Multi-dimensional Diffusion Waves for the Navier-Stokes Equations of Compressible Flow

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ABSTRACT. We derive a detailed, pointwise description of the asymptotic behavior of solutions of the Cauchy problem for the Navier-Stokes equations of compressible flow in several space dimensions, with initial data in $L^1 \cap H^{k(n)}$. We show that, asymptotically, the solution decomposes into the sum of two terms, one of which dominates in L^p for $p > 2$, the other for $p < 2$. The dominant term for $p > 2$ has constant density and divergence-free momentum field, decaying at the rate of a heat kernel. Thus, as measured in L^p for $p > 2$, *all smooth, small-amplitude solutions of the Navier-Stokes equations are asymptotically incompressible*. When $p < 2$, the dominant term reflects instead the spreading effect of convection, and decays more slowly than a heat kernel; in fact, the solution may grow without bound in L^p for p near 1. These features of the solution do not arise in one dimensional flow, nor are they apparent from previously known L^2 decay rates. An alternative interpretation of our estimates shows that the solution is asymptotically given by a diffusion about the origin whose mass is that of the initial data, convected by the fundamental solution of the linearized Euler equations. This result thus defines the correct notion of “diffusion wave” in the context of compressible, Navier-Stokes flows. Our analysis involves a number of interesting issues in Fourier multiplier and Paley-Wiener theory, required for the derivation of pointwise bounds from representations of the Fourier transforms of Green’s matrices.

1. Introduction. In this paper we derive a detailed description of the asymptotic behavior of solutions of the Cauchy problem for the Navier-Stokes equations of compressible flow in several space variables,

$$(1.1) \quad \begin{cases} \rho_t + \operatorname{div} m = 0 \\ m_t^j + \operatorname{div} \left(\frac{m^j m}{\rho} \right) + P(\rho)_{x_j} = \varepsilon \Delta \left(\frac{m^j}{\rho} \right) + \eta \operatorname{div} \left(\frac{m}{\rho} \right)_{x_j} \end{cases}$$

with smooth initial data $\begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix}$ close to a constant state $\begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$, which without loss of generality we take to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here $t \geq 0$ is time, $x \in \mathbb{R}^n$ is the spatial coordinate, and $\rho(x, t)$, $m(x, t) = (m^1(x, t), \dots, m^n(x, t))$, and $P = P(\rho)$ represent the fluid density, momentum density, and pressure; (thus m/ρ is the particle velocity). $\varepsilon > 0$ and $\eta \geq 0$ are viscosity constants, and div and Δ are the usual spatial divergence and Laplace operators.

We derive explicit pointwise bounds for the asymptotic state of the perturbation

$$(1.2) \quad u(x, t) = \begin{bmatrix} \rho(x, t) \\ m(x, t) \end{bmatrix} - \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$$

in terms of x and t ; these bounds then yield decay estimates for $D_x^\alpha u(\cdot, t)$ in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$. We find that $\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix}$ is time-asymptotic to the solution $\begin{bmatrix} \tilde{\rho}(\cdot, t) \\ \tilde{m}(\cdot, t) \end{bmatrix}$ of the following linear effective artificial viscosity system, which is associated with the nonlinear Navier-Stokes equations (1.1) in a canonical way:

$$(1.3) \quad \begin{cases} \tilde{\rho}_t + \operatorname{div} \tilde{m} = \frac{1}{2}(\varepsilon + \eta)\Delta \tilde{\rho} \\ \tilde{m}_t + c^2 \nabla \tilde{\rho} = \varepsilon \Delta \tilde{m} + \frac{1}{2}(\eta - \varepsilon)\nabla \operatorname{div} \tilde{m} \end{cases},$$

where $c^2 = P'(\rho^*) = P'(1)$. The system (1.3) is uniquely characterized by the approximation property above, by its strict parabolicity, and by the fact that the symbol of its first-order convection terms commutes with that of its second-order diffusion terms. These properties enable us to give a very detailed description of the asymptotic behavior of $\begin{bmatrix} \tilde{\rho}(\cdot, t) \\ \tilde{m}(\cdot, t) \end{bmatrix}$, and thereby that of $\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix}$. We find that, modulo the constant state $\begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$, the solution $\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix}$ of (1.1) is

asymptotically the sum of two terms, one of which dominates in L^p for $p > 2$, the other for $p < 2$. For $p > 2$ the dominant term has constant density ρ^* and a nonconstant, divergence-free momentum field, decaying at the rate of a heat kernel, which is the rate of decay of the solution of a linear, second order, strictly parabolic system with L^1 initial data. This result gives the very important conclusion that, as measured in $L^p(\mathbb{R}^n)$ for $p > 2$, *all smooth, small-amplitude solutions of the nonlinear Navier-Stokes equations (1.1) are asymptotically incompressible*. Thus, overall, the perturbation $u(\cdot, t)$ defined in (1.2)

decays at the heat kernel rate, but the difference between the solution $\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix}$

and the aforementioned incompressible flow decays more rapidly, at a rate given explicitly in terms of n and p . For $p < 2$ the situation is quite different. Here the asymptotically dominant term reflects the spreading effect of the solution operator for the standard multi-dimensional wave equation. As a result, the perturbation $u(\cdot, t)$ may actually grow without bound in $L^p(\mathbb{R}^n)$ for $p < 2$; for example, $\|u(\cdot, t)\|_{L^1} \sim t^{1/2}$ when $n = 3$. Both these effects, the decay to an incompressible flow in L^p for $p > 2$, and the retardation of the parabolic decay in L^p for $p < 2$, are truly multidimensional in character: the first is without significance in one space dimension, and the second does not occur.

Our work extends that of Zeng [22], Liu and Zeng [16], and Kawashima [8]–[9]. Zeng and Liu study general hyperbolic-parabolic systems in one space variable. Just as here, they obtain L^p decay rates for $p \in [1, \infty]$ by integrating pointwise bounds. They find that, for problems in one space dimension, the perturbation decays at the heat kernel rate in all L^p norms. Kawashima [8] studies general hyperbolic-parabolic systems in several space variables, deriving decay rates in $H^s(\mathbb{R}^n)$ by Fourier transform techniques. L^p estimates are then obtained by interpolation, but for $p \geq 2$ only. Just as here, he finds that the overall decay rate in these norms is the heat kernel rate, but he does not obtain the incompressible limit result alluded to above, which is special to the Navier-Stokes system, nor can his techniques address the question of pointwise estimates and L^p behavior for $p < 2$. Still, the initial part of our analysis parallels that of Kawashima, and we make important use of his idea of an effective artificial viscosity system. To obtain explicit pointwise bounds for the asymptotic state in terms of x and t , however, we must go considerably beyond the usual analysis based on the Hausdorff-Young inequality; this leads us to address several interesting questions in harmonic analysis related to Fourier multiplier and Paley-Wiener theory. As a by-product of this analysis we are able to reduce significantly the smoothness requirements on the initial data.

Because the artificial viscosity system (1.3) is linear and strictly parabolic, standard techniques enable us to show that, when the initial perturbation $u_0 = \begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix} - \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$ has a first moment, the corresponding solution $\begin{bmatrix} \tilde{\rho} \\ \tilde{m} \end{bmatrix} - \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$ of

(1.3) is asymptotic to the Green's matrix $\tilde{G}$ for (1.3), multiplied by the total mass vector U of u_0 . Our results therefore show that the same will be true also for the solution $u = \begin{bmatrix} \rho \\ m \end{bmatrix} - \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$ of the nonlinear Navier–Stokes system (1.1).

This asymptotic flow $\tilde{G}U$ has its mass initially concentrated at the origin, and, owing to the special properties of $\tilde{G}$ discussed above, can be interpreted as a simple diffusion convected by the fundamental solution of the linearized Euler equations. $\tilde{G}U$ is therefore a “diffusion wave,” and an important corollary of our results is therefore that the perturbation u associated with the solution of the nonlinear Navier–Stokes equations (1.1) is asymptotic to a diffusion wave whose total mass is that of the initial perturbation.

The importance of diffusion waves for problems in one space dimension was first recognized by Liu [12]: diffusion waves decay in L^p for all $p > 1$, but not in L^1 . These waves therefore “carry mass,” and so are critical for describing the asymptotic behavior of solutions of hyperbolic–parabolic systems when the initial data has differing states at $x = +\infty$ and $x = -\infty$. In particular, one-dimensional diffusion waves have played an essential role in the study of the stability properties of viscous shocks, rarefactions, and contact discontinuities. See for example Liu [12]–[13], Szepessy and Xin [20], Szepessy and Zumbrun [21], Liu and Zumbrun [14], and Liu and Xin [15]. All these analyses require in an essential way an explicit, pointwise description of the one-dimensional diffusion wave. The importance of the present work is therefore that, by identifying explicitly the *multidimensional* diffusion wave, and by studying its pointwise properties, we prepare the way for the analysis of the large-time behavior of solutions of (1.1) when the initial data is not constant at $|x| = \infty$, and the asymptotic flow includes nontrivial, nonlinear waves.

We now give a more precise formulation of our results. First, we shall assume throughout that the pressure P is smooth in a neighborhood of ρ^* , with $c^2 = P'(\rho^*) > 0$. Next we linearize the system (1.1) about the constant state $\begin{bmatrix} \rho^* \\ m^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, to obtain the approximate system

$$(1.4) \quad \begin{cases} \rho_t + \operatorname{div} m = 0 \\ m_t + c^2 \nabla \rho = \varepsilon \Delta m + \eta \nabla \operatorname{div} m . \end{cases}$$

We shall make important use of the respective Green's matrices G and $\tilde{G}$ for the linear systems (1.4) and (1.3): the entries of G and $\tilde{G}$ map the time axis $[0, \infty)$ into the set of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$; the columns of G and $\tilde{G}$ satisfy the respective systems (1.4) and (1.3); and $G(0) = \tilde{G}(0) = \delta I$, where δ is the usual Dirac distribution and I is the identity matrix. The solutions of (1.4)

and (1.3) with initial data u_0 are therefore the spatial convolutions $G * u_0$ and $\tilde{G} * u_0$.

Our analysis of the large-time behavior of the perturbation $u(\cdot, t)$ defined in (1.2) then proceeds in a sequence of successively more refined approximations. First we show that $u(\cdot, t)$ is time-asymptotic to $G * u_0$ in the sense that the difference, which is the linearization error, decays more rapidly than does $u(\cdot, t)$. Next, because of the special relationship between the linearized system (1.4) and the effective artificial viscosity system (1.3), we are able to show that the difference $G(t) - \tilde{G}(t)$ between their Green's matrices also decays more rapidly than does $u(\cdot, t)$. Thus $u(\cdot, t)$ is equally well-approximated by $\tilde{G} * u_0$ for large time. The strict parabolicity of the system (1.3), together with the commutativity of symbols property mentioned above, enables us to derive explicit, pointwise estimates for $\tilde{G}(t)$, and therefore for the perturbation $u(\cdot, t)$, by way of the approximations above. Rates of growth and decay for $\|D_x^\alpha u(\cdot, t)\|_{L^p}$ are then obtained by integration of these pointwise bounds, and our incompressible limit result follows from a very detailed study of the structure of $\tilde{G}$. Finally, we connect our results with the one-dimensional theory in a somewhat different way by showing that $\begin{bmatrix} \rho \\ m \end{bmatrix}$ is also well approximated for large time by the solution of the following nonlinear artificial viscosity system:

$$(1.5) \quad \begin{cases} \rho_t + \operatorname{div} m = \frac{1}{2}(\varepsilon + \eta)\Delta\rho, \\ m_t + \left(\frac{m^k m}{\rho}\right)_{x_k} + \nabla P(\rho) = \varepsilon\Delta m + \frac{1}{2}(\eta - \varepsilon)\nabla \operatorname{div} m. \end{cases}$$

The system (1.5) results by retaining the nonlinear convection terms in the Navier-Stokes system (1.1), but replacing its nonlinear, nondefinite diffusion terms with the strictly parabolic, linear diffusion terms of (1.3). The system (1.5) is therefore the multidimensional analogue of those approximating systems for one-dimensional problems obtained by lifting solutions of the viscous Burgers equation to the various characteristic fields.

