_1 - v_1|, |v'_2 - v_2|\}, \quad B(v, \delta) \doteq \{v'; \max_{i=1,2} |v'_i - v_i| < \delta\}.$$

Observe that if $v^\alpha = \psi_i^\varepsilon(v^{\alpha-1}, \sigma_{i,\alpha})$, then (3.1)–(3.2) imply

$$(4.2) \quad v_i^\alpha - v_i^{\alpha-1} = \sigma_{i,\alpha}, \quad |v_j^\alpha - v_j^{\alpha-1}| = O(1) \cdot ([\sigma_{i,\alpha}]_-)^3 \quad \text{for } j \neq i.$$

We consider the case (RR) first. In this case the Riemann problem (1.1)–(1.3) is solved in terms of two rarefaction waves of sizes $\tilde{\sigma}_1 = v_1^\natural - v_1^b$, $\tilde{\sigma}_2 = v_2^\natural - v_2^b$. The definitions (3.8)–(3.10) are in use.

Lemma 1. *For $\delta > 0$ sufficiently small, the following holds. Let the values $v^0, \dots, v^n \in \mathcal{U}$ satisfy (3.7), with $v^0 = v^b$, $v^n = v^\natural$, $Q(v) \leq \delta/2$, $\Upsilon(v) \leq \tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta$. Then for every α one has*

$$(4.3) \quad v^\alpha \in \left([v_1^b - \sqrt{\delta}, v_1^\natural + \sqrt{\delta}] \times [v_2^b - \sqrt{\delta}, v_2^\natural + \sqrt{\delta}] \right) \\ \cup \left([v_1^\natural - \sqrt{\delta}, v_1^\natural + \sqrt{\delta}] \times [v_2^b - \sqrt{\delta}, v_2^\natural + \sqrt{\delta}] \right)$$

Proof. Since $\tilde{\sigma}_1, \tilde{\sigma}_2 > 0$, from

$$(4.4) \quad \sum_{i=1,2} \left(\sum_{\alpha} [\sigma_{i,\alpha}]_+ \right) \left(\sum_{\alpha} [\sigma_{i,\alpha}]_- \right) \leq Q(v) \leq \frac{\delta}{2},$$

we deduce

$$(4.5) \quad \sum_{\alpha} [\![\sigma_{1,\alpha}]\!]_{-} + \sum_{\alpha} [\![\sigma_{2,\alpha}]\!]_{-} = O(1) \cdot \delta.$$

For $\delta > 0$ sufficiently small, (4.5) and (4.2) yield

$$(4.6) \quad \left| v_i^{\alpha} - v_i^{\flat} - \sum_{\beta \leq \alpha} \sigma_{i,\beta} \right| = O(1) \cdot \delta^3 \leq \delta^2.$$

If now,

$$v_1^{\alpha^*} < v_1^{\sharp} - \sqrt{\delta}, \quad v_2^{\alpha^*} > v_2^{\flat} + \sqrt{\delta}$$

for some α^* , then

$$Q(v) \geq \left(\sum_{\alpha > \alpha^*} |\sigma_{1,\alpha}| \right) \left(\sum_{\alpha \leq \alpha^*} |\sigma_{2,\alpha}| \right) \geq (\sqrt{\delta} - \delta^2) \cdot (\sqrt{\delta} - \delta^2) > \frac{\delta}{2}.$$

This contradiction proves (4.3). $\square$

Proof of Proposition 1 in Case (RR). Let τ be a time where a wave-front interaction takes place. Assuming that $u(t, \cdot) \in \mathcal{D}_{\delta}^{\varepsilon}$ for $t < \tau$, we need to show that the quantities $Q(u(t, \cdot))$, $\Upsilon(u(t, \cdot))$ do not increase across the interaction, hence $u(t, \cdot) \in \mathcal{D}_{\delta}^{\varepsilon}$ also for $t > \tau$.

Following [4], we observe that any simultaneous interaction of three or more incoming waves can be approximated by a sequence of separate interactions, each one involving exactly two incoming wave-fronts. Hence, it is not restrictive to assume that at time τ only two wave-fronts interact.

Assume first that the incoming waves belong to distinct families. For $i = 1, 2$, call σ_i^{-} , σ_i^{+} the (total) size of the waves of the i^{th} family before and after the interaction (Fig. 3a). Because of the existence of local Riemann coordinates, for some constant C_4 the following sharp interaction estimate holds.

$$(4.7) \quad |\sigma_1^{+} - \sigma_1^{-}| + |\sigma_2^{+} - \sigma_2^{-}| = C_4 |\sigma_1^{-} \sigma_2^{-}| (|\sigma_1^{-}| + |\sigma_2^{-}|).$$

If both incoming waves belong to the same family (say, the first), call σ' , σ'' the sizes of the incoming waves (Fig. 3b). Then one has the estimate

$$(4.8) \quad |\sigma_1^{+} - \sigma' - \sigma''| + |\sigma_2^{+}| \leq C_4 |\sigma' \sigma''| (|\sigma'| + |\sigma''|).$$

Recalling that $V(u) \leq \tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta$, $Q(u) \leq \delta/2$, we can assume that

$$(4.9) \quad \varepsilon < \delta, \quad |\sigma_1^{-}|, |\sigma_2^{-}|, |\sigma'|, |\sigma''| \leq C_5 \delta$$

Figure 3a

Figure 3b

for some constant C_5 independent of δ . In the two cases above, (4.9) yields

(4.10)

$$\begin{aligned} \Delta Q(\tau) &\leq -|\sigma_1^- \sigma_2^-| + C_4 |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|)(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta) \\ &\leq -\frac{|\sigma_1^- \sigma_2^-|}{2}, \end{aligned}$$

(4.11)

$$\begin{aligned} \Delta Q(\tau) &\leq -|\sigma' \sigma''| + C_4 |\sigma' \sigma''| (|\sigma'| + |\sigma''|)(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta) \\ &\leq -\frac{|\sigma' \sigma''|}{2}, \end{aligned}$$

provided that δ is sufficiently small. In both cases, $\Delta \Upsilon(\tau) \leq 0$. By Lemma 1

we conclude that $u(t, \cdot) \in \mathcal{D}_\delta^\varepsilon$ also for $t > \tau$. By induction on the sequence of interaction times $\{\tau_n\}_{n \geq 1}$ the proof is completed. $\square$

Next, consider the case (SS), recalling the definitions introduced at (3.11)–(3.15). Observe that the product $\kappa_1 \kappa_2$ in (2.10) does not change if the basis of right eigenvectors $\{r_1, r_2\}$ is replaced by any other basis, say $\{\varphi_1 r_1, \varphi_2 r_2\}$, with φ_1, φ_2 smooth scalar functions. Therefore, we can assume that these eigenvectors are chosen so that, in the local Riemann coordinates, one always has $r_1 \equiv (0, 1)$, $r_2 \equiv (1, 0)$.

Consider the 1-shock connecting u^b with u^h . If σ_1^h is the strength of a 1-wave colliding on the right of the shock, and σ_2^h denotes the size of the 2-wave produced by the interaction (Fig. 4a), from (2.4)–(2.6) and the implicit function theorem it follows

$$(4.12) \quad \left. \frac{d\sigma_2^h}{d\sigma_1^h} \right|_{\sigma_1^h=0} = - \frac{D_2 \Phi_2(u^b, u^h) \cdot r_1(u^h)}{D_2 \Phi_2(u^b, u^h) \cdot r_2(u^h)} = \kappa_1.$$

From (4.12) we now derive an estimate on wave strengths, valid for more general interactions.

Lemma 2. *Let a left and a right state, with Riemann coordinates v^l, v^r , be connected by a 1-shock. Let three wave-fronts impinge on this shock, with sizes $\sigma_2^b, \sigma_1^b, \sigma_1^h$, as in Fig. 4a. Call σ_2^h the size of the outgoing 2-wave, generated by the interaction. Assume that v^l, v^r are close to the reference states v^b, v^h and that the perturbing waves are small, i.e.,*

$$(4.13) \quad |v^l - v^b| \leq \delta, \quad |v^r - v^h| \leq \delta, \quad \max\{|\sigma_2^b|, |\sigma_1^b|, |\sigma_1^h|\} \leq \delta.$$

Then, for some constant C_6 and every $\delta > 0$ sufficiently small, one has

$$(4.14) \quad |\sigma_2^h| \leq C_6(|\sigma_1^b| + |\sigma_2^b|) + |\kappa_1|(1 + C_6\delta)|\sigma_1^h|.$$

Proof. Observing that the sizes of the outgoing waves are $\mathcal{C}^2$ functions of the incoming waves and of the left and right states u^l, u^r , the bound (4.14) is an immediate consequence of (4.12). $\square$

Recalling (2.9), an entirely similar estimate holds for interactions involving the large shock of the second family, namely (see Fig. 4b)

$$(4.15) \quad |\sigma_1^h| \leq C_6(|\sigma_1^b| + |\sigma_2^b|) + (1 + C_6\delta)|\kappa_2| \cdot |\sigma_2^h|.$$

Denoting by $v^r, \tilde{v}^r$ the states to the right of the large 1-shock before and after the interaction, from (4.14) it follows

$$(4.16) \quad \|\tilde{v}^r - v^r\| \leq C_7(|\sigma_1^b| + |\sigma_2^b| + |\sigma_1^h|)$$

for some constant C_7 . $\square$

Figure 4a

Figure 4b

Proof of Proposition 1 in Case (SS). The proof is divided in several steps.

▷ STEP 1: *Choice of the weights in (3.13)–(3.14).* By the assumption (2.12), we can choose positive constants C_1, C_2 such that

(4.17) $C_1 - C_2|\kappa_1\Theta_1| > 2, \quad C_2 - C_1|\kappa_2\Theta_2| > 2.$

Recalling that $\Theta_1, \Theta_2 < -1$, from (4.17) it follows that, for $\delta > 0$ sufficiently

small,

$$(4.18) \quad C_2|\kappa_1|(1+C_6\delta)(1+\delta)-C_1 < -2, \quad C_1|\kappa_2|(1+C_6\delta)(1+\delta)-C_2 < -2,$$

with C_6 as in (4.14)–(4.15). We let C_7 be the constant in (4.16). Recalling (4.7)–(4.8), we then choose

$$(4.19) \quad C_3 = 2C_4C_6 + 2.$$

▷ **STEP 2:** Υ decreases at interactions that do not involve the two large shocks. By an approximation argument, it again suffices to consider interactions involving only two incoming wavefronts. To fix the ideas, consider two incoming 1-waves of size σ' , σ'' , interacting in the region between the two shocks. In this case, at the interaction time τ , the quantities Υ^b , $\Upsilon^\sharp$, $\|v^{\alpha^b} - v^\sharp\|$ remain unchanged. Using (4.8) and recalling (4.9), (3.15), we have

$$(4.20) \quad \begin{aligned} \Delta\Upsilon^\sharp(\tau) &\leq (C_1|\sigma_1^+ - \sigma' - \sigma''| + C_2|\sigma_2^+|)(1 + V^\sharp(\tau-)) - C_1^2|\sigma'\sigma''| \\ &\leq \max\{C_1, C_2\} \cdot C_4|\sigma'\sigma''|(|\sigma'| + |\sigma''|) - C_1^2|\sigma'\sigma''| \\ &\leq -\frac{|\sigma'\sigma''|}{2}, \end{aligned}$$

provided that δ is sufficiently small. All other cases of interactions can be handled in the same way, using (4.7) or (4.8).