The following theorem contains the principal results of this paper. We overstate somewhat the regularity hypotheses on u_0 in order to avoid technicalities and to highlight the essential features. Complete statements, with less severe regularity conditions, are given in Theorems 5.3, 6.3, 6.7, 6.8, and 8.2. We remind the reader that, if K_ϑ is the standard heat kernel, that is, the solution of the scalar heat equation $(K_\vartheta)_t = \vartheta\Delta K_\vartheta$ with initial data $K_\vartheta(\cdot, 0) = \delta$, then

$$\|D_x^\alpha K_\vartheta(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq C(\alpha, \vartheta) t^{n/2(1/p-1)-|\alpha|/2}, \quad p \in [1, \infty].$$

We therefore define $r_{\alpha,p} = (n/2)(1 - 1/p) + |\alpha|/2$, and suggest that the various rates of decay in the following statement be compared with the rates of decay for K_ϑ above.

Theorem. Let $P, \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$, and $u_0 = \begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix} - \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$ be as above, with

$$E = \|u_0\|_{L^1} + \|u_0\|_{H^{[n/2]+l}}$$

sufficiently small, where $\ell \geq 3$ is an integer. Then the Navier-Stokes system (1.1) with initial data $\begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix}$ has a global solution satisfying the following decay estimates for all multi-indices α with $|\alpha| \leq (\ell - 3)/2$ and any positive σ :

$$(1.6) \quad \|D_x^\alpha u(\cdot, t)\|_{L^p} \leq C(\ell)E \begin{cases} (1+t)^{-r_{\alpha,p}}, & 2 \leq p \leq \infty, \\ (1+t)^{-r_{\alpha,p} + ((n-1)/4)(2/p-1)} L_\alpha(t), & 1 \leq p < 2, \end{cases}$$

where $L_\alpha(t) = \log(1+t)$ when $n = 2$ and $\alpha = 0$, and is otherwise identically one;

$$(1.7) \quad \|D_x^\alpha(u - G * u_0)(\cdot, t)\|_{L^p}, \quad \|D_x^\alpha(u - \tilde{G} * u_0)(\cdot, t)\|_{L^p}, \quad \|D_x^\alpha(u - \tilde{u})(\cdot, t)\|_{L^p} \\ \leq EL(t) \begin{cases} C(\ell)(1+t)^{-r_{\alpha,p}-1/2}, & 2 \leq p \leq \infty, \\ C(\ell, \sigma)(1+t)^{-r_{\alpha,p} + (n/4)(2/p-1)-1/2+\sigma}, & 1 \leq p < 2, \end{cases}$$

where $L(t) = \log(1+t)$ when $n = 2$ and is otherwise identically one;

$$(1.8) \quad \left\| D_x^\alpha \left(\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix} - \begin{bmatrix} \rho^* \\ K_\varepsilon * m_{0f} \end{bmatrix} \right) \right\|_{L^p} \\ \leq C(\ell)E(1+t)^{-r_{\alpha,p} + ((n-1)/4)(2/p-1)}, \quad p \in [2, \infty],$$

where m_{0f} is the divergence-free part of m_0 (see (1.12) below); and finally if $\int |x| |u_0(x)| dx \equiv \mu$ is finite, then

$$(1.9) \quad \|D_x^\alpha(u - \tilde{G}U)(\cdot, t)\|_{L^p} \\ \leq (E + \mu) \begin{cases} C(|\alpha|)(1+t)^{-r_{\alpha,p}-1/2}, & 2 \leq p \leq \infty, \\ C(|\alpha|, \sigma)(1+t)^{-r_{\alpha,p} + (n/4)(2/p-1)-1/2+\sigma}, & 1 \leq p < 2, \end{cases}$$

where $U \equiv \int u_0(x) dx$.

(1.6) therefore shows that the perturbation $u(\cdot, t)$ decays at the rate of a heat kernel in L^p for $p \geq 2$, but less rapidly for $p < 2$; in fact, the bound in (1.6) for $u(\cdot, t)$ may actually grow with increasing t . (1.6) and (1.7) then show that $u(\cdot, t)$ is time-asymptotic to each of $G(t) * u_0$, $\tilde{G}(t) * u_0$, and $\tilde{u}(\cdot, t)$, in the sense

that the respective differences are negligible compared to the estimated size of $u(\cdot, t)$.

We shall discuss (1.8) and (1.9) in more detail, in order to highlight some of the important aspects of our analysis which do not appear explicitly in the statements above. First, we let G^B be the Green's matrix for the strictly parabolic system

$$(1.10) \quad \begin{cases} \rho_t = \frac{1}{2}(\varepsilon + \eta)\Delta\rho \\ m_t = \varepsilon\Delta m + \frac{1}{2}(\eta - \varepsilon)\nabla \operatorname{div} m, \end{cases}$$

and W the Green's matrix for the hyperbolic system

$$(1.11) \quad \begin{cases} \rho_t + \operatorname{div} m = 0 \\ m_t + c^2 \nabla \rho = 0. \end{cases}$$

As we shall see, G^B and W commute, so that the Green's matrix $\tilde{G}$ for the artificial viscosity system (1.3) is simply the convolution $\tilde{G} = G^B * W$. Now, because the system (1.10) is strictly parabolic, G^B will have all the decay properties of an ordinary heat kernel. The hyperbolic system (1.11), on the other hand, which consists of the linearized Euler equations, is somewhat more complex. First observe that (1.11) has nontrivial steady-state solutions, namely, those flows $\begin{bmatrix} \rho \\ m \end{bmatrix}$ for which ρ is constant and $\operatorname{div} m = 0$; these are time-independent, incompressible flows. This leads us to consider the projection Π defined by

$$\Pi * \begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix} = \begin{bmatrix} 0 \\ m_{0f} \end{bmatrix},$$

where m_{0f} is the divergence-free part of m_0 ,

$$(1.12) \quad \hat{m}_{0f}(\xi) = \left(I - \frac{\xi \xi^t}{|\xi|^2} \right) \hat{m}_0(\xi).$$

We then write $W_f = W * \Pi$ and $W = W_f + W_\perp$. Then since W_f has nontrivial fixed points, $W_f * u_0$ cannot in general decay. On the other hand, we find that $W_\perp$ has precisely the growth and decay characteristics of the fundamental solution for the standard wave equation. These observations lead us to decompose the perturbation $u(\cdot, t)$ as follows:

$$(1.13) \quad \begin{aligned} u &= \tilde{G} * u_0 + (G - \tilde{G}) * u_0 + (u - G * u_0) \\ &= G^B * W_f * u_0 + G^B * W_\perp * u_0 + (G - \tilde{G}) * u_0 + (u - G * u_0). \end{aligned}$$

Now, (1.7) shows that the last two terms on the right here are negligible for large time. Estimating the convolutions in the first two terms, we find that the first decays at precisely the heat kernel rate in all L^p , which is exactly what is expected, since W_f has nontrivial fixed points and (1.10) is strictly parabolic; and the second term $G^B * W_\perp * u_0$ decays at the heat kernel rate times the factor $(1+t)^{((n-1)/4)(2/p-1)}$. This latter factor is neutral when $p = 2$, enhances the decay rate for $p > 2$, and retards it for $p < 2$. Thus it is the diffusing, incompressible flow $G^B * W_f * u_0$ which dominates in L^p for $p > 2$; expressing this result in terms of ρ and m , we thus obtain statement (1.8) of the theorem. When $p < 2$, the situation reverses itself, and it is the second term in (1.13) which dominates, and which accounts for the overall rate of decay of $u(\cdot, t)$ given in (1.6).

The statement in (1.9) gives an alternative interpretation of our asymptotic description of $u(\cdot, t)$. Applying the above factorization $\tilde{G} = W * G^B$ to the result (1.9), we find that $u(\cdot, t)$ is time-asymptotic to the flow $W * (G^B U)$, where U is the total mass of u_0 . Now, since G^B is the fundamental solution of a strictly parabolic system, $G^B U$ may be regarded as a diffusion about the origin, with total mass U . The asymptotic state $u \sim W * (G^B U)$ is therefore the diffusion from a single point, convected by the fundamental solution W of the linearized Euler equations; that is, $u(\cdot, t)$ is asymptotically a “diffusion wave.”

The two points of view expressed in (1.8) and (1.9) are reflected in the following pointwise bounds for $\tilde{G}$, which are derived from a detailed analysis of the convolutions $\tilde{G} = G^B * W_f + G^B * W_\perp$: we find that, when $n = 3$,

$$|(\tilde{G}^B * W_f)(x, t)| \leq C \min\{t^{-3/2}, |x|^{-3}\}$$

and

$$|(\tilde{G}^B * W_\perp)(x, t)| \leq Ct^{-2} \exp\left(-\frac{(|x| - ct)^2}{Ct}\right).$$

There is no hyperbolic effect in the bound for $G^B * W_f$ above since, as we saw above, the operator $W_f *$ has nontrivial fixed points. On the other hand, the bound for $G^B * W_\perp$ reflects the interaction of the contractive hyperbolic operator $W_\perp$ with the three-dimensional diffusion $G^B \sim t^{-3/2} e^{-|x|^2/Ct}$: the bulk of the mass of G^B is convected along the light cone $\{x : |x| = ct\}$, and its pointwise decay rate is increased by the factor $t^{-1/2}$, which is precisely the term $t^{((n-1)/4)(2/p-1)}$ with $n = 3$ and $p = \infty$.

We remark that, in one space dimension, the system (1.1) can be simplified by the introduction of Lagrangean coordinates (v, q) , where v is the specific volume and q is the velocity. By working in these coordinates and applying the arguments of Theorems 5.3 and 8.3, together with the energy estimates of

Hoff [2], we can extend our estimates to the case of fairly rough initial data $v_0 - v^* \in L^\infty \cap L^1$ and $q_0 - q^* \in L^1 \cap L^2$. A corresponding existence and regularity theory for solutions of (1.1) in two and three space dimensions has been developed in Hoff [3] and [4] and in Lions [11], but it remains an open question whether the smoothness requirements of the present theorem can be relaxed for the multidimensional case.

To provide a context for the results above, we give a survey of some related work, and a brief history of some of the ideas related to those of the present paper. Aside from its importance in applications, the Navier-Stokes system (1.1) is of particular interest mathematically because of its hyperbolic-parabolic character: there is no smoothing in the mass equation, and in most applications the dominant physical forces are those represented by the hyperbolic terms. One of the most important aspects of this system is therefore the interaction between convective and diffusive effects. As we shall see, this interaction results in a weak smoothing of the density, and this fact turns out to be crucial in nearly all the known results.

A precise sense for this observation has evolved gradually, through a series of interesting and related results. The first of these was Kan'el's proof [7] of global existence of solutions of (1.1) in one space dimension, wherein the compensating effect of the hyperbolic terms is revealed indirectly, via nonstandard energy estimates augmenting the usual entropy-type bounds. The result is that both the density ρ and the velocity q are controlled in all derivatives, but smoothing of the initial data is apparent only in the q -coordinates. These ideas were then extended to two and three space dimensions by Matsumura and Nishida [17].

Later, Hoff [2] proved global existence of weak solutions of (1.1), in one space dimension, in the case of rough initial data. His analysis in the setting of weak solutions clarifies the nature of the smoothing of the density ρ : discontinuities in ρ are shown to persist for all time, convecting along particle trajectories; however, their strengths decay exponentially in time, more rapidly for smaller viscosities. This exponential decay is crucial for the analysis, and is shown to be a consequence of the hyperbolic character of the corresponding inviscid equations; thus these interesting, new phenomena are shown to result precisely from the interaction of hyperbolic and parabolic effects.

Kawashima was the first to study rates of decay and specific asymptotic behavior of solutions of (1.1), in the process offering further insight into the compensative mechanism of hyperbolic effects. In [8], he identified a general class of "stable" hyperbolic-parabolic systems exhibiting such compensation, characterized primarily by the requirement that the associated constant coefficient system,

$$u_t + A_j u_{x_j} + B_{jk} u_{x_j x_k} = 0 ,$$

obtained by linearizing around a constant state, be asymptotically stable in L^2 . By studying the spectral properties of the Fourier transform of the corresponding Green's matrix

$$\widehat{G}(\xi, t) = e^{(-i\xi_j A_j - \xi_j \xi_k B_{jk})t},$$

Kawashima proved that L^2 -linearized stability is equivalent to the condition that

$$(1.14) \quad |\widehat{G}(\xi, t)| \leq C_1 e^{-\mu|\xi|^2 t} + C_2 e^{-\gamma t},$$

for constants $\gamma, \mu > 0$. The estimate (1.14) is obtained by perturbation expansions around $|\xi| = 0$ and $|\xi| = \infty$, and results in the decay rates

$$\|D_x^\alpha (G(\cdot, t) * u_0)\|_2 \leq C_1 t^{-n/4 - |\alpha|/2} \|u_0\|_1 + C_2 e^{-\gamma t} \|D_x^\alpha u_0\|_2.$$

Estimates for solutions of the original nonlinear system then follow by an iteration argument. Kawashima also determined the asymptotic behavior of solutions in one space dimension and for $p \geq 2$, by applying similar techniques. Observe that we see here in (1.14) the same phenomenon of exponential decay in time that was observed for the one-dimensional case in [2].

The Fourier methods of Kawashima suffice to give information about asymptotic behavior in L^2 , and then via Sobolev bounds, in L^p for $p \geq 2$. As for decay in lower L^p spaces, Zeng [22] has given an analysis which completes the picture for problems in one space dimension by proving decay estimates for $p \in [1, 2]$. Her analysis involves the decomposition of the Green's matrix G into the sum

$$(1.15) \quad G(x, t) = e^{-bt} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \delta(x - q^* t) + G_1(x, t),$$

where the first term is a δ -function decaying exponentially in time, and $G_1 \in L^1$ is a matrix function for which sharp, pointwise bounds can be obtained. The explicit description (1.15) of the Green's function unifies the various perspectives above, and gives a precise sense to the nature of hyperbolic compensation. In particular, we now see an explicit connection with the results of Hoff [2]: the solution operator supports discontinuities in the ρ -coordinate, convected along particle paths at the background speed q^* , and decaying as e^{-bt} . These results have recently been extended to general hyperbolic-parabolic systems in one space dimension by Liu and Zeng [16].

Now, as we shall see in Lemma 3.1 and Theorem 3.2 for the linearized Navier-Stokes system, the lowest order term arising in the expansion of $\widehat{G}$ near $|\xi| = \infty$ is again a delta function, just as in the one-dimensional case. On the other hand, the complex analytic argument used in [22] to bound the remainder G_1 cannot be applied here, owing to a failure of analyticity. More specifically, in the one-dimensional case, Zeng writes a spectral decomposition for $\widehat{G}_1$ in the

Fourier inversion integral for G , moves the contour of integration into the upper-half complex plane in an appropriate way, then derives the necessary pointwise bounds. This device depends upon the fact that the projection operators in the spectral decomposition for $\hat{G}$ are analytic in the complex variable ξ , except at two resonant points on the real axis, which can be avoided in the integration. This argument appears to fail in several space variables, however, because the projection operators associated with the complexified transform $\hat{G}$ now have an entire surface $(\xi + i\eta) \cdot (\xi + i\eta) = \text{constant}$ of singularities extending throughout the imaginary half-space; these cannot be avoided by moving the contour of integration. Moreover, there appears to be no particular advantage to such a decomposition in higher dimensions, because the support of G is no longer concentrated near the characteristic cone. For example, setting ε and η to zero, we find that G becomes the Green's matrix for the linearized Euler equations, which in the density component is exactly that of the scalar wave equation; in even dimensions, the sharp Huygen's principle fails for the wave equation. The best we can salvage from the complex-analytic approach is the analog of a weak Huygen's principle, that the support of G is concentrated in the interior of the characteristic cone. Even this requires some extra work, due to loss of integrability in error terms of order $|\xi|^{-2}$.

Accordingly, we follow a different route in estimating G . Namely, we separate out from $\hat{G}$ not just its behavior at $\xi = \infty$ (corresponding to a delta function), but its behavior in a neighborhood of infinity by setting $G = G_1 + G_2$, where $\hat{G}_1(\xi, t) = \hat{G}(\xi, t)\chi(\xi)$, and $\chi(\xi)$ is a smooth cutoff function supported on $|\xi| \leq R$. G_1 can then be estimated by comparison with the Green's matrix $\tilde{G}$ of the associated artificial viscosity system; the difficulty is therefore shifted to the term G_2 . To be more specific, the transform $\hat{G}$ of the Green's matrix in question (and indeed that for any second order, isotropic system) has the form

$$\hat{G}(\xi, t) = h_1(|\xi|, t)I + h_2(|\xi|, t)M \left(\frac{\xi}{|\xi|} \right),$$

where the entries of the matrix M are quadratic polynomials in $\xi/|\xi|$. Now, the Fourier transform of the function $\xi/|\xi|$ is the distribution given by the Riesz kernel

$$\mathcal{R} \stackrel{\text{def}}{=} \text{P.V. } C_n \frac{x}{|x|^{n+1}},$$

so that the Green's matrix G includes the convolution operator $u \mapsto \mathcal{R} * u$, which is not bounded from L^1 to L^1 , nor from L^∞ to L^∞ . Though it happens that $h_2(\infty, t) \equiv 0$, this operator must be reckoned with in higher order terms, $\mathcal{O}(|\xi|^{-1})$. A careful analysis of the interplay between this operator and other, diffusive, effects in G_2 is needed to deal with these terms. The requisite Fourier

multiplier theory developed in Section 4 and its application in Section 5, we regard as one of the most interesting aspects of this paper. Using these tools, we develop a result roughly analogous to that of Zeng: that

$$G_2 = e^{-bt} \delta(|x| - q^*t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{-bt} m(x, t),$$

where $m \in L^1(\mathbb{R}^n)$ satisfies

$$|m(x)| \leq \begin{cases} C|x|^{-n+1} & |x| \leq R \\ C(N)|x|^{-N} & |x| > R, \end{cases}$$

for any integer N . Thus, G_2 is concentrated primarily along particle paths. The fact that the decay of G_2 is algebraic as $|x| \rightarrow 0$ appears to be not a technical artifact but rather a natural consequence of the decomposition $W = W_f + W_\perp$ mentioned earlier in connection with the decay to an incompressible flow.

Plan of the paper. In Section 2 we give a complete theory of nonlinear, “artificial viscosity” systems. These are equations of the form

$$u_t + f_j(u)_{x_j} = B_{jk} u_{x_j x_k},$$

which are strictly parabolic and for which the symbols $e^{-\xi_j \xi_k B_{jk} t}$ and $e^{-i \xi_j f'_j(0) t}$ commute. We are primarily interested in the nonlinear artificial viscosity system (1.5); general systems may be treated with little increase in complexity, however, and this we do. The results of Section 2 extend in a significant way the work of Hoff and Smoller [5].

In Section 3, we begin our study of the Navier-Stokes system (1.1) by deriving a complete frequency-domain description of the Green’s matrix G for the corresponding linearized equations. By uncoupling these equations, we can factor out the rotational invariance of the problem to obtain an explicit description of $\widehat{G}$ and bounds on all ξ -derivatives, $D_\xi^\alpha \widehat{G}(\xi, t)$, at both $|\xi| = 0$ and $|\xi| = \infty$. In Section 4, we develop certain facts about Fourier multipliers, and in Section 5 we combine these facts with the bounds obtained in Section 3 to derive a decomposition of the Green’s matrix into the sum

$$G = G_1 + G_2,$$

where $|\widehat{G}_1| \leq C e^{-\vartheta|\xi|^2 t}$ is a strictly parabolic term, and $G_2 = e^{-bt} \widehat{M}(\xi, t)$ is an exponentially decaying L^p multiplier. Then in Theorem 5.3 we apply this decomposition to establish the L^p decay (1.6)–(1.7) of solutions of (1.1) for $p \in [2, \infty]$. In Section 6, we give a definition of effective artificial viscosity, and we show

that (1.3) is the unique effective artificial viscosity system whose solutions are asymptotically equivalent to those of (1.1). Combining this with the estimates of Section 5, we then obtain our results concerning the incompressible limit and the decay to diffusion waves discussed earlier.

In Section 7, we begin the study of L^p behavior in the low norms $1 \leq p < 2$, by deriving pointwise bounds on $G(x, t)$. First we give some results for fairly general systems which are strictly parabolic and whose convective parts are symmetrizable. We establish for such systems the pointwise estimate

$$(1.16) \quad |G(x, t)| \leq C e^{-((|x| - \Lambda t)^+)^2 / 4\mu t},$$

where Λ is a bound for the characteristic speeds of the associated hyperbolic system. (1.16) is thus a weaker version of the Paley-Wiener theorem. When the system additionally exhibits the commutativity of an artificial viscosity system, we can factor out the hyperbolic effects and derive even stronger estimates. On the other hand, if the system is not strictly parabolic, such as is the case for the linearized Navier-Stokes equations, this entire complex-analytic approach fails. We can still obtain the somewhat weaker bounds

$$(1.17) \quad |G(x, t)| \leq C_n \left(\frac{|x|}{1+t} \right)^{-n}, \quad |x| > 0,$$

however, directly from the derivative estimates for $D_\xi \hat{G}$ derived in Section 3, together with standard properties of the Fourier transform. Finally in Section 8, we apply the results of Section 7 to derive rates of decay in L^p for $p \in [1, 2)$. To do this we first describe the behavior of the smooth part G_1 of the Green's function G by comparing with the Green's function $\tilde{G} = G^B * W$ of the associated effective artificial viscosity system (1.3). The residual $(G_1 - \tilde{G})$ satisfies only the weak pointwise bound (1.17), but decays much faster than $\tilde{G}$ in L^2 . Combining these facts, we find that $\|G_1 - \tilde{G}\|_p$ is still dominated by $\|\tilde{G}\|_p$, for $1 \leq p < 2$. This information, together with the previous description of G_2 derived in Sections 3 and 4, is sufficient to give decay and asymptotic behavior of the solution in L^p for $p \in [1, 2)$.

2. Nonlinear systems with artificial viscosity. The nonlinear artificial viscosity system (1.5) is of particular importance for the Navier-Stokes equations (1.1). We therefore collect together here various properties of its solutions. Actually, much more general systems may be treated, with little increase in complexity. We therefore study in this section nonlinear systems with constant viscosity terms:

$$(2.1) \quad u_t + f_j(u)_{x_j} = B_{jk} u_{x_j} u_{x_k}.$$

Here $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^m$ is an unknown vector function of $x \in \mathbb{R}^n$ and $t \in [0, \infty)$; B_{jk} is a constant $m \times m$ matrix; and f_j maps a ball in $\mathbb{R}^m$ into $\mathbb{R}^m$. In Theorem 2.2 below we prove the global existence of solutions for initial data which is bounded in L^∞ and small in L^2 , requiring only that (2.1) be symmetrizable and strictly parabolic. Then in Theorem 2.3, under the further assumption that the symbol associated with the hyperbolic part of the equation commutes with that of its parabolic part, we show that the solution u decays at the optimal rate in $W^{k,p}$ for $p \in [2, \infty]$, and, moreover, that $u(\cdot, t)$ is well-approximated by the solution of the corresponding linearized equation, for large t .

We assume that f_j is defined and C^1 in a closed ball $\overline{B_{r_0}(0)} \subseteq \mathbb{R}^m$, and we denote by A_j the Jacobian matrix $f'_j(0)$. The linearization of (2.1) is therefore the system

$$(2.2) \quad u_t + A_j u_{x_j} = B_{jk} u_{x_j x_k}.$$

In the following theorem we derive a number of important properties of the Green's matrix corresponding to the linearized system (2.2).