▷ **STEP 3:** Υ decreases at interactions involving one of the large shocks. Assume that, at time τ , an interaction of the type sketched in Fig. 4a takes place, involving the large 1-shock. Observing that no outgoing wave emerges on the left of the 1-shock, the decrease in Υ^b is estimated by

$$(4.21) \quad \Delta\Upsilon^b(\tau) \leq -|\sigma_1^b| - |\sigma_2^b|.$$

By (4.14) and (4.18), the change in $\Upsilon^\sharp$ is bounded by

$$(4.22) \quad \begin{aligned} \Delta\Upsilon^\sharp(\tau) &\leq C_2|\sigma_2^\sharp| - C_1|\sigma_1^\sharp| + C_2|\sigma_2^\sharp|V^\sharp(\tau-) \\ &\leq C_2[C_6(|\sigma_1^b| + |\sigma_2^b|) + |\kappa_1|(1+C_6\delta)|\sigma_1^\sharp|] \\ &\quad \times (1 + V^\sharp(\tau-)) - C_1|\sigma_1^\sharp| \\ &\leq C_2C_6(|\sigma_1^b| + |\sigma_2^b|)(1+\delta) \\ &\quad + (C_2|\kappa_1|(1+C_6\delta)(1+\delta) - C_1)|\sigma_1^\sharp| \\ &\leq 2C_2C_6(|\sigma_1^b| + |\sigma_2^b|) - |\sigma_1^\sharp|. \end{aligned}$$

From (4.19) and (4.22) it now follows

$$\begin{aligned}
(4.23) \quad \Delta \Upsilon(\tau) &\leq C_3 \Delta \Upsilon^b(\tau) + \Delta \Upsilon^\sharp(\tau) + C_7^{-1} \|\tilde{v}^r - v^r\| \\
&\leq -C_3(|\sigma_1^b| + |\sigma_2^b|) + 2C_2 C_6(|\sigma_1^b| + |\sigma_2^b|) \\
&\quad - 2|\sigma_1^\sharp| + (|\sigma_1^b| + |\sigma_2^b| + |\sigma_1^\sharp|) \\
&\leq -(|\sigma_1^b| + |\sigma_2^b| + |\sigma_1^\sharp|).
\end{aligned}$$

The analysis of an interaction involving the large 2-shock is entirely similar.

▷ STEP 4: $v(t, \cdot)$ remains in the domain $\mathcal{D}_\delta^\varepsilon$ as long as $\Upsilon(v)$ is non-increasing. Recalling the notations in (3.12), because of the finite propagation speed one always has

$$v^0 = \lim_{x \rightarrow -\infty} v(t, x) = b, \quad v^n = \lim_{x \rightarrow \infty} v(t, x) = d.$$

Hence

$$\begin{aligned}
\text{if } \beta = 1, \dots, \alpha^b, \quad & \|v^\beta - b\| \leq \sum_{\alpha=1}^{\bar{\alpha}} \|v^\alpha - v^{\alpha-1}\| \leq \Upsilon^b(t) \leq \delta, \\
\text{if } \beta = \alpha^b + 1, \dots, \alpha^\sharp, \quad & \|v^\beta - q\| \leq \|v^{\alpha^b} - q\| + \sum_{\alpha=\alpha^b+1}^{\alpha^\sharp} \|v^\alpha - v^{\alpha-1}\| \leq C_7 \Upsilon(t) \leq C_7 \delta, \\
\text{if } \beta = \alpha^\sharp + 1, \dots, n, \quad & \|v^\beta - d\| \leq \sum_{\alpha=\alpha^\sharp+1}^n \|v^\alpha - v^{\alpha-1}\| \leq \Upsilon^\sharp(t) \leq \delta.
\end{aligned}$$

For $\delta > 0$ sufficiently small, the inequalities above imply (3.12), completing the proof. $\square$

Proof of Proposition 1 in Case (SR). Recall the definitions at (3.16)–(3.20). As in the previous cases, we have to choose the weights C_i in (3.18)–(3.19) so that, for $\delta > 0$ sufficiently small, the quantity $\Upsilon(v)$ does not increase at interaction times.

Consider an interaction involving the large 1-shock, as in Fig. 4a. By the estimates (4.14), (4.16), still valid in this case, there exists a constant C_8 such that

$$(4.24) \quad |\sigma_2^\sharp| + \|\tilde{v}^r - v^r\| \leq C_8(|\sigma_1^b| + |\sigma_2^b| + |\sigma_1^\sharp|).$$

We then choose

$$(4.25) \quad C_1 = C_3 = C_8(2 + \tilde{\sigma}_2), \quad C_2 = 1.$$

With the choices above, consider a time τ where a wave-front interaction takes place. If the interaction does not involve the large 1-shock, by (4.7)–(4.8), the same arguments used in the previous cases show that the quantities Υ^b , $\Upsilon^\sharp$ do not increase, provided that δ is sufficiently small. Next, let the interaction involve the large 1-shock, with the incoming waves as in Fig. 4a. We then have

$$(4.26) \quad \Delta \Upsilon^b(\tau) \leq -|\sigma_1^b| - |\sigma_2^b|,$$

$$(4.27) \quad \begin{aligned} \Delta \Upsilon^\sharp(\tau) &\leq |\sigma_2^\sharp|(1 + V^\sharp(\tau-)) - C_1|\sigma_1^\sharp| + \|\tilde{v}^r - v^r\| \\ &\leq C_8(|\sigma_1^b| + |\sigma_2^b| + |\sigma_1^\sharp|)(2 + \tilde{\sigma}_2) - C_1|\sigma_1^\sharp|. \end{aligned}$$

By (4.25), the two previous estimates imply $\Delta \Upsilon(\tau) \leq 0$. $\square$

5. Estimates on weighted lengths. This section contains a proof of Proposition 3. With a judicious choice of the various constants K_i which define the metric, in the three cases (RR), (SS), (SR), we will show that the “infinitesimal path length” Υ_ξ decreases at every *simple interaction*, i.e., at every interaction involving just two incoming wave-fronts. The same limiting argument used in the proof of Proposition 7 of [4] will then imply that Υ_ξ still decreases, when any number of wave-fronts interact together.

For convenience, we shall often denote a wave-front simply by its size, writing for example “the wave σ_i ” in place of “the wave whose size is σ_i ”. Some basic estimates on the strength and on the shift rates of waves involved in a simple interaction are collected below.

Assume that a 1-wave σ_1 , shifting at rate ξ_1^- , collides with a 2-wave σ_2^- , shifting at rate ξ_2^- . Call $\sigma_{1,\ell}^+$, $\sigma_{2,\ell}^+$ the sizes of the outgoing wave-fronts of the first and second family, respectively, and let $\xi_{1,\ell}^+$, $\xi_{2,\ell}^+$ be their shift rates. By Lemma 2 and 21 in [4], one has

$$(5.1) \quad \left| \sigma_1^- - \sum_{\ell} \sigma_{1,\ell}^+ \right| + \left| \sigma_2^- - \sum_{\ell} \sigma_{2,\ell}^+ \right| \leq C_4 |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|),$$

$$(5.2) \quad \sum_{\ell} |\sigma_{1,\ell}^+ \xi_{1,\ell}^+| - |\sigma_1^- \xi_1^-| + \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| - |\sigma_2^- \xi_2^-| \leq C_9 |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|),$$

for suitable constants C_4 , C_9 . Next, assume that two waves σ' and σ'' of the first family collide, shifting at rates ξ' , ξ'' , respectively. Call σ_1^+ the (single) outgoing wave-front of the first family and $\sigma_{2,\ell}^+$ the outgoing fronts of the second family,

and let ξ_1^+ , $\xi_{2,\ell}^+$ be their shift rates. From Lemma 3 and Lemma 22 in [4] it follows

$$(5.3) \quad |\sigma_1^+ - (\sigma' + \sigma'')| + \sum_{\ell} |\sigma_{2,\alpha}^+| \leq C_4 |\sigma' \sigma''| (|\sigma'| + |\sigma''|),$$

$$(5.4) \quad |\sigma_1^+ \xi_1^+| - (|\sigma' \xi'| + |\sigma'' \xi''|) + \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| \leq C_9 |\sigma' \sigma''| (|\xi'| + |\xi''|).$$

We assume that entirely similar estimates hold (with the same constants) for interactions of two waves of the second characteristic family.

Let the domain $\mathcal{D}_{\delta}^{\varepsilon}$ be defined as in (3.10), (3.15) or (3.20). Then, in all three cases, the assumption $u \in \mathcal{D}_{\delta}^{\varepsilon}$ implies that all wave-fronts of u (except possibly the one or two large shocks) satisfy

$$(5.5) \quad |\sigma_{i,\alpha}| = O(1) \cdot \delta.$$

Proof of Proposition 3 in Case (RR).

▷ **STEP 1: Choice of the weights in (3.26).** Choose the constants K_1 , K_2 so that

$$(5.6) \quad K_2 > 4C_9, \quad K_1 > 4C_9 e^{K_2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + 1)}$$

▷ **STEP 2: Estimates at interactions of waves of the same family.** Let the interaction take place at the point $(\tau, \bar{x})$. It is convenient to estimate the change in Υ_{ξ} across the interaction time τ as

$$(5.7) \quad \Delta \Upsilon_{\xi}(\tau) = [\Delta \Upsilon_{\xi}]_{\text{int}} + [\Delta \Upsilon_{\xi}]_{\text{nonint}}.$$

The first summand on the right hand side of (5.7) accounts for the wave-fronts directly involved in the interaction, while the second term collects the contributions to (3.26) of the wave-fronts not related with the interaction (i.e., with $x_{\alpha} \neq \bar{x}$).