Theorem 2.1. (a). *Assume that there is a symmetric, positive definite matrix P such that PA_j is symmetric for each j , and let $\kappa = |P^{1/2}| |P^{-1/2}|$. Then for $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$,*

$$(2.3) \quad \kappa^{-1} |w| \leq |e^{-i\xi_j A_j t} w| \leq \kappa |w|.$$

(b) *Assume, in addition, that there is a positive number ε such that*

$$(2.4) \quad \langle \xi_j \xi_k P B_{jk} w, w \rangle_{\mathbb{R}^m} \geq \varepsilon |\xi|^2 |w|^2$$

for all $\xi \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. Then for $t > 0$ the components of the matrix functions $\widehat{G}(\xi, t) = e^{-(i\xi_j A_j + \xi_j \xi_k B_{jk})t}$ and $\widehat{G}^B(\xi, t) = e^{-\xi_j \xi_k B_{jk} t}$ are in the Schwartz class $\mathcal{S}(\mathbb{R}^m)$, and therefore so are those of the transforms $G(\cdot, t)$ and $G^B(\cdot, t)$. Moreover, for each nonnegative integer k there is a positive constant $C(k)$ such that

$$(2.5) \quad \|D_x^\alpha G^B(\cdot, t)\|_p \leq C(k) t^{n/2(1/p-1)-|\alpha|/2}, \quad |\alpha| \leq k, \quad p \in [1, \infty], \quad t > 0;$$

and

$$(2.6) \quad \|D_x^\alpha G(\cdot, t)\|_p \leq C(k) t^{n/2(1/p-1)-|\alpha|/2},$$

$$|\alpha| \leq k, \quad \begin{cases} p \in [2, \infty] & t > 0, \quad \text{or} \\ p \in [1, 2], & 0 < t \leq 1. \end{cases}$$

Finally, there is a constant C_1 such that

$$(2.7) \quad \|G^B(\cdot, t)\|_1 = C_1, \quad t > 0.$$

Proof: To prove (a) we observe that the matrix $S_j = P^{-1/2}(PA_j)P^{-1/2}$ is symmetric and that $P^{-1/2}S_jP^{1/2} = A_j$. The matrix $e^{-i\xi_j S_j t}$ is therefore unitary, and $e^{-i\xi_j A_j t} = P^{-1/2}e^{-i\xi_j S_j t}P^{1/2}$. This proves (2.3).

Next we show that there are positive constants C and ϑ such that

$$(2.8) \quad |\widehat{G}(\xi, t)|, |\widehat{G}^B(\xi, t)| \leq Ce^{-\vartheta|\xi|^2 t}.$$

For the proof of (2.8) we temporarily let $\widehat{G}$ denote either $\widehat{G}$ or $\widehat{G}^B$. Fix $\xi \in \mathbb{R}^n$ and $w_0 \in \mathbb{C}^m$ and set $w(t) = \widehat{G}(\xi, t)w_0$. Then

$$\dot{w} = -(i\xi_j A_j + \xi_j \xi_k B_{jk})w.$$

A routine computation based on the symmetry of S_j and on (2.4) shows that $v = P^{1/2}w$ satisfies

$$\frac{d}{dt}|v|^2 = \langle \dot{v}, v \rangle_{\mathbb{C}^m} + \langle v, \dot{v} \rangle_{\mathbb{C}^m} \leq -2\varepsilon|\xi|^2|w|^2.$$

It follows that

$$|\widehat{G}(\xi, t)w_0| = |w(t)| \leq Ce^{-\vartheta|\xi|^2 t}|w_0|,$$

which proves (2.8). The statements in (2.5) and (2.6) for $p \geq 2$ follow immediately, for if q is the Hölder conjugate of p , then

$$\|D_x^\alpha G(\cdot, t)\|_p \leq C \|\xi^\alpha \widehat{G}(\xi, t)\|_q \leq C \left[\int |\xi|^{|\alpha|q} e^{-q\vartheta|\xi|^2 t} d\xi \right]^{1/q}.$$

The $p < 2$ bounds in (2.5), as well as (2.7), follow from the fact that $G^B(\cdot, 1) \in \mathcal{S} \subseteq L^1$ and the scaling property

$$G^B(x, t) = t^{-n/2} G^B(t^{-1/2}x, 1),$$

which is evident from the Fourier integral representation of G^B in terms of its transform $\widehat{G}^B = e^{-\xi_j \xi_k B_{jk} t}$.

To prove the $p < 2$ estimates in (2.6) for G , we first show that there is a constant C such that

$$(2.9) \quad \|G(\cdot, t)\|_1 \leq C, \quad 0 < t \leq 1.$$

To see this, we choose $u_0 \in \mathcal{S}$ and define $u(\cdot, t) = G(\cdot, t) * u_0 \in \mathcal{S}$. u is a solution of (2.2) and so satisfies the alternative representation

$$(2.10) \quad u(\cdot, t) = G^B(\cdot, t) * u_0 - \int_0^t G_{x_j}^B(\cdot, t-s) * (A_j u(\cdot, s)) ds.$$

Now, an easy argument based on (2.8) and the fact that $u_0 \in \mathcal{S}$ shows that $\mathcal{B}(t) = \sup_{0 \leq s \leq t} \|u(\cdot, s)\|_1$ is finite. On the other hand, from (2.5) and (2.10), we find that, for $s \in [0, t]$,

$$\|u(\cdot, s)\|_1 \leq C \left[\|u_0\|_1 + t^{1/2} \mathcal{B}(t) \right].$$

A simple Gronwall-type argument then shows that

$$\|G(\cdot, t) * u_0\|_1 = \|u(\cdot, t)\|_1 \leq C \|u_0\|_1, \quad 0 < t \leq 1.$$

We now replace u_0 by a sequence of mollifying kernels converging to the δ -distribution, multiplied by standard basis vectors, to obtain (2.9). A similar argument, based on (2.5) and (2.9), then shows that

$$(2.11) \quad \|G_{x_k}(\cdot, t)\|_1 \leq C t^{-1/2}, \quad 0 < t \leq 1.$$

To obtain estimates for higher-order derivatives, we simply write

$$D_x^\alpha G(\cdot, t) = \frac{\partial}{\partial x_{j_1}} G(\cdot, |\alpha|^{-1}t) * \frac{\partial}{\partial x_{j_2}} G(\cdot, |\alpha|^{-1}t) * \cdots * \frac{\partial}{\partial x_{j_{|\alpha|}}} G(\cdot, |\alpha|^{-1}t)$$

and apply (2.11) to each factor. This proves (2.6) for $p = 1$; the bounds for $1 < p < 2$ then follow by interpolation. $\square$

We now make a few remarks concerning the symmetrizability hypothesis in Theorem 2.1a. We say that $S \in C^2(\overline{B_{r_0}(0)})$ is a convex entropy for (2.1) if $S''(u) > 0$ for $u \in \overline{B_{r_0}(0)}$ and if each of the vector functions $(f'_j)^t \nabla_u S$ is the gradient with respect to u of a scalar function in $\overline{B_{r_0}(0)}$. The fundamental observation of Lax and Friedrichs [10] and Godunov [1] is that the existence of a convex entropy implies the symmetrizability of the nonlinear system (2.1): if S is the entropy, then each of the matrices $S''(u) f'_j(u)$ is symmetric. In particular, setting $u = 0$, we find that the symmetric, positive definite matrix $P = S''(0)$ symmetrizes the A_j in the sense of Theorem 2.1a. As is well known, this symmetrizability implies a global L^2 bound for solutions of (2.1); this fact will be crucial in extending local solutions for all time.

Given initial data $u_0 \in L^\infty(\mathbb{R}^n)$ with $\|u_0\|_\infty \leq r_0$, we say that u is a solution of (2.1) on $[0, T]$ with data u_0 provided

$$(2.12) \quad \begin{cases} u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^n)), \quad u(\cdot, 0) = u_0; \\ u(\cdot, t) \in \overline{B_{r_0}(0)} \quad \text{a.e.}; \\ u(\cdot, t) = G^B(\cdot, t) * u_0 - \int_0^t G^B_{x_j}(\cdot, t-s) * f_j(u(\cdot, s)) ds, \end{cases}$$

where G^B is the Green's matrix defined in Theorem 2.1b. In the following theorem we prove the global existence of such solutions when there is a convex entropy and when the initial data is small in L^2 .

Theorem 2.2. *Let k be a positive integer, and assume that:*

$$(2.13) \quad f_j \in C^k(\overline{B_{r_0}(0)}),$$

$$(2.14)$$

there is a convex entropy S for the system (2.1) satisfying $S'' > 0$ on $\overline{B_{r_0}(0)}$,

$$(2.15) \quad \text{the parabolicity (2.4) holds with } P = S''(0).$$

Let C_1 be as in (2.7). Then given $r < r_0$ there are positive constants C and μ such that, if

$$\|u_0\|_\infty \leq \frac{r}{C_1} \text{ and } \|u_0\|_2 \leq \mu,$$

then there is a unique global solution u of the system (2.1). u satisfies

$$(2.16) \quad u \in C((0, \infty); W^{k,p}(\mathbb{R}^m)), \quad p \in [2, \infty],$$

and

$$(2.17) \quad \|D_x^\alpha u(\cdot, t)\|_p \leq C \max\{1, t^{-|\alpha|/2}\} \|u_0\|_p$$

for $|\alpha| \leq k$ and $p \in [2, \infty]$. If u_0 is additionally in L^{p_0} for $1 \leq p_0 \leq 2$, then (2.16) and (2.17) hold for $p \in [p_0, \infty]$.

Proof. The proof is fairly routine, given the properties of G^B listed in Theorem 2.1. We therefore omit most of the technical details. First we define a sequence of approximate solutions u^ℓ by taking $u^0 = 0$ and $u^{\ell+1} = \mathcal{L}(u^\ell)$, where $\mathcal{L}(u)$ is the expression on the right-hand side of the third condition in (2.12). Standard estimates based on (2.5) and (2.7) then show that, if $\tau > 0$ is sufficiently small, then each $u^\ell(\cdot, t)$ has values in $\overline{B_{r_0}(0)}$ for $0 \leq t \leq \tau$, and that the u^ℓ converge uniformly on $\mathbb{R}^n \times (0, \tau]$ to a unique solution $u \in C(\mathbb{R}^n \times (0, \tau])$. An easy induction argument proves that

$$(2.18) \quad \|D_x^\alpha u^\ell(\cdot, t)\|_p \leq C_2 t^{-|\alpha|/2} \|u_0\|_p,$$

for $t \leq \tau$, $|\alpha| \leq k$, and $p \in [p_0, \infty]$, and straightforward weak convergence considerations show that the solution u inherits these properties. We fix the time τ and the constant C_2 in (2.18), and we extend the solution to all of $t > 0$ as follows. First, as is well known, the existence of the entropy S implies a time-independent L^2 energy estimate: there is a constant C_3 such that, if u is a solution on $[0, T]$ with values in $\overline{B_{r_0}(0)}$, then

$$(2.19) \quad \|u(\cdot, t)\|_2 \leq C_3 \|u_0\|_2, \quad 0 \leq t \leq T.$$

Then by (2.18) and (2.19), for $p > n$,

$$\begin{aligned} \|u(\cdot, T)\|_\infty &\leq C \|u(\cdot, T)\|_{W^{1,p}} \\ &\leq C \left[\|u(\cdot, T)\|_p + C_2 \left(\frac{1}{2}\tau\right)^{-1/2} \|u(\cdot, T - \frac{1}{2}\tau)\|_p \right] \\ &\leq C r_0^{(p-2)/p} \left[\|u(\cdot, T)\|_2^{2/p} + C_2 \left(\frac{1}{2}\tau\right)^{-1/2} \|u(\cdot, T - \frac{1}{2}\tau)\|_2^{2/p} \right] \\ &\leq C C_3^{2/p} r_0^{(p-2)/p} \left[1 + C_2 \left(\frac{1}{2}\tau\right)^{-1/2} \right] \|u_0\|_2^{2/p} \\ &\leq r \end{aligned}$$

if $\|u_0\|_2$ is sufficiently small. We may then reapply our local existence result at the new initial time T , and in this way extend the solution to all of $t > 0$. The estimate (2.17) follows from (2.18), and the continuity assertion (2.16) follows from the representation $u = \mathcal{L}(u)$. $\square$

To obtain more detailed information about the large time behavior of u , we shall have to take into account more of the structure of the flux functions f_j . We therefore linearize each f_j about $u = 0$:

$$(2.20) \quad f_j(u) = A_j u - Q_j(u)$$

where, as before, $A_j = f'_j(0)$, and

$$|Q_j(u)| \leq C|u|^2, \quad |u| \leq r_0.$$

Note that Q_j is C^k if f is C^{k+2} . We substitute (2.20) into the representation (2.12), take Fourier transforms, then solve the resulting ode in terms of the fundamental matrix $e^{-(i\xi_j A_j + \xi_j \xi_k B_{jk})t}$. Transforming back, we obtain the following alternative representation for the solution u of Theorem 2.2:

$$(2.21) \quad u(\cdot, t) = G(\cdot, t) * u_0 + \int_0^t G_{x_j}(\cdot, t-s) * Q_j(u(\cdot, s)) ds.$$

In the following theorem, we shall assume that, for each $\xi \in \mathbb{R}^n$, the matrices $\xi_j A_j$ and $\xi_j \xi_k B_{jk}$ commute, so that

$$(2.22) \quad \widehat{G}(\xi, t) = e^{-(i\xi_j A_j + \xi_j \xi_k B_{jk})t} = e^{-i\xi_j A_j t} \widehat{G}^B(\xi, t).$$

Now, (2.3) shows that, for each fixed t , the function $\xi \mapsto e^{-i\xi_j A_j t}$ is in $L^\infty(\mathbb{R}^n)$, and therefore is the Fourier transform of a tempered distribution, which we denote by $(2\pi)^{n/2}W(t)$. Evidently,

$$(2.23) \quad \kappa^{-1} \|\varphi\|_2 \leq \|W(t) * \varphi\|_2 \leq \kappa \|\varphi\|_2$$

for all $\varphi \in L^2$, where κ is as in (2.3). In addition, the convolution theorem applied in (2.22) gives the very useful factorization

$$(2.24) \quad G(\cdot, t) = W(t) * G^B(\cdot, t).$$

We now give the main theorem of this section, which establishes optimal rates of decay in $W^{k,p}$ for small-data solutions of (2.1):

Theorem 2.3. *Let k be a positive integer, assume that (2.14) and (2.15) hold, that*

$$(2.25) \quad f_j \in C^{k+2}(\overline{B_{r_0}(0)}),$$

and that

$$(2.26) \quad \xi_j A_j \text{ and } \xi_j \xi_k B_{jk} \text{ commute, for all } \xi \in \mathbb{R}^n.$$

Let C_1 be as in (2.7). Then, given $r < r_0$, there are positive constants C and μ such that, if

$$\|u_0\|_\infty \leq \frac{r}{C_1} \text{ and } \|u_0\|_1 \leq \mu,$$

then the corresponding solution u of (2.1) satisfies

$$(2.27) \quad \|D_x^\alpha u(\cdot, t)\|_p \leq C t^{n/2(1/p-1)-|\alpha|/2} \|u_0\|_1$$

and

$$(2.28) \quad \|D_x^\alpha (u - G(\cdot, t) * u_0)\|_p \leq C t^{n/2(1/p-1)-|\alpha|/2-1/2} \\ \times \left[\|u_0\|_2^2 + L(t) \|u_0\|_1^2 \right],$$

where L is the function defined by $L(t) = \log t$ when $n = 2$, and $L(t) = 1$ when $n \geq 3$. (2.27) and (2.28) hold for $t \geq 1$, $p \in [2, \infty]$, and $|\alpha| \leq k$.

Proof. First we let $u_I = G * u_0$ and $u_{II} = u - u_I$. For $T \geq 1$ we define

$$A(T) = \sup_{1 \leq t \leq T} \sup_{\substack{0 \leq |\alpha| \leq k \\ 2 \leq p \leq \infty}} t^{r_{|\alpha|,p}} \|D_x^\alpha u(\cdot, t)\|_p,$$

where

$$r_{\ell,p} = \frac{n}{2} \left(1 - \frac{1}{p}\right) + \frac{\ell}{2}.$$

We shall estimate $A(T)$ in terms of $\|u_0\|_1$ and $A(T)^2$, treating u_I and u_{II} separately. The required bounds for u_I follow directly from (2.6):

$$(2.29) \quad \|D_x^\alpha u_I(\cdot, t)\|_p \leq \|D_x^\alpha G(\cdot, t)\|_p \|u_0\|_1 \leq C t^{-r_{|\alpha|,p}} \|u_0\|_1.$$

The estimates for the term

$$(2.30) \quad D_x^\alpha u_{II}(\cdot, t) = \int_0^t D_x^\alpha \left[G_{x_j}^B(\cdot, t-s) * W(t-s) * Q_j(u(\cdot, s)) \right] ds$$

are somewhat more involved. First, if $t \in [1, 2]$, then straightforward arguments based on the bounds (2.17) of Theorem 2.2 show that

$$\|D_x^\alpha u_{II}(\cdot, t)\|_p \leq C \|u_0\|_2^2 \leq C \|u_0\|_2^2 t^{-r_{|\alpha|,p}-1/2}.$$

Now assume that $t \in [2, T]$. We let $\mathcal{I}(x, t)$ denote the integrand in (2.30), and we estimate $\int_0^{t/2} \mathcal{I}$ and $\int_{t/2}^t \mathcal{I}$ separately. Applying (2.23), (2.26), and (2.5), we may bound the first of these by

$$\begin{aligned} & \int_0^{t/2} \left\| G_{x_j}^B(\cdot, \tfrac{1}{2}(t-s)) * W(t-s) * D_x^\alpha G^B(\cdot, \tfrac{1}{2}(t-s)) * Q_j(u(\cdot, s)) \right\|_p ds \\ & \leq \int_0^{t/2} \left\| G_{x_j}^B(\cdot, \tfrac{1}{2}(t-s)) \right\|_{2p/(p+2)} \left\| D_x^\alpha G^B(\cdot, \tfrac{1}{2}(t-s)) \right\|_2 \|Q_j(u(\cdot, s))\|_1 ds \\ & \leq \int_0^{t/2} (t-s)^{-r_{|\alpha|,p}-1/2} \|Q_j(u(\cdot, s))\|_1 ds. \end{aligned}$$

For $s \in [0, 1]$, $\|Q_j\|_1 \leq C\|u_0\|_2^2$, by (2.17); and for $s \in [1, t/2]$, $\|Q_j\|_1 \leq C\|u(\cdot, s)\|_2^2 \leq Cs^{-n/2}A(T)^2$. Thus

$$\begin{aligned}
 (2.31) \quad & \left\| \int_0^{t/2} \mathcal{I}(\cdot, s) ds \right\|_p \\
 & \leq C \left[\|u_0\|_2^2 \int_0^1 (t-s)^{-r|\alpha|, p-1/2} + A(T)^2 \int_1^{t/2} (t-s)^{-r|\alpha|, p-1/2} s^{-n/2} ds \right] \\
 & \leq Ct^{-r|\alpha|, p-1/2} \left[\|u_0\|_2^2 + L(T)A(T)^2 \right],
 \end{aligned}$$

where L is as in the statement of the theorem.

Before deriving estimates for $\int_{t/2}^t \mathcal{I}$, we first observe that, for $|\alpha| \leq k$, each component of the vector $D_x^\alpha Q_j(u)$ is a sum of terms of the form $a(u)(D_x^{\gamma_1} u^{j_1}) \cdots (D_x^{\gamma_r} u^{j_r})$, where $a \in L^\infty(\overline{B_{r_0}(0)})$, $2 \leq r \leq k$, and $|\gamma_1| + \cdots + |\gamma_r| = |\alpha|$. Thus

$$\begin{aligned}
 (2.32) \quad & \|D_x^\alpha Q_j(u(\cdot, t))\|_p \leq C\|D_x^{\gamma_1} u\|_\infty \cdots \|D_x^{\gamma_{r-1}} u\|_\infty \|D_x^{\gamma_r} u\|_p \\
 & \leq Ct^{n/2(1/p-2)-|\alpha|/2} [A(T)^2 + A(T)^k].
 \end{aligned}$$

We shall give estimates for $\int_{t/2}^t \mathcal{I}$ in L^2 and L^∞ , and then interpolate. First, we obtain from (2.32), (2.23), and (2.5) that

$$\begin{aligned}
 (2.33) \quad & \left\| \int_{t/2}^t \mathcal{I}(\cdot, s) ds \right\|_2 \leq \int_{t/2}^t \|G_{x_j}^B(\cdot, t-s)\|_1 \|D_x^\alpha Q_j(u(\cdot, s))\|_2 ds \\
 & \leq C \left[\int_{t/2}^t (t-s)^{-1/2} s^{-3n/4-|\alpha|/2} ds \right] [A(T)^2 + A(T)^k] \\
 & \leq Ct^{-r|\alpha|, 2-1/2} [A(T)^2 + A(T)^k],
 \end{aligned}$$

since $n \geq 2$. To estimate $\|\int_{t/2}^t \mathcal{I}\|_\infty$ we split further: first, by (2.23), (2.32), and (2.5),

$$\begin{aligned}
 (2.34) \quad & \left\| \int_{t/2}^{t-1} \mathcal{I}(\cdot, s) ds \right\|_\infty \leq \int_{t/2}^{t-1} \|G_{x_j}^B(\cdot, t-s)\|_2 \|D_x^\alpha Q_j(u(\cdot, s))\|_2 ds \\
 & \leq C \left[\int_{t/2}^{t-1} (t-s)^{-n/4-1/2} s^{-3n/4-|\alpha|/2} ds \right] [A(T)^2 + A(T)^k] \\
 & \leq Ct^{-r|\alpha|, \infty-1/2} [A(T)^2 + A(T)^k];
 \end{aligned}$$

and by (2.6) and (2.32),

$$\begin{aligned}
 (2.35) \quad \left\| \int_{t-1}^t \mathcal{I}(\cdot, s) ds \right\|_{\infty} &\leq \int_{t-1}^t \|G_{x_j}(\cdot, s) * D_x^{\alpha} Q_j(u(\cdot, s))\|_{\infty} ds \\
 &\leq \int_{t-1}^t \|G_{x_j}(\cdot, t-s)\|_1 \|D_x^{\alpha} Q_j(u(\cdot, s))\|_{\infty} ds \\
 &\leq C \left[\int_{t-1}^t (t-s)^{-1/2} s^{-n-|\alpha|/2} ds \right] [A(T)^2 + A(T)^k] \\
 &\leq Ct^{-r_{|\alpha|, \infty}-1/2} [A(T)^2 + A(T)^k].
 \end{aligned}$$

Adding (2.34) and (2.35), we then obtain

$$(2.36) \quad \left\| \int_{t/2}^t \mathcal{I}(\cdot, s) ds \right\|_{\infty} \leq Ct^{-r_{|\alpha|, \infty}-1/2} [A(T)^2 + A(T)^k].$$

Interpolating (2.33) and (2.36), we then conclude that

$$\left\| \int_{t/2}^t \mathcal{I}(\cdot, s) ds \right\|_p \leq Ct^{-r_{|\alpha|, p}-1/2} [A(T)^2 + A(T)^k].$$

This together with (2.31) shows that

$$(2.37) \quad \|D_x^{\alpha} u_{II}(\cdot, t)\|_p \leq Ct^{-r_{|\alpha|, p}-1/2} [\|u_0\|_2^2 + L(T)A(T)^2 + A(T)^k].$$

Combining (2.37) with the estimate (2.29) for u_I and taking the appropriate sup over t , we then obtain that

$$A(T) \leq C [\|u_0\|_1 + A(T)^2 + A(T)^k].$$

A routine argument based on the continuity (2.16) of A in T then shows that $A(T) \leq C\|u_0\|_1$, for all T , provided that $\|u_0\|_1$ is sufficiently small. This proves (2.27), and (2.28) follows by substituting this bound for A into (2.37). $\square$

We conclude this section with an example showing that the decay rate (2.27) is sharp for $p = 2$. We simply take $u_0 = G^B(\cdot, 1)\bar{u}$, where $\bar{u} \neq 0$ is a constant vector, which can be chosen so that $\|u_0\|_1$ and $\|u_0\|_2$ are arbitrarily small. Then, by (2.23) and (2.24), for $t \geq 1$,

$$\begin{aligned}
 \|D_x^{\alpha} [G(\cdot, t) * u_0]\|_2 &= \|W(t) * D_x^{\alpha} G^B(\cdot, t+1)\bar{u}\|_2 \\
 &\geq C^{-1} \left[\int |\xi^{\alpha} e^{-\xi_j \xi_k B_{jk}(t+1)} \bar{u}|^2 d\xi \right]^{1/2} \\
 &= C^{-1} t^{-n/4-|\alpha|/2}.
 \end{aligned}$$

On the other hand, if $u(\cdot, t)$ is the solution of the nonlinear system (2.1) with this same initial data u_0 , then by (2.28) ,

$$\|D_x^\alpha[u(\cdot, t) - G(\cdot, t) * u_0]\|_2 \leq Ct^{-n/4-|\alpha|/2-1/2}L(t).$$

We conclude that

$$\|D_x^\alpha u(\cdot, t)\|_2 \geq C^{-1}t^{-n/4-|\alpha|/2},$$

which shows that (2.27) is sharp for $p = 2$, as well as that

$$\|D_x^\alpha[u(\cdot, t) - G(\cdot, t) * u_0]\|_2 \leq Ct^{-1/2}L(t)\|u(\cdot, t)\|_2.$$

Thus in the limit as $t \rightarrow \infty$, small-data solutions of the nonlinear system (2.1) are well approximated by solutions of the corresponding linearized system (2.2).