For a wave-front located at a point $x_{\alpha} \neq \bar{x}$, not involved in the interaction, call R_{α}^{-} , R_{α}^{+} , $S_{i,\alpha}^{-}$, $S_{i,\alpha}^{+}$ the corresponding quantities in (3.25) before and after the interaction. From the estimate (5.3) it follows

$$(5.8) \quad R_{\alpha}^{+} - R_{\alpha}^{-} \leq \begin{cases} |\sigma_1^+| - |\sigma'| - |\sigma''| & \text{if } x_{\alpha} < \bar{x}, \\ \sum |\sigma_{2,\ell}^+| & \text{if } x_{\alpha} > \bar{x}, \end{cases}$$

$$\leq C_4 |\sigma' \sigma''| (|\sigma'| + |\sigma''|),$$

$$(5.9) \quad S_{i,\alpha}^{+} - S_{i,\alpha}^{-} \leq 2C_4 |\sigma' \sigma''| (|\sigma'| + |\sigma''|).$$

Concerning those terms in (3.26) which correspond to non-interacting waves, recalling (4.11) and (5.5), we thus have

$$\begin{aligned}
 (5.10) \quad [\Delta \Upsilon_\xi]_{\text{nonint}} &= \\
 &= \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha} \xi_\alpha| (e^{K_2 R_\alpha^+ + Q^+} - e^{K_2 R_\alpha^- + Q^-}) (1 + K_1 S_{i,\alpha}^+) \\
 &\quad + K_1 \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha} \xi_\alpha| e^{K_2 R_\alpha^- + Q^-} (S_{i,\alpha}^+ - S_{i,\alpha}^-) \\
 &\leq \left[-\frac{1}{2} + C_4 K_2 (|\sigma'| + |\sigma''|) \right] |\sigma' \sigma''| \times \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha} \xi_\alpha| e^{K_2 R_\alpha^- + Q^-} (1 + K_1 S_{i,\alpha}^+) \\
 &\quad + 2C_4 K_1 |\sigma' \sigma''| (|\sigma'| + |\sigma''|) \times \sum_{x_\alpha \neq \bar{x}} |\sigma_{i,\alpha} \xi_\alpha| e^{K_2 R_\alpha^- + Q^-} \\
 &\leq 0,
 \end{aligned}$$

provided that δ , and hence $|\sigma'|$, $|\sigma''|$, are sufficiently small.

To derive estimates on the waves directly involved in the interaction, call R' , R'' , S' , S'' the quantities in (3.25) corresponding to the incoming wave-fronts, and R_1^+ , $R_{2,\ell}^+$, S_1^+ , $S_{2,\ell}^+$ those connected with the outgoing fronts. Since at least one of the incoming waves is negative, we now have the estimates

$$(5.11) \quad R_1^+ = R'' = R' - |\sigma''|, \quad S_1^+ - S' \leq -\frac{|\sigma''|}{2}, \quad S_1^+ - S'' \leq -\frac{|\sigma'|}{2}.$$

More generally, one has

$$(5.12) \quad R_{j,\beta} \leq \Upsilon(u) \leq (\tilde{\sigma}_1 + \tilde{\sigma}_2 + \delta), \quad S_{j,\beta} = O(1) \cdot \delta \quad \forall j, \beta.$$

Using (5.11), (5.12) and (5.4), and recalling (4.11), for δ sufficiently small, we obtain

$$\begin{aligned}
 (5.13) \quad [\Delta \Upsilon_\xi]_{\text{int}} &= \\
 &= |\sigma_1^+ \xi_1^+| e^{K_2 R_1^+ + Q^+} (1 + K_1 S_1^+) + \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| e^{K_2 R_{2,\ell}^+ + Q^+} (1 + K_1 S_{2,\ell}^+) \\
 &\quad - |\sigma' \xi'| e^{K_2 R' + Q^-} (1 + K_1 S') - |\sigma'' \xi''| e^{K_2 R'' + Q^-} (1 + K_1 S'') \\
 &= (|\sigma_1^+ \xi_1^+| - |\sigma' \xi'| - |\sigma'' \xi''|) e^{K_2 R' + Q^-} (1 + K_1 S_1^+)
 \end{aligned}$$

$$\begin{aligned}
& + |\sigma_1^+ \xi_1^+| (e^{K_2 R_1^+ + Q^+} - e^{K_2 R' + Q^-}) (1 + K_1 S_1^+) \\
& + \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| e^{K_2 R_{2,\ell}^+ + Q^+} (1 + K_1 S_{2,\ell}^+) + K_1 |\sigma' \xi'| e^{K_2 R' + Q^-} (S_1^+ - S') \\
& + |\sigma'' \xi''| (e^{K_2 R''} - e^{K_2 R'}) e^{Q^-} (1 + K_1 S_1^+) + K_1 |\sigma'' \xi''| e^{K_2 R' + Q^-} (S_1^+ - S'') \\
& \leq (-K_1 + 4C_9 e^{K_2 \Upsilon^- + Q^-}) |\sigma' \sigma''| (|\xi'| + |\xi''|) \\
& \leq 0,
\end{aligned}$$

provided that (5.6) holds. The case of two interacting wave-fronts both of the second family is entirely similar.

▷ **STEP 3: Estimates at interactions between waves of different families.** Again, consider first the quantities related to non-interacting waves. By (5.1) and (5.5), the outgoing waves have the same sign of the corresponding incoming ones. From (5.1) it also follows

$$\begin{aligned}
(5.14) \quad R_{\alpha}^+ - R_{\alpha}^- & \leq \begin{cases} |\sigma_1^+| - |\sigma_1^-| & \text{if } x_{\alpha} > \bar{x}, \\ |\sigma_2^+| - |\sigma_2^-| & \text{if } x_{\alpha} < \bar{x}, \end{cases} \\
& \leq C_4 |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|),
\end{aligned}$$

$$(5.15) \quad S_{i,\alpha}^+ - S_{i,\alpha}^- \leq 2C_4 |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|).$$

Hence, by (4.10) and (5.1),

$$\begin{aligned}
[\Upsilon_{\xi}]_{\text{nonint}} & \leq \\
& \leq \sum_{i,\alpha} |\sigma_{i,\alpha} \xi_{\alpha}| (e^{K_2 R_{\alpha}^+ + Q^+} - e^{K_2 R_{\alpha}^- + Q^-}) (1 + K_1 S_{i,\alpha}^+) \\
& \quad + K_1 \sum_{i,\alpha} |\sigma_{i,\alpha} \xi_{\alpha}| e^{K_2 R_{\alpha}^- + Q^-} (S_{i,\alpha}^+ - S_{i,\alpha}^-) \\
& \leq \left[-\frac{1}{2} + (2K_2 C_4 + 2K_1 C_4) (|\sigma_1^-| + |\sigma_2^-|) \right] |\sigma_1^- \sigma_2^-| \sum_{i,\alpha} |\sigma_{i,\alpha} \xi_{\alpha}| e^{K_2 R_{\alpha}^- + K_3 Q^-} \\
& \leq 0
\end{aligned}$$

provided that δ , and hence also $|\sigma_1^-|$, $|\sigma_2^-|$, are sufficiently small.

To estimate the terms related to the interacting waves, from (5.1) it follows

$$(5.16) \quad \begin{aligned} R_{1,\ell}^+ - R_1^- &\leq -\frac{|\sigma_2^-|}{2}, \quad S_{1,\ell}^+ - S_1^- \leq 2C_4|\sigma_1^- \sigma_2^-|(|\sigma_1^-| + |\sigma_2^-|), \\ R_{2,\ell}^+ - R_2^- &\leq -\frac{|\sigma_1^-|}{2}, \quad S_2^+ - S_2^- \leq 2C_4|\sigma_1^- \sigma_2^-|(|\sigma_1^-| + |\sigma_2^-|). \end{aligned}$$

Recalling (4.10), by (5.1), (5.2) and (5.16) we deduce

$$\begin{aligned} &[\Delta \Upsilon_\xi]_{\text{int}} \leq \\ &\leq \sum_{i,\ell} |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| e^{K_2 R_{i,\ell}^+ + Q^+} (1 + K_1 S_{i,\ell}^+) - \sum_i |\sigma_i^- \xi_i^-| e^{K_2 R_i^- + Q^-} (1 + K_1 S_i^-) \\ &\leq \sum_{i \neq j} \left(\sum_\ell |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| \right) e^{K_2(R_i^- - |\sigma_j^-|/2) + Q^+} \times \{1 + K_1[S_i^- + 2C_4|\sigma_1^- \sigma_2^-|(|\sigma_1^-| + |\sigma_2^-|)]\} \\ &\quad - \sum_i |\sigma_i^- \xi_i^-| e^{K_2 R_i^- + Q^-} (1 + K_1 S_i^-) \\ &= \sum_{i \neq j} \left[\left(\sum_\ell |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| \right) - |\sigma_i^- \xi_i^-| \right] e^{K_2(R_i^- - |\sigma_j^-|/2) + Q^+} \\ &\quad \times \{1 + K_1[S_i^- + 2C_4|\sigma_1^- \sigma_2^-|(|\sigma_1^-| + |\sigma_2^-|)]\} \\ &\quad + \sum_{i \neq j} |\sigma_i^- \xi_i^-| (e^{K_2(R_i^- - |\sigma_j^-|/2) + Q^+} - e^{K_2 R_i^- + Q^-}) \\ &\quad \times \{1 + K_1[S_i^- + 2C_4|\sigma_1^- \sigma_2^-|(|\sigma_1^-| + |\sigma_2^-|)]\} + \\ &\quad + 2C_4 K_1 |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|) \sum_i |\sigma_i^- \xi_i^-| e^{K_2 R_i^- + Q^-} \\ &\leq \left(2C_9 - \frac{1}{2} K_2 \right) |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|) \sum_i e^{K_2(R_i^- - |\sigma_j^-|/2) + Q^+} \\ &\quad + O(1) \cdot |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|) (|\sigma_1^- \xi_1^-| + |\sigma_2^- \xi_2^-|) \\ &\leq 0, \end{aligned}$$

provided that (5.6) holds and δ is small. This completes the proof of Proposition 3 in the case (RR). $\square$

Proof of Proposition 3 in Case (SS). In the following, those waves that do not take part in the interaction are denoted by $\sigma_{i,\alpha}$, $\sigma_{j,\beta}$; their shift rates by $\xi_{i,\alpha}$, $\xi_{j,\beta}$. The set of wave-fronts approaching $\sigma_{i,\alpha}$ is denoted by $\mathcal{A}_{i,\alpha}$. The

quantities related to the small interacting waves are $\sigma_i^\pm, p_i^\pm, \xi_i^\pm$ for $i = 1, 2$, or $\sigma', p', \xi', \sigma'', \text{etc.}\dots$ The shift rate and the speed of the large i -shock are denoted by $\hat{\xi}_i^\pm, \hat{\Lambda}_i^\pm$. By $\mathcal{A}', \mathcal{A}'', \mathcal{A}_i^-$, we indicate the set of wave-fronts approaching $\sigma', \sigma'', \sigma_i^-$, respectively.