3. Fourier analysis of the Green's matrix for the linearized Navier-Stokes system. In this section we begin the systematic study of the large-time behavior of solutions of the Navier-Stokes equations (1.1). In Lemma 3.1 below we give an explicit derivation of the Fourier transform $\widehat{G}(\xi, t)$ of the Green's matrix corresponding to the linearized system (1.4). Then in Theorem 3.2 we prove various pointwise bounds for $\widehat{G}$ and its ξ -derivatives, required for the decay analysis of Sections 4–8.

Lemma 3.1. *The Fourier transform $\widehat{G}$ of the Green's matrix for the linearized Navier-Stokes system (1.4) is given by*

$$(3.1) \quad \widehat{G}(\xi, t) = \begin{bmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -i \left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \xi^t \\ -ic^2 \left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \xi & e^{-\varepsilon|\xi|^2 t} I + \left(\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-\varepsilon|\xi|^2 t} \right) \frac{\xi \xi^t}{|\xi|^2} \end{bmatrix},$$

where

$$(3.2) \quad \lambda_{\pm} = -\frac{1}{2}\nu|\xi|^2 \pm \frac{1}{2}\sqrt{\nu^2|\xi|^4 - 4c^2|\xi|^2}$$

and $\nu = \varepsilon + \eta$. The representation above holds for $|\xi| \neq 0, 2c/\nu$.

Proof. We solve the linearized system (1.4) with initial data (ρ_0, m_0) as follows. First, applying the operator $\sum_j \partial/\partial x_j$ to the second equation in (1.4), we obtain

$$D_t + c^2 \Delta \rho = \nu \Delta D,$$

where $D = \operatorname{div} m$. Combining this with the first equation in (1.4), we can isolate ρ :

$$\rho_{tt} = -D_t = c^2 \Delta \rho + \nu \Delta \rho_t.$$

Taking Fourier transforms, we then obtain the following initial-value problem for $\hat{\rho}(\xi, \cdot)$:

$$(3.3) \quad \begin{cases} \hat{\rho}_{tt} + \nu |\xi|^2 \hat{\rho}_t + c^2 |\xi|^2 \hat{\rho} = 0 \\ \hat{\rho}(\xi, 0) = \hat{\rho}_0(\xi) \\ \hat{\rho}_t(\xi, 0) = -i\xi \cdot \hat{m}_0(\xi). \end{cases}$$

It is easily seen that $\lambda_{\pm}(\xi)$ are the roots of the corresponding indicial equation, so that, when $\lambda_- \neq \lambda_+$,

$$\hat{\rho}(\xi, t) = A(\xi)e^{\lambda_-(\xi)t} + B(\xi)e^{\lambda_+(\xi)t}.$$

The initial conditions give that

$$A = \frac{\lambda_+ \hat{\rho}_0 + i\xi \cdot \hat{m}_0}{\lambda_+ - \lambda_-} \quad \text{and} \quad B = \frac{-i\xi \cdot \hat{m}_0 - \lambda_- \hat{\rho}_0}{\lambda_+ - \lambda_-},$$

so that

$$(3.4) \quad \hat{\rho}(\xi, t) = \left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \right) \hat{\rho}_0(\xi) - i \left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \xi \cdot \hat{m}_0(\xi).$$

This determines $\widehat{G^{11}}$ and $\widehat{G^{12}}$.

To compute $\widehat{m}$ we take transforms in the second equation in (1.4) to obtain

$$(3.5) \quad \widehat{m}_t = (-\varepsilon |\xi|^2 I - \eta \xi \xi^t) \widehat{m} - ic^2 \hat{\rho} \xi.$$

To facilitate the computation, we introduce components parallel to and orthogonal to ξ . We thus write

$$(3.6) \quad \widehat{m}(\xi, t) = a(\xi, t) \frac{\xi}{|\xi|} + b(\xi, t),$$

where the vector b is perpendicular to ξ , and where a is the scalar $a = \widehat{m} \cdot \xi / |\xi|$. Substituting (3.6) into (3.5), we find that

$$(3.7) \quad \begin{cases} a_t = -\nu |\xi|^2 a - ic^2 |\xi| \hat{\rho} \\ b_t = -\varepsilon |\xi|^2 b. \end{cases}$$

Thus

$$(3.8) \quad b(\xi, t) = e^{-\varepsilon|\xi|^2 t} \left(I - \frac{\xi \xi^t}{|\xi|^2} \right) \widehat{m}_0$$

and

$$(3.9) \quad a(\xi, t) = e^{-\nu|\xi|^2 t} \left[a(\xi, 0) - ic^2 |\xi| \int_0^t e^{\nu|\xi|^2 s} \widehat{\rho}(\xi, s) ds \right].$$

We substitute (3.4) into (3.9), perform the integration, and simplify, applying the relations $\lambda_{\pm} + \nu|\xi|^2 = -\lambda_{\mp}$ and $\lambda_- \lambda_+ = c^2 |\xi|^2$. The result is that

$$(3.10) \quad a(\xi, t) = -ic^2 |\xi| \left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \widehat{\rho}_0 + \left(\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \frac{\xi \cdot \widehat{m}_0}{|\xi|}.$$

Substituting (3.8) and (3.10) into (3.6), we obtain finally that

$$\begin{aligned} \widehat{m}(\xi, t) &= -ic^2 \left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \widehat{\rho}_0 \xi \\ &\quad + \left[e^{-\varepsilon|\xi|^2 t} I + \left(\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-\varepsilon|\xi|^2 t} \right) \frac{\xi \xi^t}{|\xi|^2} \right] \widehat{m}_0. \end{aligned}$$

This determines $\widehat{G^{21}}$ and $\widehat{G^{22}}$. □

In the following theorem we derive pointwise estimates for $\widehat{G}$ and its ξ -derivatives, as well as a series expansion for $\widehat{G}$ valid for large $|\xi|$.

Theorem 3.2. *Let $G(x, t)$ be the Green's matrix of Lemma 3.1.*

- (a) *Given $R > 0$, there is a positive number ϑ such that, for multi-indices α and for $|\xi| \leq R$,*

$$(3.11) \quad |D_{\xi}^{\alpha} \widehat{G}(\xi, t)| \leq C e^{-\vartheta|\xi|^2 t} (1 + |\xi|)^{|\alpha|} (1 + t)^{|\alpha|},$$

where $C = C(R, |\alpha|)$.

- (b) *There is a positive constant R such that the following series expansion is valid for $|\xi| \geq R$:*

$$\begin{aligned}
 (3.12) \quad \widehat{G}(\xi, t) &= e^{-\vartheta_1 |\xi|^2 t} \mathcal{M}_1 + \dots + e^{-\vartheta_\ell |\xi|^2 t} \mathcal{M}_\ell + e^{-at} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
 &+ \left(\sum_{j=0}^{\infty} |\xi|^{-2j-1} q_j(|\xi|, t) \right) \begin{bmatrix} 0 & \frac{\xi^t}{|\xi|} \\ \frac{c^2 \xi}{|\xi|} & 0 \end{bmatrix} \\
 &+ \left(\sum_{j=0}^{\infty} |\xi|^{-2j-2} D_j(|\xi|, t) \right) \begin{bmatrix} 1 & 0 \\ 0 & \frac{\xi \xi^t}{|\xi|^2} \end{bmatrix},
 \end{aligned}$$

where $a = c^2/\nu$, ϑ_k are positive constants, $\mathcal{M}_k$ are matrices whose entries are linear combinations of constants and homogenous quadratic polynomials in $\xi/|\xi|$, q_j are scalar functions, and D_j are diagonal matrices. q_j and D_j satisfy the estimates

$$(3.13) \quad |q_j(r, t)|, |D_j(r, t)| \leq C \left(e^{-at/2} + e^{-\nu r^2 t/2} \right) r_0^{2j}$$

for positive constants C and r_0 . Finally, there is a positive constant ϑ such that, for $|\xi| \geq R$ and $\alpha \neq 0$,

$$(3.14) \quad |D_\xi^\alpha \widehat{G}(\xi, t)| \leq C(\alpha) \left(e^{-at/2} |\xi|^{-|\alpha|-1} + e^{-\vartheta |\xi|^2 t} |\xi|^{-|\alpha|} \right).$$

Proof. We first prove the $\alpha = 0$ case of (a), that is, that

$$(3.15) \quad |\widehat{G}(\xi, t)| \leq C e^{-\vartheta |\xi|^2 t}, \quad |\xi| \leq R.$$

Observe from (3.2) that $\operatorname{Re}(\lambda_-) \leq -\frac{1}{2}\nu|\xi|^2$ always, and $\operatorname{Re}(\lambda_+) \leq -\frac{1}{2}\nu|\xi|^2$ when $|\xi| \leq 2c/\nu$ and λ_+ is complex. It is easy to check that there is also a positive constant ϑ depending on R such that $\operatorname{Re}(\lambda_+) \leq -\vartheta|\xi|^2$ for $2c/\nu < |\xi| \leq R$. Thus

$$(3.16) \quad \operatorname{Re}(\lambda_\pm(\xi)) \leq -\vartheta|\xi|^2,$$

and therefore

$$(3.17) \quad |e^{\lambda_\pm(\xi)t}| \leq e^{-\vartheta|\xi|^2 t}$$

when $0 \leq |\xi| \leq R$.

Now let $r = 2c/\nu$ be the resonant value of $|\xi|$, and choose a small positive number μ . Then for $r + \mu \leq |\xi| \leq R$,

$$|\lambda_+ - \lambda_-| = \nu|\xi|^2 \sqrt{1 - r^2|\xi|^{-2}} \geq \nu|\xi|^2 \sqrt{1 - \left(\frac{r}{r+\mu}\right)^2} \geq C^{-1};$$

and for $0 \leq |\xi| \leq r - \mu$,

$$|\lambda_+ - \lambda_-| = rc|\xi| \sqrt{1 - \left(\frac{r-\mu}{r}\right)^2} \geq C^{-1}|\xi|.$$

These bounds, together with (3.17), the fact that $|\lambda_{\pm}| \leq C|\xi| \leq C$ for $|\xi| \leq R$, and the explicit representation (3.1), combine to prove (3.15) when $|\xi| \leq R$ and $||\xi| - r| \geq \mu$.