Observe that from the definitions (3.13)–(3.14) it follows

$$(5.17) \quad V^b, V^\natural, V^\sharp = \mathcal{O}(1)\delta, \quad Q^b, Q^\natural, Q^\sharp = \mathcal{O}(1)\delta.$$

Together with the bounds (5.1)–(5.4), concerning interactions between small waves, we now list some estimates valid at interactions involving one of the large shocks. Assume first that a small wave-front with size σ^- and shift rate ξ^- hits the large 1-shock from the left. Let $\hat{\xi}_1^-, \hat{\xi}_1^+$ be the shift rates of the large shock before and after the interaction, and call $\sigma_{2,\ell}^+, \xi_{2,\ell}^+$ the sizes and shift rates of the outgoing 2-waves. Then, for suitable constants $C_i > 0$, one has

$$(5.18) \quad \begin{aligned} \sum_{\ell} |\sigma_{2,\ell}^+| &\leq C_8 |\sigma^-|, \\ |\hat{\xi}_1^+ - \hat{\xi}_1^-| &\leq C_{10} |\sigma^-| (|\xi^-| + |\hat{\xi}_1^-|), \\ \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| &\leq C_{10} |\sigma^-| (|\xi^-| + |\hat{\xi}_1^-|), \\ \Delta \Upsilon &\leq -|\sigma^-|. \end{aligned}$$

Next, let a small 1-wave σ_1^- , shifting at rate ξ_1^- hit the large 1-shock from the right. With the notation above, calling $\Lambda_1^-, \Lambda_{2,\ell}^+$ the speeds of the small wave-fronts, one has

$$\begin{aligned}
(5.19) \quad & \sum_{\ell} |\sigma_{2,\ell}^+| \leq (1 + C_6 \delta) |\kappa_1| |\sigma_1^-|, \\
& |\hat{\xi}_1^+ - \hat{\xi}_1^-| \leq C_{10} |\sigma_1^-| (|\xi_1^-| + |\hat{\xi}_1^-|), \\
& \Delta \Upsilon \leq -|\sigma_1^-|,
\end{aligned}$$

$$\begin{aligned}
(5.20) \quad & \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| = \sum_{\ell} \left| \sigma_{2,\ell}^+ \frac{(\Lambda_{2,\ell}^+ - \hat{\Lambda}_1^-) \xi_1^- + (\Lambda_{2,\ell}^+ - \Lambda_1^-) \hat{\xi}_1^-}{\Lambda_1^- - \hat{\Lambda}_1^-} \right| \\
& \leq C_{10} |\sigma_1^-| |\hat{\xi}_1^-| + (1 + C_{10} \delta) |\kappa_1 \Theta_1| |\sigma_1^- \xi_1^-|.
\end{aligned}$$

▷ *STEP 1: Choice of the weights.* The strong non-resonance condition (2.12) ensures that there exist two weights C_1, C_2 such that (4.17) holds. Referring to the constants C_i in (5.1)–(5.4) and in (5.18)–(5.20), we then choose

$$\begin{aligned}
(5.21) \quad & K_3 \doteq 1 + 4C_9, \quad K_4 \doteq \frac{1}{(8C_{10})}, \\
& K_8 \doteq 2 + 4C_2 C_{10}, \quad K_9 \doteq 1 + C_2 C_6 C_9 + 8C_{10}(10 + 2C_2 C_{10} + C_{10}),
\end{aligned}$$

Finally, we choose $\delta > 0$ sufficiently small and $\varepsilon_0 \doteq \sqrt{\delta}$.

▷ *STEP 2: Interaction between two small waves of different families.* To fix the ideas, assume that the waves σ_1^- and σ_2^- collide in the region on the left of the large 1-shock, producing the outgoing waves $\sigma_{1,1}^+, \dots, \sigma_{1,n_1}^+$ of the first family and $\sigma_{2,1}^+, \dots, \sigma_{2,n_2}^+$ of the second family. Observe that the waves $\sigma_1^-, \sigma_{1,\ell}^+$ all have the same sign, and the same holds for the waves $\sigma_2^-, \sigma_{2,\ell}^+$. Hence the corresponding coefficients in (3.27) satisfy $p_{i,\ell}^+ = p_i^-$. The only terms in (3.31) that change at the time of the interaction are Υ^b and Υ . The latter decreases, as shown in the proof of Proposition 1. Recalling (4.10) and (5.2), we now have

$$\begin{aligned}
\Delta V_{\xi}^b &= \sum_{i,\ell} p_{i,\ell}^+ |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| - \sum_i p_i^- |\sigma_i^- \xi_i^-| = \sum_i p_i^- \left(\sum_{\ell} |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| - |\sigma_i^- \xi_i^-| \right) \\
&\leq (1 + \varepsilon_0) C_9 |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|),
\end{aligned}$$

$$\begin{aligned}
\Delta Q_{\xi}^b &= \sum_{i,\ell} \sum_{(j,\beta) \in \mathcal{A}_{i,\ell}^+} |\sigma_{i,\ell}^+ \sigma_{j,\beta}| (p_{i,\ell}^+ |\xi_{i,\ell}^+| + p_{j,\beta} |\xi_{\beta}|) \\
&\quad - \sum_i \sum_{(j,\beta) \in \mathcal{A}_i^-} |\sigma_i^- \sigma_{j,\beta}| (p_i^- |\xi_i^-| + p_{j,\beta} |\xi_{\beta}|) - |\sigma_1^- \sigma_2^-| (p_1^- |\xi_1^-| + p_2^- |\xi_2^-|)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_i p_i^- \left| \sum_\ell |\sigma_{i,\ell}^+ \xi_{i,\ell}^+| - |\sigma_i^- \xi_i^-| \right| V^{b-} + \sum_i p_i^- \left| \sum_\ell |\sigma_{i,\ell}^+| - |\sigma_i^-| \right| V_\xi^{b-} \\
&\quad - \frac{2}{3} |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|) \\
&\leq (1 + \varepsilon_0) C_9 |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|) V^{b-} + \mathcal{O}(1) |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|) V_\xi^{b-} \\
&\quad - \frac{2}{3} |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|) \\
&\leq \mathcal{O}(1) |\sigma_1^- \sigma_2^-| (|\sigma_1^-| + |\sigma_2^-|) V_\xi^{b-} - \frac{1}{2} |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|),
\end{aligned}$$

$$\Delta Q^b \leq -\frac{1}{2} |\sigma_1^- \sigma_2^-|,$$

provided that (5.5) and (5.17) hold, with $\delta > 0$ small enough. Recalling (3.29), from the above inequalities we obtain

$$(5.22) \quad \Delta \Upsilon_\xi^b = \Delta V_\xi^b + K_3 \Delta Q_\xi^b + (\Delta V_\xi^b) Q^{b+} + V_\xi^{b-} \Delta Q^b \leq 0.$$

An entirely similar argument applies to interactions taking place in the region between the two large shocks, or in the region at the right of the 2-shock.

▷ **STEP 3: Interaction between two small shocks of the same family.** To fix the ideas, let $\sigma', \sigma'' < 0$ be two incoming shocks of the first family, and denote by $\sigma_1^+, \sigma_{2,1}^+, \dots, \sigma_{2,n_2}^+$ the outgoing waves. Assume that the interaction takes place on the left of the large 1-shock. Then by (5.3),

$$\Delta V_\xi^b \leq C_9 (1 + \varepsilon_0) |\sigma' \sigma''| (|\xi'| + |\xi''|)$$

while

$$\begin{aligned}
\Delta Q_\xi^b &= \sum_{(j,\beta) \in \mathcal{A}_1^+} |\sigma_1^+ \sigma_{j,\beta}| (p_1^+ |\xi_1^+| + p_{j,\beta} |\xi_\beta|) \\
&\quad + \sum_\ell \sum_{(j,\beta) \in \mathcal{A}_{2,\ell}^+} |\sigma_{2,\ell}^+ \sigma_{j,\beta}| (p_{2,\ell}^+ |\xi_{2,\ell}^+| + p_{j,\beta} |\xi_\beta|) \\
&\quad - \sum_{(j,\beta) \in \mathcal{A}'} |\sigma' \sigma_{j,\beta}| (p' |\xi'| + p_{j,\beta} |\xi_\beta|) - \sum_{(j,\beta) \in \mathcal{A}''} |\sigma'' \sigma_{j,\beta}| (p'' |\xi''| + p_{j,\beta} |\xi_\beta|) \\
&\quad - |\sigma' \sigma''| (p' |\xi'| + p'' |\xi''|) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \left((1-\varepsilon_0)|\sigma_1^+ \xi_1^+| - |\sigma' \xi'| - |\sigma'' \xi''| + (1+\varepsilon_0) \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| \right) V^{\flat-} \\
&\quad + \left((1-\varepsilon_0)|\sigma_1^+ - (\sigma' + \sigma'')| + (1+\varepsilon_0) \sum_{\ell} |\sigma_{2,\ell}^+| \right) V_{\xi}^{\flat-} - \frac{2}{3} |\sigma' \sigma''| (|\xi'| + |\xi''|) \\
&\leq C_9(1+\varepsilon_0) |\sigma' \sigma''| (|\xi'| + |\xi''|) V^{\flat-} \\
&\quad + \mathcal{O}(1) |\sigma' \sigma''| (|\sigma'| + |\sigma''|) V_{\xi}^{\flat-} - \frac{2}{3} |\sigma' \sigma''| (|\xi'| + |\xi''|) \\
&\leq \mathcal{O}(1) |\sigma' \sigma''| (|\sigma'| + |\sigma''|) V_{\xi}^{\flat-} - \frac{1}{2} |\sigma' \sigma''| (|\xi'| + |\xi''|),
\end{aligned}$$

$$\Delta Q^{\flat} \leq -\frac{1}{2} |\sigma' \sigma''|.$$

By the choice of K_3 in (5.21), we again obtain (5.22) for δ sufficiently small.

▷ **STEP 4: Interaction between two small waves of the same family but of different sign.** To fix the ideas, assume that $\sigma' > 0$ and $\sigma'' < 0$, both belong to the first family, the other case being similar. Observe that in this case one must have $\sigma_1^+ \leq 0$, otherwise the two incoming wavefronts would have exactly the same speed, and could not interact. We now estimate

$$\begin{aligned}
\Delta V_{\xi}^{\flat} &= (1-\varepsilon_0)|\sigma_1^+ \xi_1^+| + \sum_{\ell} p_{2,\ell}^+ |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| - (1+\varepsilon_0)|\sigma' \xi'| - (1-\varepsilon_0)|\sigma'' \xi''| \\
&= C_9(1+\varepsilon_0) |\sigma' \sigma''| (|\xi'| + |\xi''|) - 2\varepsilon_0 |\sigma' \xi'|, \\
\Delta Q_{\xi}^{\flat} &= \sum_{(j,\beta) \in \mathcal{A}_1^+} (|\sigma_1^+ \sigma_{j,\beta}| (p_1^+ |\xi_1^+| + p_{j,\beta} |\xi_{\beta}|)) \\
&\quad + \sum_{\ell} \sum_{(j,\beta) \in \mathcal{A}_{2,\ell}^+} (|\sigma_{2,\ell}^+ \sigma_{j,\beta}| (p_{2,\ell}^+ |\xi_{2,\ell}^+| + p_{j,\beta} |\xi_{\beta}|)) \\
&\quad - \sum_{(j,\beta) \in \mathcal{A}'} (|\sigma' \sigma_{j,\beta}| (p' |\xi'| + p_{j,\beta} |\xi_{\beta}|)) \\
&\quad - \sum_{(j,\beta) \in \mathcal{A}''} (|\sigma'' \sigma_{j,\beta}| (p'' |\xi''| + p_{j,\beta} |\xi_{\beta}|)) - |\sigma' \sigma''| (p' |\xi'| + p'' |\xi''|) \\
&\leq \left((1-\varepsilon_0)|\sigma_1^+ \xi_1^+| - |\sigma'' \xi''| + (1+\varepsilon_0) \sum_{\ell} |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| \right) V^{\flat-} \\
&\quad + (1+\varepsilon_0) \sum_{\ell} |\sigma_{2,\ell}^+| V_{\xi}^{\flat-} - \frac{2}{3} |\sigma' \sigma''| (|\xi'| + |\xi''|) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{O}(1)|\sigma'\sigma''|(|\xi'|+|\xi''|)V^{b-}+|\sigma'\xi'|V^{b-} \\
&\quad +\mathcal{O}(1)|\sigma'\sigma''|(|\sigma'|+|\sigma''|)V_{\xi}^{b-}-\frac{2}{3}|\sigma'\sigma''|(|\xi'|+|\xi''|) \\
&\leq \mathcal{O}(1)|\sigma'\sigma''|(|\sigma'|+|\sigma''|)V_{\xi}^{b-}+|\sigma'\xi'|V^{b-}-\frac{1}{2}|\sigma'\sigma''|(|\xi'|+|\xi''|).
\end{aligned}$$

Hence (5.22) still holds for δ sufficiently small.

▷ **STEP 5:** *Interaction between the large 1-shock and a small wave coming from the left.* Let the small incoming wave σ^- have shift rate ξ^- . By the estimates (5.18) and (5.21) one has

$$\begin{aligned}
\Delta V_{\xi}^b &= -p^-|\sigma^-\xi^-| \leq -\frac{2}{3}|\sigma^-\xi^-|, \\
\Delta Q_{\xi}^b &= -\sum_{(j,\beta)\in\mathcal{A}^-} |\sigma^-\sigma_{j,\beta}|(p^-|\xi^-|+p_{j,\beta}|\xi_{\beta}|) \leq 0, \\
\Delta Q^b &= -\sum_{(j,\beta)\in\mathcal{A}^-} |\sigma^-\sigma_{j,\beta}| \leq 0, \\
\Delta \Upsilon_{\xi}^b &\leq -\frac{2}{3}|\sigma^-\xi^-|.
\end{aligned}$$

By (4.14) and (5.18)

$$\begin{aligned}
\Delta V_{\xi}^{\natural} &= \sum_{\ell} C_2 p_{2,\ell}^+ |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| \leq (1+\varepsilon_0) C_2 C_{10} |\sigma^-| (|\xi^-| + |\widehat{\xi}_1^-|), \\
\Delta Q_{\xi}^{\natural} &= \sum_{\ell} \sum_{(j,\beta)\in\mathcal{A}_{2,\ell}^+} C_2 C_j |\sigma_{2,\ell}^+ \sigma_{j,\beta}| (p_{2,\ell}^+ |\xi_{2,\ell}^+| + p_{j,\beta} |\xi_{j,\beta}|) \\
&\leq \mathcal{O}(1) |\sigma^-| (|\xi^-| + |\widehat{\xi}_1^-|) V^{\natural-} + (1+\varepsilon_0) C_2 C_8 V_{\xi}^{\natural-} |\sigma^-|, \\
\Delta Q^{\natural} &= \sum_{\ell} \sum_{(j,\beta)\in\mathcal{A}_{2,\ell}^+} C_2 C_j |\sigma_{2,\ell}^+ \sigma_{j,\beta}| = \mathcal{O}(1) |\sigma^-| V^{\natural-}, \\
\Delta \Upsilon_{\xi}^{\natural} &\leq 2C_2 C_{10} |\sigma^-| (|\xi^-| + |\widehat{\xi}_1^-|) + 2K_3 C_2 C_8 |\sigma^-| V_{\xi}^{\natural-}.
\end{aligned}$$

By (3.30) and the choice of the weights in (5.21), for δ small enough it now follows

$$\begin{aligned}
\Delta \Upsilon_{\xi} &= (\Delta \Upsilon_{\xi}^{\natural} + K_8 \Delta \Upsilon_{\xi}^b + K_4 \Delta |\widehat{\xi}_1^-|) e^{K_9 \Upsilon^+} \\
&\quad + (\Upsilon_{\xi}^{\natural-} + K_8 (\Upsilon_{\xi}^{b-} + \Upsilon_{\xi}^{\natural-}) + K_4 (|\widehat{\xi}_1^-| + |\widehat{\xi}_2^-|)) (e^{K_9 \Upsilon^+} - e^{K_9 \Upsilon^-}) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \left[2C_2C_{10}|\sigma^-|(|\xi^-| + |\hat{\xi}_1^-|) + 2K_3C_2C_8\Upsilon_\xi^{\natural-}|\sigma^-| \right. \\
&\quad \left. - \frac{K_8}{2}|\sigma^- \xi^-| + K_4C_{10}|\sigma^-|(|\xi^-| + |\hat{\xi}_1^-|) \right] e^{K_9\Upsilon^+} \\
&\quad - K_9|\sigma^-|(\Upsilon_\xi^{\natural-} + K_8(\Upsilon_\xi^{\flat-} + \Upsilon_\xi^{\natural-}) + K_4|\hat{\xi}_1^-|) e^{K_9\Upsilon^+} \\
&\leq 0.
\end{aligned}$$

▷ **STEP 6:** *Interaction between the large 1-shock and a small 1-wave coming from the right.* Call σ_1^- the 1-wave that hits the large 1-shock and let ξ_1^- be its shift rate. Let $\sigma_{2,1}^+, \dots, \sigma_{2,n_2}^+$ be the outgoing small waves. From (4.17), (5.19) and (5.20) it follows

$$\begin{aligned}
\Delta V_\xi^{\natural} &= \sum_{\ell} C_2 p_{2,\ell}^+ |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| - C_1 p_1^- |\sigma_1^- \xi_1^-| \\
&\leq ((1 + \varepsilon_0)(1 + \mathcal{O}(1)\delta)C_2|\kappa_1\Theta_1| - (1 - \varepsilon_0)C_1)|\sigma_1^- \xi_1^-| + C_{10}|\sigma_1^- \widehat{\xi}_1^-| \\
&\leq -|\sigma_1^- \xi_1^-| + C_{10}|\sigma_1^- \widehat{\xi}_1^-|, \\
\Delta Q_\xi^{\natural} &\leq \sum_{\ell} \sum_{(j,\beta) \in \mathcal{A}_{2,\ell}^+} C_2 C_j |\sigma_{2,\ell}^+ \sigma_{j,\beta}| (p_{2,\ell}^+ |\xi_{2,\ell}^+| + p_{j,\beta} |\xi_\beta|) \\
&\leq \mathcal{O}(1)|\sigma_1^- \xi_1^-| V^{\natural-} + \mathcal{O}(1)|\sigma_1^- \widehat{\xi}_1^-| V^{\natural-} + (1 + C_6\delta)|\kappa_1| |\sigma_1^-| V_\xi^{\natural-}, \\
\Delta Q^{\natural} &= \sum_{\ell} \sum_{(j,\beta) \in \mathcal{A}_{2,\ell}^+} C_2 C_j p_{2,\ell}^+ |\sigma_{2,\ell}^+ \sigma_{j,\beta}| = \mathcal{O}(1)|\sigma_1^-| V^{\natural-}, \\
\Delta \Upsilon_\xi^{\natural} &\leq \left(-\frac{1}{2} + \mathcal{O}(1)\delta \right) |\sigma_1^- \xi_1^-| + (C_{10} + \mathcal{O}(1)\delta) |\sigma_1^- \widehat{\xi}_1^-| \\
&\quad + (K_3 + 1)(1 + C_6\delta) |\kappa_1| |\sigma_1^-| V_\xi^{\natural-}.
\end{aligned}$$

The previous estimates, together with the choice of the weights K_i in (5.21), yield

$$\begin{aligned}
\Delta \Upsilon_\xi &\leq (\Delta \Upsilon_\xi^{\natural} + K_4 \Delta |\hat{\xi}_1^-|) e^{K_9\Upsilon^+} + (\Upsilon_\xi^{\natural-} + K_4 |\hat{\xi}_1^-|) (e^{K_9\Upsilon^+} - e^{K_9\Upsilon^-}) \\
&\leq \left[\left(-\frac{1}{2} + \mathcal{O}(1) \cdot \delta \right) |\sigma_1^- \xi_1^-| + (C_{10} + \mathcal{O}(1) \cdot \delta) |\sigma_1^- \widehat{\xi}_1^-| \right. \\
&\quad \left. + (K_3 + 1)(1 + C_6\delta) |\kappa_1| |\sigma_1^-| V_\xi^{\natural-} + K_4 C_{10} |\sigma_1^-| (|\xi_1^-| + |\hat{\xi}_1^-|) \right] e^{K_9\Upsilon^+} \\
&\quad - K_9 |\sigma_1^-| (\Upsilon_\xi^{\natural-} + K_4 |\hat{\xi}_1^-|) e^{K_9\Upsilon^+} \leq 0.
\end{aligned}$$

The remaining cases, concerning interactions involving the second large shock, are entirely similar. $\square$

Proof of Proposition 3 in Case (SR). As in the previous cases, the estimates (5.1)–(5.4) hold at interactions between two small waves. Moreover, if a wave-front σ^- with shift rate ξ^- hits the large 1-shock either from the left or from the right, for some constants C_8, C_{10} we can assume that the estimates (5.18) still hold. Observe that the quantity Υ is strictly decreasing at every interaction.