To prove (3.15) when $||\xi| - r| \leq \mu$, we apply (3.16) again to find that

$$\begin{aligned} (3.18) \quad \left| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| &\leq t \sup_{0 \leq s \leq 1} e^{t \operatorname{Re}(s\lambda_- + (1-s)\lambda_+)} \\ &\leq te^{-\vartheta|\xi|^2 t} \\ &\leq Ce^{-\vartheta|\xi|^2 t/2}, \quad ||\xi| - r| \leq \eta. \end{aligned}$$

This establishes the required bounds for $\widehat{G^{12}}$ and $\widehat{G^{21}}$. For $\widehat{G^{11}}$ we simply observe that

$$\begin{aligned} |\widehat{G^{11}}(\xi)| &\leq \left| \lambda_+ \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| + |e^{\lambda_+ t}| \\ &\leq Ce^{-\vartheta|\xi|^2 t} \end{aligned}$$

by (3.16) and (3.18). A similar argument applies to $\widehat{G^{22}}$. This completes the proof of (3.15).

The general statement (3.11) now follows easily by induction on $|\alpha|$. Since $\widehat{G}(\xi, t) = e^{-(i\xi_j A_j + \xi_j \xi_k B_{jk})t}$ for appropriate matrices A_j and B_{jk} , we may write

$$\begin{cases} \frac{\partial}{\partial t} \widehat{G}(\xi, t) = H(\xi) \widehat{G}(\xi, t) \\ \widehat{G}(\xi, 0) = I, \end{cases}$$

where the entries of H are quadratic polynomials in ξ . Thus for $\alpha \neq 0$,

$$\begin{cases} \frac{\partial}{\partial t} D_\xi^\alpha \widehat{G} = H D_\xi^\alpha \widehat{G} + \sum_{\substack{0 < |\beta| \leq 2\wedge |\alpha| \\ \beta + \gamma = \alpha}} \frac{\alpha!}{\beta! \gamma!} D_\xi^\beta H D_\xi^\gamma \widehat{G} \\ D_\xi^\alpha \widehat{G}(\xi, 0) = 0, \end{cases}$$

so that

$$\begin{aligned}
 |D_\xi^\alpha \widehat{G}(\xi, t)| &\leq \sum \frac{\alpha!}{\beta! \gamma!} \int_0^t \left| \widehat{G}(\xi, t-s) D_\xi^\beta H(\xi) D_\xi^\gamma \widehat{G}(\xi, s) \right| ds \\
 &\leq C(R, |\alpha|) \int_0^t e^{-\vartheta |\xi|^2 (t-s)} \left[(1+|\xi|)(1+|\xi|)^{|\alpha|-1} (1+s)^{|\alpha|-1} \right. \\
 &\quad \left. + (1+|\xi|)^{|\alpha|-2} (1+s)^{|\alpha|-2} \right] e^{-\vartheta |\xi|^2 s} ds \\
 &\leq C e^{-\vartheta |\xi|^2 t} (1+|\xi|)^{|\alpha|} (1+t)^{|\alpha|},
 \end{aligned}$$

as required. This completes the proof of (a).

To prove (b) we shall express $\widehat{G}$ in terms of certain fundamental solutions associated with the ode (3.3) satisfied by $\hat{\rho}(\xi, \cdot)$. These fundamental solutions admit contour integral representations, which are easily expanded and differentiated. Thus, let p be the symbol

$$p(z, r) = z^2 + \nu r^2 z + c^2 r^2,$$

and observe that, by (3.2), the roots $\lambda_\pm(r)$ of $p(\cdot, r)$ satisfy

$$\lambda_+(r) = -a + O(r^{-2}),$$

$$\lambda_-(r) = -\nu r^2 + a + O(r^{-2})$$

for large $r = |\xi|$. We define contours Γ_+, Γ_- , and Γ_0 in the complex plane to be the circles of radius $a/2$ centered respectively at $-a$, $-\nu r^2 + a$, and 0, and we define

$$(3.19) \quad \begin{cases} E(r, t) = \frac{1}{2\pi i} \int_{\Gamma_- \cup \Gamma_+} \frac{e^{tz}}{p(z, r)} dz, \\ F(r, t) = E_t(r, t) + \nu r^2 E(r, t). \end{cases}$$

Then

$$(3.20) \quad \begin{cases} p(D_t, r)E = p(D_t, r)F = 0, \\ E(r, 0) = 0, \quad F(r, 0) = 1, \\ E_t(r, 0) = 1, \quad F_t(r, 0) = 0. \end{cases}$$

We shall also make use of the auxiliary function

$$(3.21) \quad H(r, t) = e^{-\nu r^2 t} \int_0^t e^{\nu r^2 s} E(r, s) ds.$$

By expanding $1/p$ under the integral sign in powers of r^{-1} , we then obtain the following series expansions for large r :

$$(3.22) \quad \begin{cases} E(r,t) = \sum_{j=0}^{\infty} E_j(r,t) r^{-2j-2} \\ F(r,t) = e^{-at} + \sum_{j=0}^{\infty} F_j(r,t) r^{-2j-2} \\ H(r,t) = \sum_{j=0}^{\infty} H_j(r,t) r^{-2j-4}, \end{cases}$$

where

$$(3.23) \quad |E_j(r,t)|, |F_j(r,t)|, |H_j(r,t)| \leq C \left(e^{-at/2} + e^{-\nu r^2 t/2} \right) r_0^{2j}$$

for suitable constants C and r_0 . For example, making the change of variable $w = z + \nu r^2 - a$, we compute that

$$(3.24) \quad \begin{aligned} \int_{\Gamma_-(r)} \frac{e^{tz}}{p(z,r)} dz &= -e^{(a-\nu r^2)t} \int_{\Gamma_0} \frac{e^{tw}}{\nu r^2 w} \left[1 - \frac{(w+a)^2}{\nu r^2 w} \right]^{-1} dw \\ &= \sum_{j=0}^{\infty} r^{-2j-2} \left[-e^{(a-\nu r^2)t} \int_{\Gamma_0} \frac{(w+a)^{2j}}{(\nu w)^{j+1}} e^{tw} dw \right]. \end{aligned}$$

When r is sufficiently large, the term in brackets here is bounded by

$$C e^{(a-\nu r^2)t} e^{at/2} r_0^{2j} \leq C e^{-\nu r^2 t/2} r_0^{2j}.$$

Expanding the integral over Γ_+ in a similar way, we thus obtain the series (3.22) for E . The expansions for F and H then follow from the definitions (3.19) and (3.21).

We shall also require bounds for the r -derivatives of the fundamental solutions E, F , and H . For example, letting b_j denote the contour integral on the right side of (3.24), we compute that

$$\begin{aligned} \left(\frac{\partial}{\partial r} \right)^k \int_{\Gamma_-} \frac{e^{tz}}{p(z,r)} dz \\ = -e^{at} \sum_{j=0}^{\infty} b_j(t) \sum_{\ell=0}^k \binom{k}{\ell} \left[\left(\frac{\partial}{\partial r} \right)^{\ell} e^{-\nu r^2 t} \right] \left[\left(\frac{\partial}{\partial r} \right)^{k-\ell} r^{-2j-2} \right]. \end{aligned}$$

The first brackets is bounded by $Cr^{-\ell}e^{-2\nu r^2 t/3}$ for large r , so that

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial r} \right)^k \int_{\Gamma_-} \frac{e^{tz}}{p(z,r)} dz \right| \\ & \leq C e^{(a-2\nu r^2/3)t} \sum_{j=0}^{\infty} |b_j(t)| r^{-2j-2-k} \sum_{\ell=0}^k \binom{k}{\ell} \frac{(2j+2)!}{(2j+2-k+\ell)!} \\ & \leq C(k) e^{-\nu r^2 t/2} r^{-k-2}. \end{aligned}$$

Applying similar estimates to the integral over Γ_+ and passing these estimates through the definitions (3.19) and (3.21) of F and H , we obtain that, for $k > 0$,

$$(3.25) \quad \left| \frac{\partial^k E}{\partial r^k} \right|, \quad \left| \frac{\partial^k F}{\partial r^k} \right| \leq C(k) \left(e^{-at/2} + e^{-\nu r^2 t/2} \right) r^{-k-2},$$

$$(3.26) \quad \left| \frac{\partial^k H}{\partial r^k} \right| \leq C(k) \left(e^{-at/2} + e^{-\nu r^2 t/2} \right) r^{-k-4}.$$

We now let $\hat{\rho}$ and $\hat{m}$ be as in the proof of Lemma 3.1. $\hat{\rho}(\xi, \cdot)$ satisfies the initial-value problem (3.3), so that, by (3.20),

$$(3.27) \quad \hat{\rho}(\xi, t) = F(|\xi|, t) \hat{\rho}_0(\xi) + E(|\xi|, t) (-i\xi \cdot \hat{m}_0(\xi)).$$

Writing

$$\hat{G} = \begin{bmatrix} \hat{G}^{11} & \hat{G}^{12} \\ \hat{G}^{21} & \hat{G}^{22} \end{bmatrix},$$

where $\hat{G}^{11}$ is a scalar and $\hat{G}^{22}$ is $n \times n$, we then conclude that

$$(3.28) \quad \hat{G}^{11}(\xi, t) = F(|\xi|, t) \quad \text{and} \quad \hat{G}^{12}(\xi, t) = -iE(|\xi|, t)\xi^t.$$

(3.1) shows that $\hat{G}^{21} = c^2(\hat{G}^{12})^t$, so that

$$(3.29) \quad \hat{G}^{21}(\xi, t) = -ic^2 E(|\xi|, t)\xi.$$

To represent $\hat{G}^{22}$ in terms of the fundamental solutions, we again decompose $\hat{m}$ into its components parallel to and orthogonal to ξ , as in (3.6). The orthogonal component $b(\xi, t)$ is given by (3.8), and to compute the radial component $a(\xi, t)$ we now substitute the representation (3.27) for $\hat{\rho}$ into the ode (3.7) and apply the definition (3.21) of H . The result is that, except for terms $e^{-\vartheta_k |\xi|^2 t} \mathcal{M}_k$ as in (3.12),

$$(3.30) \quad \hat{G}^{22}(\xi, t) = -c^2 |\xi|^2 H(|\xi|, t) \frac{\xi \xi^t}{|\xi|^2}.$$

Substituting the series (3.22) into the representations (3.28)–(3.30), we then obtain the required expansion (3.12) for $\hat{G}$. The derivative estimates (3.14) follow in a straightforward way from (3.25)–(3.26) by way of these same representations. $\square$

4. Fourier multipliers. To derive a decomposition analogous to (1.15) from the estimates (3.11) and (3.12), we will need some results about L^p multipliers. Most fundamental of these is the following classical result:

Proposition 4.1 (Marcinkiewicz Multiplier Theorem). *Let $\hat{f}(\xi) \in L^\infty \cap C^k(\mathbb{R}^n/\{0\})$, with $k = [n/2] + 1$, and*

$$(4.1) \quad |D_\xi^\alpha \hat{f}(\xi)| \leq C' |\xi|^{-|\alpha|}, \quad |\alpha| \leq k.$$

Then $\hat{f}(\xi)$ is an L^p multiplier for $1 < p < \infty$; that is, if we define $\widehat{f * g} = (2\pi)^{n/2} \hat{f} \hat{g}$, then

$$(4.2) \quad \|f * g\|_p \leq C_p \|g\|_p \text{ for } g \in L^2 \cap L^p, \quad 1 < p < \infty,$$

where C_p depends only on p and the constant C' , and is bounded on any closed interval $[p_1, p_2] \subset (1, \infty)$. (See Stein [18], p. 96.)

Typical examples of L^p multipliers are given by the class of functions

$$(4.3) \quad \hat{f}(\xi) = \chi(\xi) h\left(\frac{\xi}{|\xi|}\right),$$

where χ is a C^∞ cutoff function, constant except on a set bounded away from 0 and ∞ , and $h \in C^k(S^n)$.

For our purposes, it is useful to sharpen Proposition 4.1 somewhat. The following two propositions, complementary to Proposition 4.1, turn out to be natural for the study of second-order hyperbolic-parabolic systems.