▷ **STEP 1: Choice of the weights in (3.33)–(3.34).** Introduce the constants

$$(5.23) \quad \begin{aligned} K_1 &\doteq 1 + 3C_{10}, & K_2 &\doteq 1, & K_3 &\doteq 1 + 4C_9, \\ K_4 &\doteq 1 + 4C_9e^{(1+4C_9)\tilde{\sigma}_2+1}, & K_5 &\doteq 1 + 4C_9, & K_8 &\doteq 1 + 6C_{10}, \\ K_9 &\doteq 1 + 3C_{10} + 2(2K_4 + K_5 + (\tilde{\sigma}_2 + 1) + (1 + C_6)|\kappa_1|)C_8. \end{aligned}$$

Then choose $\delta > 0$ sufficiently small and set $\varepsilon_0 = \sqrt{\delta}$.

▷ **STEP 2: Interactions between two small waves on the left of the large 1-shock.** In this case, we clearly have $\Delta\Upsilon_\xi^\natural = \Delta|\hat{\xi}_1| = 0$. The fact that $\Upsilon_\xi^\natural$ decreases is proved exactly as in Steps 2, 3 and 4 of Case (SS).

▷ **STEP 3: Υ_ξ decreases at interactions on the right of the large shock.** In this case we have $\Delta\Upsilon_\xi^\flat = \Delta|\hat{\xi}_1| = 0$. The fact that $\Upsilon_\xi^\natural$ decreases across the interaction is proved as in Steps 2 and 3 of Case (RR).

▷ **STEP 4: Υ_ξ decreases when the wave σ^- shifting with speed ξ^- and coming from the left hits the large 1-shock.** With the same notation used in (5.18), for any wave-front not involved in the interaction, located on the right of the large 1-shock, one has

$$S_{i,\alpha} \leq 2\delta, \quad \Delta S_{i,\alpha} \leq 2 \sum_\ell |\sigma_{2,\ell}^+| \leq 2C_8|\sigma^-|, \quad \Delta R_\alpha \leq \sum_\ell |\sigma_{2,\ell}^+| \leq C_8|\sigma^-|.$$

Moreover, since

$$Q^\natural = O(1) \cdot \delta, \quad R_{2,\ell}^+ = O(1) \cdot \delta, \quad S_{2,\ell}^+ = O(1) \cdot \delta, \quad 0 \leq \Delta Q^\natural \leq (\tilde{\sigma}_2 + \delta)C_8|\sigma^-|,$$

choosing $\delta > 0$ small enough, we can assume

$$\begin{aligned}
(5.24) \quad & (1 + K_4 S_{2,\ell}^+) e^{K_5 R_{2,\ell}^+ + Q^{\natural+}} \leq 2, \\
& \Delta((1 + K_4 S_{i,\alpha}) e^{K_5 R_\alpha + Q^{\natural}}) \\
& \leq (1 + K_4 S_{i,\alpha}^-) e^{K_5 R_\alpha^- + Q^{\natural-}} \cdot (2K_4 \Delta S_{i,\alpha} + 2K_5 \Delta R_\alpha + 2\Delta Q^{\natural}) \\
& \leq 2(1 + K_4 S_{i,\alpha}^-) e^{K_5 R_\alpha^- + Q^{\natural-}} (2K_4 + K_5 + (\tilde{\sigma}_2 + 1)) C_8 |\sigma^-|.
\end{aligned}$$

Recalling (3.31)₁ and (3.33)₁, since $p^- \geq 1 - \varepsilon_0 > \frac{1}{2}$, one can see that

$$(5.25) \quad \Delta \Upsilon_\xi^b(\tau) \leq -\frac{1}{2} |\sigma^- \xi^-|.$$

Using (5.18) together with (5.24)–(5.25), we obtain

$$\begin{aligned}
(5.26) \quad \Delta \Upsilon_\xi^{\natural} & \leq \sum_{\ell} K_2 |\sigma_{2,\ell}^+ \xi_{2,\ell}^+| (1 + K_4 S_{2,\ell}^+) e^{K_5 R_{2,\ell}^+ + Q^{\natural+}} \\
& \quad + \sum_{\alpha > \alpha^b} \sum_{i=1}^2 K_i |\sigma_{i,\alpha} \xi_\alpha| \left[(1 + K_4 S_{i,\alpha}^+) e^{K_5 R_\alpha^+ + Q^{\natural+}} \right. \\
& \quad \left. - (1 + K_4 S_{i,\alpha}^-) e^{K_5 R_\alpha^- + Q^{\natural-}} \right] \\
& \leq 2K_2 C_{10} |\sigma^-| (|\xi^-| + |\hat{\xi}_1^-|) + \Upsilon_\xi^{\natural-} (4K_4 + 2K_5 + 2(\tilde{\sigma}_2 + 1)) C_8 |\sigma^-|.
\end{aligned}$$

The change in the right hand side of (3.34) across the interaction can now be estimated by

$$\begin{aligned}
\Delta \Upsilon_\xi & \leq \left\{ 2K_2 C_{10} |\sigma^-| (|\xi^-| + |\hat{\xi}_1^-|) + 2\Upsilon_\xi^{\natural-} (2K_4 + K_5 + \tilde{\sigma} + 1) C_8 |\sigma^-| \right. \\
& \quad \left. + K_6 C_{10} |\sigma^-| (|\xi^-| + |\hat{\xi}_1^-|) - \frac{K_8}{2} |\sigma^- \xi^-| \right\} e^{K_9 \Upsilon^+} \\
& \quad - (K_8 \Upsilon_\xi^{b-} + \Upsilon_\xi^{\natural-} + K_6 |\hat{\xi}_1^-|) K_9 |\sigma^-| e^{K_9 \Upsilon^+} \\
& \leq 0,
\end{aligned}$$

because of the choices of the constants K_i in (5.23).

▷ STEP 5: Υ_ξ decreases when a small 1-wave σ^- , with shift rate ξ^- , hits the large 1-shock from the right. In this case we have $\Delta\Upsilon_\xi^b = 0$, while all estimates in (5.19)–(5.20) remain valid. Concerning Υ_ξ^b , in place of (5.26) we now have

$$\begin{aligned}\Delta Q^b &\leq (1 + C_6)|\kappa_1| |\sigma^-|, \\ \Delta\Upsilon_\xi^b &\leq 2K_2C_{10}|\sigma_1^-|(|\xi^-| + |\hat{\xi}_1^-|) + 2\Upsilon_\xi^{b-}(2K_4 + K_5 + (1 + C_6)|\kappa_1|)C_8|\sigma^-| \\ &\quad - K_1|\sigma^-\xi^-|.\end{aligned}$$

The previous estimates together yield

$$\begin{aligned}\Delta\Upsilon_\xi &\leq \left\{ 2K_2C_{10}|\sigma^-|(|\xi^-| + |\hat{\xi}_1^-|) + 2\Upsilon_\xi^{b-}(2K_4 + K_5 + (1 + C_5)|\kappa_1|)C_8|\sigma^-| \right. \\ &\quad \left. - K_1|\sigma^-\xi^-| + K_6C_{10}|\sigma^-|(|\xi^-| + |\hat{\xi}_1^-|) \right\} e^{K_9\Upsilon^+} \\ &\quad (K_8\Upsilon_\xi^{b-} + \Upsilon_\xi^{b-} + K_6|\hat{\xi}_1^-|)K_9|\sigma^-|e^{K_9\Upsilon^+} \\ &\leq 0\end{aligned}$$

because of the choices of the constants K_i in (5.23). This completes the proof of Proposition 3 in Case (SR). $\square$

6. Proof of Theorem 1. Relying on the estimates proved in the previous two sections, we complete here the proof of Theorem 1.

Let $\varepsilon > 0$ be given. In all three cases (RR), (SS), (SR), for $\delta > 0$ sufficiently small, the wave-front tracking algorithm described in Section 3 constructs a globally defined ε -approximate solution for every initial condition $\bar{u} \in \mathcal{D}_\delta^\varepsilon$. The properties (i), (ii) and (iv) in Proposition 1 follow from the estimates on the total strength of waves, proved in Section 4, while (iii) follows by the same argument as Proposition 3 in [4].

Proposition 2 is proved here exactly as Proposition 5 in [4]. Because of the estimates obtained in Section 5, the length of a pseudopolygonal is a non-increasing function of time, i.e., Proposition 3 holds.

To prove Proposition 4, let $u, u' \in \mathcal{D}_{\delta'}^\varepsilon$ be given, with $\delta' > 0$ small. Observe that the construction of a pseudopolygonal joining u with u' must be performed with some care. For example, by taking

$$\gamma(\vartheta) = u \cdot \chi_{]-\infty, \vartheta]} + u' \cdot \chi_{] \vartheta, \infty[},$$

it would certainly follow $\|\gamma\| = \|u - u'\|_{\mathbf{L}^1}$. However, for certain values of ϑ , the waves u^ϑ may be much larger than those in u or u' , so that $u^\vartheta \notin \mathcal{D}_\delta^\varepsilon$, in general. We describe below the construction of a suitable pseudopolygonal, in the three distinct cases.

Case (RR). We follow here the same technique used in Section 9 of [4]. Assume that

$$u(x) = u'(x) = u^b \text{ for } x < a, \quad u(x) = u'(x) = u^\sharp \text{ for } x > b.$$

Recalling (3.8), let $(v_1, v_2)(x)$ be the coordinates of $u(x)$, and $(v'_1, v'_2)(x)$ those of $u'(x)$. Introduce the function w whose coordinates are

$$w_1 = \min\{v_1, v'_1\}, \quad w_2 = v_2.$$

The functions u and w can be joined by the pseudopolygonal $\gamma_1 : [a, b] \mapsto \mathcal{D}_\delta^\varepsilon$,

$$\gamma_1(\vartheta) \doteq w \cdot \chi_{]-\infty, \vartheta]} + u \cdot \chi_{] \vartheta, \infty[}.$$

Next, define w^* as

$$w_1^* = \min\{v_1, v'_1\}, \quad w_2^* \doteq \min\{v_2, v'_2\},$$

and connect w with w^* by the path $\gamma_2 : [a, b] \mapsto \mathcal{D}_\delta^\varepsilon$,

$$\gamma_2(\vartheta) \doteq w \cdot \chi_{]-\infty, \vartheta]} + w^* \cdot \chi_{] \vartheta, \infty[}.$$

Since u and u' play here a symmetric role, we can construct a pseudopolygonal joining w^* with u' , in the same way. It is now easy to check that, for $u, u' \in \mathcal{D}_\delta^\varepsilon$, with $\delta' > 0$ sufficiently small, all these pseudopolygons take values within $\mathcal{D}_\delta^\varepsilon$ and the weighted length of their concatenation is bounded by a constant multiple of $\|u - u'\|_{\mathbf{L}^1}$.

Case (SS). For $i = 1, 2$, call x_i, x'_i the positions of the i^{th} large shock in u and u' , respectively. One can then connect u with u' in such a way that each intermediate state u^ϑ contains exactly two large shocks. For example, assume $x_1 < x'_1 < x_2 < x'_2$, the other cases being entirely similar. We first define the path $\gamma_1 : [x_2, x'_2] \mapsto \mathcal{D}_\delta^\varepsilon$,

$$\gamma_1(\vartheta) \doteq u \cdot \chi_{]-\infty, x_2] \cup] \vartheta, \infty[} + u' \cdot \chi_{] x_2, \vartheta]}$$

joining u with the intermediate function

$$w(x) = \begin{cases} u(x) & \text{if } x \in]-\infty, x_2] \cup] x'_2, \infty[, \\ u'(x) & \text{if } x \in] x_2, x'_2]. \end{cases}$$

We then connect w with u' by setting

$$\gamma_2(\vartheta) \doteq u' \cdot \chi_{]-\infty, \vartheta]} + w \cdot \chi_{] \vartheta, \infty[}.$$

The concatenation of γ_1 and γ_2 yields the desired path.

Case (SR). To fix the ideas, let $x_1 < x'_1$ be the locations of the large shocks in u and u' , respectively. In this case, we first define the path $\gamma_1 : [x_1, x'_1] \mapsto \mathcal{D}_\delta^\varepsilon$,

$$\gamma_1(\vartheta) \doteq u' \cdot \chi_{]-\infty, \vartheta] \cup]x'_1, \infty[} + u^\sharp \cdot \chi_{] \vartheta, x'_1]}$$

connecting u' with the intermediate function

$$w(x) = \begin{cases} u'(x) & \text{if } x \in]-\infty, x_1] \cup]x'_1, \infty[, \\ u^\sharp & \text{if } x \in]x_1, x'_1]. \end{cases}$$

Next, we consider the function $w' = (w'_1, w'_2)$ whose coordinates are

$$w'_1 = \min\{w_1, u_1\}, \quad w'_2 = \min\{w_2, u_2\}.$$

Both u and w can be connected with w' by the paths

$$\gamma_2(\vartheta) = w' \cdot \chi_{]-\infty, \vartheta]} + u \cdot \chi_{] \vartheta, \infty[},$$

$$\gamma_3(\vartheta) = w' \cdot \chi_{]-\infty, \vartheta]} + w \cdot \chi_{] \vartheta, \infty[}.$$

A concatenation of $\gamma_1, \gamma_2, \gamma_3$ now provides the required path, joining u with u' . This completes the proof of Proposition 4.

Proposition 5 is an immediate consequence of the two previous ones.

Concerning Proposition 6, to show that the right hand side of (3.36) is a Cauchy sequence, we use the estimate

$$(6.1) \quad \|u(\tau) - S_{\tau}^{\varepsilon_n} \bar{u}_n\|_{\mathbf{L}^1} \leq L \cdot \left\{ \|u(0) - \bar{u}_n\|_{\mathbf{L}^1} + \int_0^\tau \limsup_{h \rightarrow 0^+} \frac{1}{h} \|u(t+h) - S_h^{\varepsilon_n} u(t)\|_{\mathbf{L}^1} dt \right\},$$

valid for any piecewise constant function u constructed by a wave-front tracking algorithm. When $u(t) = S_t^{\varepsilon_m} \bar{u}_m$, the right hand side of (6.1) tends to zero, as $m, n \rightarrow \infty$. Hence the limit semigroup $S = \lim_{\varepsilon \rightarrow \infty} S^\varepsilon$ is well defined by (3.36). The properties (i)–(v) in Theorem 1 are now proved exactly as in Proposition 9 of [4].

7. Proof of Theorems 2 and 3. Let $\bar{u} : \mathbb{R} \mapsto \mathbb{R}^2$ be a BV function satisfying all of the assumptions in Theorem 1. Applying Theorem 1, we can cover the real line with finitely many open intervals, say

$$I_1 =]-\infty, b_1[, \quad I_2 =]a_2, b_2[, \quad \dots, \quad I_N =]a_N, \infty[,$$

such that the following holds. For each $j = 1, \dots, N$, the Cauchy problem with initial data

$$(7.1) \quad u(0, x) = \bar{u}_j(x) = \begin{cases} \bar{u}(a_j) & \text{if } x \leq a_j, \\ \bar{u}(x) & \text{if } x \in]a_j, b_j[, \\ \bar{u}(b_j) & \text{if } x \geq b_j, \end{cases}$$

admits a viscosity solution u_j taking values within the domain of a suitable semigroup S^j . More precisely, $u_j(t, \cdot) = S_t^j \bar{u}_j$. Let λ^* be an upper bound for all characteristic speeds, as in (2.3). For $t \geq 0$, $x \in \mathbb{R}$, set

$$(7.2) \quad u(t, x) = u_j(t, x) \quad \text{if } [x - t\lambda^*, x + t\lambda^*] \subset]a_j, b_j[.$$

We claim that the function u is well defined on a strip $[0, T] \times \mathbb{R}$, and provides a viscosity solution to (1.1)–(1.2) satisfying (A3) at every point.

As a preliminary, to show the local uniqueness of viscosity solutions, consider a domain of the form

$$(7.3) \quad \Delta \doteq \{(t, x); t \in [0, \rho'], x \in J_t \doteq]a + t\lambda^*, b - t\lambda^*[\},$$

with $0 < \rho' < (b - a)/2$. For any function $u : \Delta \mapsto \mathbb{R}^2$, define the extension

$$(7.4) \quad u_\Delta(t, x) \doteq \begin{cases} u^b & \text{if } (t, x) \notin \Delta, x < \frac{a+b}{2}, \\ u(t, x) & \text{if } (t, x) \in \Delta, \\ u^\# & \text{if } (t, x) \notin \Delta, x > \frac{a+b}{2}. \end{cases}$$

Here the states u^b , $u^\#$ are as in (1.3).

Lemma 6. *Let $S : [0, \infty[\times \mathcal{D} \mapsto \mathcal{D}$ be a semigroup constructed as in Theorem 1. With the notations above, let $u : \Delta \mapsto \mathbb{R}^2$ satisfy (2.17)–(2.18) at every point of Δ . Moreover, assume that the map $t \mapsto u_\Delta(t, \cdot)$ is continuous in $\mathbf{L}_{\text{loc}}^1$, with values inside $\mathcal{D}$. Then, for each $\tau \in [0, \rho'[$, the function $u(\tau, \cdot)$ coincides with $S_\tau u_\Delta(0)$, restricted to the interval J_τ .*

Proof. The proof relies on the same techniques used for Theorem 2 and Corollary 3 in [2]. Denote by L the Lipschitz constant of the semigroup. Because of the finite propagation speed, for any $u, v \in \mathcal{D}$ and $0 \leq \tau < \tau' < \rho'$ there holds

$$\int_{J_{\tau'}} |S_{\tau' - \tau} u - S_{\tau' - \tau} v| dx \leq L \cdot \int_{J_\tau} |u - v| dx,$$

with J_t as in (7.3). One then establishes the bound

$$\begin{aligned}
 (7.5) \quad & \|S_\tau u_\Delta(0) - u(\tau)\|_{\mathbf{L}^1(J_\tau)} \\
 & \leq L \cdot \int_0^\tau \limsup_{h \rightarrow 0^+} \frac{1}{h} \|S_h u_\Delta(t) - u(t+h)\|_{\mathbf{L}^1(J_{t+h})} dt.
 \end{aligned}$$

Using the fact that u is a viscosity solution, we check that the integrand on the right hand side of (7.5) vanishes for all t , proving the lemma. $\square$

If now $a_i < a_j < b_i < b_j$ for some $i, j \in \{1, \dots, N\}$, consider the overlap triangle

$$\Delta_{ij} \doteq \{(t, x); t \geq 0, a_j + t\lambda^* < x < b_i - t\lambda^*\}.$$

Since u_i, u_j are both viscosity solutions, an application of Lemma 6 now yields $u_i = u_j$ on Δ_{ij} . Therefore, the function u in (7.2) is well defined.

To prove that the property (A3) holds, let $(\bar{t}, \bar{x}) \in [0, T[$ and $\varepsilon > 0$ be given, and set $u^\flat = u(\bar{t}, \bar{x}-)$, $u^\sharp = u(\bar{t}, \bar{x}+)$. Then there exists $\rho > 0$ and a semigroup $S : [0, \infty[\times \mathcal{D} \mapsto \mathcal{D}$ with all properties listed in Theorem 1, such that

$$(7.6) \quad Q(u_\Delta(\bar{t}, \cdot)) < \varepsilon, \quad u_\Delta(\bar{t}, \cdot) \in \mathcal{D}.$$

with u_Δ defined as in (2.22). By Lemma 6 we have

$$(7.7) \quad u(t, x) = (S_{t-\bar{t}} u_\Delta(\bar{t}))(x) \quad \forall (t, x) \in \Delta.$$

By (7.6), the estimates proved in Section 4 now imply

$$(7.8) \quad Q(u_\Delta(t)) \leq Q(S_{t-\bar{t}} u_\Delta(\bar{t})) \leq Q(u_\Delta(\bar{t})) < \varepsilon.$$

This completes the proof of Theorem 2. $\square$

In order to establish Theorem 3, let two solutions u, v be given, satisfying all requirements but with $u \neq v$. By continuity and finite speed of propagation, there exists a point $(\bar{t}, \bar{x}) \in [0, T[\times \mathbb{R}$ such that $u(\bar{t})$ coincides with $v(\bar{t})$ on the interval $[\bar{x}-1, \bar{x}+1]$, while

$$(7.9) \quad \int_0^{\rho/\lambda^*} \int_{\bar{x}-\rho+(t-\bar{t})\lambda^*}^{\bar{x}+\rho-(t-\bar{t})\lambda^*} |u(t, x) - v(t, x)| dx dt > 0$$

for all $\rho > 0$. By the assumptions (A1)–(A3), there exists $\rho \in]0, 1]$, $\rho' \in]0, \rho/\lambda^*]$ and a semigroup $S : [0, \infty[\times \mathcal{D} \mapsto \mathcal{D}$ such that, with u_Δ, v_Δ as in (2.22), one has

$$u_\Delta(t) \in \mathcal{D}, v_\Delta(t) \in \mathcal{D} \quad \forall t \in [\bar{t}, \bar{t} + \rho'].$$

We can thus apply Lemma 6 and deduce

$$u(t, x) = (S_{t-\bar{t}} u_\Delta(\bar{t}))(x) (S_{t-\bar{t}} v_\Delta(\bar{t}))(x) = v(t, x) \quad \forall (t, x) \in \Delta, t \in [\bar{t}, \bar{t} + \rho'].$$

This yields a contradiction with (7.9), proving Theorem 3. $\square$

8. Remarks on the non-resonance condition. Assume that the Riemann problem (1.1), (1.3) admits a self-similar solution $\tilde{u}$ containing two shocks, say

$$(8.1) \quad \tilde{u}(t, x) = \begin{cases} u^b & \text{if } x < \Lambda_1 t, \\ u^\natural & \text{if } \Lambda_1 t < x < \Lambda_2 t, \\ u^\sharp & \text{if } x > \Lambda_2 t, \end{cases}$$

and let the stability conditions (2.6), (2.7) and (2.14) hold. Aim of this section is to show that, if the non-resonance assumption (2.12) is replaced by the opposite inequality

$$(8.2) \quad |\kappa_1 \Theta_1| \cdot |\kappa_2 \Theta_2| > 1,$$

then the conclusion of Theorem 1 cannot hold. More precisely, recalling (2.15), for every $\rho > 0$ the system (1.1) cannot generate a Lipschitz continuous flow, on any non-trivial closed domain $\mathcal{D} \supset \tilde{\mathcal{D}}_\rho$.

Indeed, let $T > 0$ and $n \geq 1$ be given. Call $x_1(t) \doteq \Lambda_1 t$, $x_2(t) \doteq \Lambda_2 t$ the positions of the two shocks in the self-similar solution (8.1). Then there exists $\tau > 0$ and two polygonal lines

$$t \mapsto y_1(t), y_2(t), \quad t \in [\tau, T]$$

bouncing back and forth exactly n times between the large shocks, with

$$\Lambda_1 \tau < y_1(\tau) < y_2(\tau) < \Lambda_2 \tau, \quad \Lambda_1 T < y_1(T) < y_2(T) < \Lambda_2 T,$$

$$\dot{y}_i(t) \in \lambda_1(u^\natural) \quad \text{or} \quad \dot{y}_i(t) \in \lambda_2(u^\natural),$$

and with $\dot{y}_i(t) = \lambda_1(u^\natural)$ at the initial and at the final time (see Fig. 5 for the case $n = 2$).

We now construct a family of perturbed solutions u^ε as follows. Let φ be a non-negative $\mathcal{C}^\infty$ function such that

$$\text{Supp}(\varphi) \subseteq [y_1(\tau), y_2(\tau)], \quad \int \varphi(s) ds = 1.$$

For $\varepsilon \geq 0$ small, let $u^\varepsilon : [\tau, T] \times \mathbb{R} \mapsto \mathbb{R}^2$ be the solution of (1.1) with initial condition

$$(8.3) \quad u^\varepsilon(\tau, x) = \tilde{u}(\tau, x) + \varepsilon \varphi(x) r_1(u^\natural).$$

For $\varepsilon \in [0, \varepsilon_0]$ sufficiently small, the perturbed solution u^ε remains piecewise smooth on its domain, with two large shocks located at $x_1^\varepsilon(t) < x_2^\varepsilon(t)$. Therefore,

Figure 5

it admits a first-order approximation in terms of a generalized tangent vector, according to the theory developed in [5].

Let $(v, \xi) \in \mathbf{L}^1 \times \mathbb{R}^2$ be the solution of the following linear hyperbolic mixed problem:

(8.4)
$$v_t + DF(\tilde{u})v_x = 0$$

outside the shocks, together with the boundary conditions

(8.5)
$$\begin{aligned} \langle Dl_2(u^\natural, u^b) \cdot (v(t, t\Lambda_1+), v(t, t\Lambda_1-)), u^\natural - u^b \rangle \\ + \langle l_2(u^\natural, u^b), v(t, t\Lambda_1+) - v(t, t\Lambda_1-) \rangle = 0, \end{aligned}$$

(8.6)
$$\dot{\xi}_1 = D\lambda_1(u^\natural, u^b) \cdot (v(t, t\Lambda_1+), v(t, t\Lambda_1-)),$$

along the 1-shock,

$$(8.7) \quad \langle Dl_1(u^\sharp, u^\natural) \cdot (v(t, t\Lambda_2+), v(t, t\Lambda_2-)), u^\sharp - u^\natural \rangle \\ + \langle l_1(u^\sharp, u^\natural), v(t, t\Lambda_2+) - v(t, t\Lambda_2-) \rangle = 0,$$

$$(8.8) \quad \dot{\xi}_2 = D\lambda_2(u^\sharp, u^\natural) \cdot (v(t, t\Lambda_2+), v(t, t\Lambda_2-)),$$

along the 2-shock, and with initial conditions

$$(8.9) \quad v(\tau, x) = \varphi(x)r_1(u^\natural), \quad \xi_1(\tau) = \xi_2(\tau) = 0.$$

We can now introduce the first-order approximation

$$(8.10) \quad w^\varepsilon(t, x) \doteq \tilde{u}(t, x) + \varepsilon v(t, x) + \sum_{\xi_i(t) < 0} \Delta \tilde{u}(t, x_i(t)) \chi_{[x_i(t) - \varepsilon \xi_i, x_i(t)]} \\ - \sum_{\xi_i(t) > 0} \Delta \tilde{u}(t, x_i(t)) \chi_{[x_i(t), x_i(t) + \varepsilon \xi_i]}.$$

By Theorem 2.2 in [5] we have

$$(8.11) \quad \lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon} \|u^\varepsilon(t, \cdot) - w^\varepsilon(t, \cdot)\| = 0 \quad \forall t \in [\tau, T].$$

Observe that, at time $t = T$, the component v of the solution to (8.4)–(8.9) can be readily computed, namely

$$(8.12) \quad v(T, x) = (\kappa_1 \kappa_2)^n \varphi \left(y_1(\tau) + \frac{x - y_1(T)}{|\Theta_1 \Theta_2|^n} \right) r_1(u^\natural).$$

From (8.11)–(8.12) it now follows

$$(8.13) \quad \lim_{\varepsilon \rightarrow 0+} \frac{\|u^\varepsilon(T, \cdot) - \tilde{u}(T, \cdot)\|_{\mathbf{L}^1}}{\|u^\varepsilon(\tau, \cdot) - \tilde{u}(\tau, \cdot)\|_{\mathbf{L}^1}} \\ = \lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon} \|w^\varepsilon(T, \cdot) - \tilde{u}(T, \cdot)\|_{\mathbf{L}^1} \\ \geq \|v(T, \cdot)\| + |u^\natural - u^\sharp| |\xi_1(T)| + |u^\sharp - u^\natural| |\xi_2(T)| \\ \geq \|v(T, \cdot)\|_{\mathbf{L}^1} = |\kappa_1 \kappa_2 \Theta_1 \Theta_2|^n.$$

By (8.13), any Lipschitz constant for the flow generated by (1.1) in a neighborhood of $\tilde{u}$ must be greater than $|\kappa_1 \kappa_2 \Theta_1 \Theta_2|^n$. Since n is arbitrary, if (8.2) holds, no Lipschitz constant can exist.

The remainder of this section is concerned with the p -system (1.4), assuming $p'(v) < 0$, $p''(v) > 0$. We claim that, for any Riemann problem solved in terms of two shocks as in (7.1), the non-resonance conditions (2.12) are always satisfied.

Indeed, calling

$$q(v', v'') \doteq \frac{p(v'') - p(v')}{v'' - v'},$$

after some elementary computations one finds

$$(8.14) \quad \kappa_1 = \frac{D_1 \Phi_1(u^\natural, u^\sharp) \cdot r_2(u^\natural)}{D_1 \Phi_1(u^\natural, u^\sharp) \cdot r_1(u^\natural)} = - \left(\frac{\sqrt{-q(u_1^\natural, u_1^\sharp)} - \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\natural, u_1^\sharp)} + \sqrt{-p'(u_1^\natural)}} \right)^2,$$

$$\kappa_2 = \frac{D_2 \Phi_2(u^\flat, u^\natural) \cdot r_1(u^\natural)}{D_2 \Phi_2(u^\flat, u^\natural) \cdot r_2(u^\natural)} = - \left(\frac{\sqrt{-q(u_1^\flat, u_1^\natural)} - \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\flat, u_1^\natural)} + \sqrt{-p'(u_1^\natural)}} \right)^2.$$

Furthermore,

$$(8.15) \quad \Theta_1 = \frac{\Lambda_1 - \lambda_2(u^\natural)}{\Lambda_1 - \lambda_1(u^\natural)} = \frac{\sqrt{-q(u_1^\flat, u_1^\natural)} + \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\flat, u_1^\natural)} - \sqrt{-p'(u_1^\natural)}},$$

$$\Theta_2 = \frac{\Lambda_2 - \lambda_1(u^\natural)}{\Lambda_2 - \lambda_2(u^\natural)} = \frac{\sqrt{-q(u_1^\natural, u_1^\sharp)} + \sqrt{-p'(u_1^\natural)}}{\sqrt{-q(u_1^\natural, u_1^\sharp)} - \sqrt{-p'(u_1^\natural)}}.$$

Together, (8.14) and (8.15) yield the identities

$$|\kappa_1| = \frac{1}{|\Theta_1|^2} < 1, \quad |\kappa_2| = \frac{1}{|\Theta_2|^2} < 1,$$

which imply (2.12).

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