Proposition 4.2. *Let $n \geq 1$ and assume that $\hat{f}(\xi) \in L^\infty \cap C^{n+1}(\mathbb{R}^n/\{0\})$, with*

$$(4.4) \quad |D_\xi^\alpha \hat{f}(\xi)| \leq C' \begin{cases} |\xi|^{-|\alpha|+\sigma_1} & |\xi| \leq R; \quad |\alpha| = n, \\ |\xi|^{-|\alpha|-\sigma_2} & |\xi| \geq R; \quad |\alpha| = n-1, n, n+1, \end{cases}$$

where $\sigma_1, \sigma_2 > 0$ and

$$(4.5) \quad n > 2 - 2\sigma_2.$$

Then $\hat{f}(\xi)$ is continuous at 0 and ∞ , and

$$(4.6) \quad f = m_1 + m_2 \delta,$$

where $m_1 \in L^1(\mathbb{R}^n)$ satisfies $\|m_1\|_1 \leq C(C')$, m_2 is the constant

$$m_2 = (2\pi)^{-n/2} \lim_{|\xi| \rightarrow \infty} \hat{f}(\xi),$$

and δ is the Dirac distribution. In particular, $\hat{f}(\xi)$ is a strong L^p multiplier, $1 \leq p \leq \infty$, in the sense that, for any $g \in L^p$,

$$(4.7) \quad \|f * g\|_p \leq C \|g\|_p, \quad 1 \leq p \leq \infty,$$

where C depends only on $|m_2| \leq \|\hat{f}\|_\infty$ and the constant C' above.

Proof. Set

$$(4.8) \quad \begin{aligned} \hat{f}_1(\xi) &\equiv \chi(\xi) \hat{f}(\xi), \\ \hat{f}_2(\xi) &\equiv (1 - \chi(\xi)) \hat{f}(\xi), \end{aligned}$$

where

$$(4.9) \quad \chi(\xi) = \begin{cases} 1, & |\xi| \leq R-1, \\ 0, & |\xi| \geq R, \end{cases}$$

is a smooth cutoff function, so that $f = f_1 + f_2$.

Since $\chi \hat{f} \in L^\infty$, with compact support, the Hausdorff-Young inequality gives

$$\|f_1\|_\infty \leq \|\hat{f}_1\|_1 \leq C,$$

so that $f_1 \in L^1(|x| \leq R)$.

For $|\alpha| = n+1$, the Hausdorff-Young inequality gives

$$(4.10) \quad \begin{aligned} \|x^\alpha f_2\|_\infty &\leq \|\widehat{x^\alpha f_2}\|_1 \\ &= \|D^\alpha(1 - \chi)\hat{f}\|_1 \\ &\leq C \left(\sum_{1 \leq |\beta| \leq |\alpha|} \|D_\xi^\beta \chi\|_1 + \|(1 - \chi)D_\xi^\alpha \hat{f}\|_1 \right) \\ &\leq C \left(1 + \|(1 - \chi(\xi))|\xi|^{-(n+1)-\sigma_2}\|_1 \right) \\ &\leq C. \end{aligned}$$

Thus

$$|f_2(x)| \leq |x|^{-(n+1)},$$

so that $f_2 \in L^1(|x| \geq R)$.

Let $q_1 = n/(n - \vartheta_1 \sigma_1)$ and $q_2 = n/(n - 1 + \vartheta_2 \sigma_2)$, with $0 < \vartheta_j < 1$. Since $n + 2\sigma_2 > 2$, we can choose ϑ_j so that $1 < q_1 < 2$, $1 < q_2 \leq 2$. Let p_1, p_2 be the Holder conjugates, $1/q_j + 1/p_j = 1$. Then, as above, for $|\alpha| = n$,

$$(4.11) \quad \begin{aligned} \|\chi^\alpha f_1\|_{p_1} &\leq \|D_\xi^\alpha(\chi \hat{f})\|_{q_1} \\ &\leq C \left(1 + \|\chi(\xi)|\xi|^{-n+\sigma_1}\|_{q_1} \right) \\ &\leq C, \end{aligned}$$

so that $|x|^n f_1 \in L^{p_1}$. By Hölder's inequality,

$$(4.12) \quad \|f_1\|_{L^1(|x| \geq R)} \leq \| |x|^n f_1 \|_{L^{p_1}(|x| \geq R)} \| |x|^{-n} \|_{L^{q_1}\{|x| \geq R\}} \leq C,$$

and $f_1 \in L^1(|x| \geq R)$.

Similarly, $\|x^\alpha f_2\|_{p_2} \leq C$ for $|\alpha| = n-1$, so that $|x|^{n-1} f_2 \in L^{p_2}$, and, by Hölder's inequality,

$$(4.13) \quad \| |x|^{1-n} (|x|^{n-1} f_2) \|_{L^1(|x| \leq R)} \leq \| |x|^{n-1} f_2 \|_{L^{p_2}(|x| \leq R)} \| |x|^{1-n} \|_{L^{q_2}(|x| \leq R)} \leq C.$$

Thus, the distribution $\tilde{f}_2(x) \stackrel{\text{def}}{=} |x|^{1-n} (|x|^{n-1} f_2(x))$ is in $L^1(|x| \leq R)$. Since $\tilde{f}_2 \equiv f_2$ away from the origin, $\tilde{f}_2$ is in fact in $L^1(\mathbb{R}^n)$.

Combining results, we find that

$$(4.14) \quad m_1(x) \stackrel{\text{def}}{=} f_1(x) + \tilde{f}_2(x)$$

is in L^1 , with $\|m_1\|_1 \leq C(C')$. On the other hand, the distribution $f - m_1 = f_2 - \tilde{f}_2$ is supported at the point $x = 0$, and its transform is bounded, since

$$(4.15) \quad \|\hat{f} - \hat{m}_1\|_\infty \leq \|\hat{f}\|_\infty + \|m_1\|_1.$$

It follows (cf., e.g., [Donahue, p. 103]) that $f - m_1$ is a multiple of the delta distribution,

$$(4.16) \quad f - m_1 = m_2 \delta,$$

verifying (4.6). Since $\hat{m}_1$, as the transform of an L^1 function, is in $C_0(\mathbb{R}^n)$, it follows that $\hat{f} = \hat{m}_1 + (2\pi)^{-n/2} m_2$ is continuous at 0 and ∞ , and that $\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = (2\pi)^{-n/2} m_2$, as claimed.

Finally, for $g \in L^p$, $1 \leq p < \infty$, we have

$$(4.17) \quad \begin{aligned} \|f * g\|_p &= \|(m_1 + m_2 \delta) * g\|_p \\ &\leq \|m_1 * g\|_p + |m_2| \|g\|_p \\ &\leq \|m_1\|_1 \|g\|_p + |m_2| \|g\|_p \\ &\leq \left(\frac{C}{2} + |m_2| \right) \|g\|_p \\ &\leq \left(\frac{C}{2} + \|\hat{f}\|_\infty \right) \|g\|_p, \end{aligned}$$

proving (4.7). □

Note that Proposition 4.2 is truly multi-dimensional in character, in the sense that the most interesting applications and also the strongest conclusions

(see (4.5)) of the result are for the case $n \geq 2$. Another instance in which Proposition 4.1 can be sharpened is given in the following result:

Proposition 4.3. *Let $\hat{g}(\xi, t) = \widehat{K}(\xi, t)\hat{f}(\xi)$, where $\widehat{K}(\xi, t) = e^{-\vartheta|\xi|^2 t}$, $\hat{f}(\xi) \in L^\infty \cap C^{n+1}(\mathbb{R}^n)$, and*

$$|D_\xi^\beta \hat{f}(\xi)| \leq C' |\xi|^{-|\beta|}, \quad |\beta| \leq n+1.$$

Then $D_x^\alpha g(\cdot, t) \in L^p$ for $t > 0$, for all α , and for all $p \in [1, \infty]$, and

$$(4.18) \quad \|D_x^\alpha g(\cdot, t)\|_p \leq C(|\alpha|) t^{(n/2)(1/p-1)-|\alpha|/2}.$$

In particular, $\widehat{D_x^\alpha g(x, t)} = (i\xi)^\alpha \hat{g}(\xi, t)$ is a strong L^p multiplier, with norm bounded by $C(|\alpha|, \vartheta) C' t^{-|\alpha|/2}$, where the constant $C(|\alpha|, \vartheta)$ depends only on $|\alpha|$ and ϑ .

Proof. By Hausdorff-Young, for $|\beta| = n+1$,

$$\begin{aligned} (4.19) \quad \|x^\beta D_x^\alpha g\|_\infty &\leq C \|D_\xi^\beta \xi^\alpha \widehat{K} \hat{f}\|_1 \\ &\leq C \sum_{\beta=(\beta_1, \beta_2, \beta_3)} \| |\xi|^{|\alpha|-|\beta_1|} |D_\xi^{\beta_2} \widehat{K}| |D_\xi^{\beta_3} \hat{f}| \|_1 \\ &\leq C C' \sum_{\beta} \| |\xi|^{|\alpha|-|\beta_1|-|\beta_3|} (|\xi t|^{|\beta_2|} + |\xi|^{|\beta_2|-2} t^{|\beta_2|-1} + \dots) e^{-\vartheta|\xi|^2 t} \|_1 \\ &= C C' t^{-n/2-|\alpha|/2+|\beta|/2}. \end{aligned}$$

Thus

$$(4.20) \quad |D_x^\alpha g| \leq C(|\alpha|) C' t^{-n/2-|\alpha|/2} \left(\frac{|x|}{\sqrt{t}} \right)^{-(n+1)},$$

for any α , from which (4.18) follows, by scaling. $\square$

The following very relevant example distinguishes the classes of L^p and strong L^p multipliers.

Example 4.4. Let $\hat{f}(\xi) = P_N(\xi/|\xi|)$, with P_N an N th degree polynomial, and define

$$\begin{aligned} (4.21) \quad \hat{f}_1(\xi) &= \chi(\xi) \hat{f}(\xi), \\ \hat{f}_2(\xi) &= (1 - \chi(\xi)) \hat{f}(\xi), \end{aligned}$$

where $\chi(\xi)$ is the cutoff function of (4.9) with support in $\{\xi : |\xi| \leq R\}$. Then each of $\hat{f}_1, \hat{f}_2, \hat{f}$ is an L^p multiplier, $1 < p < \infty$, but is not an L^1 or L^∞ multiplier in the sense of (4.2).

Proof. We first observe that $f(x)$ has the simple form

$$(4.22) \quad f(x) = PV q_N \left(\frac{x}{|x|} \right) |x|^{-n},$$

where q_N is a nontrivial N th order polynomial. This follows by induction on monomials. For $N = 1$, we have $\widehat{\xi_j}/|\xi| = C PV(x_j/|x|^{n+1})$ by the Riesz formula [Stein]. For $N = 2$,

$$(4.23) \quad \frac{\widehat{\xi_j \xi_k}}{|\xi|^2} = D_{x_j} D_{x_k} \Delta^{-1} \delta = C D_{x_j} D_{x_k} |x|^{2-n},$$

and we obtain (4.22) by direct calculation. We extend to $N > 2$ by writing

$$\frac{\xi_j \xi_k}{|\xi|^2} \cdot \left(\frac{\xi}{|\xi|} \right)^\alpha = C_1 \xi_j D_{\xi_k} \left(\frac{\xi}{|\xi|} \right)^\alpha - C_2 P_{|\alpha|} \left(\frac{\xi}{|\xi|} \right),$$

whose Fourier transform, by induction, is

$$\begin{aligned} C'_1 D_{x_j} \left(x_k \left(\frac{\xi}{|\xi|} \right)^\alpha \right) - PV C'_2 q_{|\alpha|} \left(\frac{x}{|x|} \right) |x|^{-n} \\ = PV q_{|\alpha|+2} \left(\frac{x}{|x|} \right) |x|^{-n}. \end{aligned}$$

To see that $\hat{f}$ is not an L^∞ multiplier, set

$$(4.24) \quad g^a(x) = \left[1 - \chi \left(\frac{x}{Ra} \right) \right] \Gamma_W \left(\frac{x}{|x|} \right),$$

where Γ_W is the characteristic function of a set $W \subseteq S^{n-1}$ on which q_N is positive. That is, $q_N(\vartheta) \geq C^{-1}$ for $-\vartheta \in W$. Then for $|x| < a/2$,

$$\begin{aligned} f * g^a(x) &= PV \int_0^\infty r^{-1} \int_{|\vartheta|=1} q_N(\vartheta) \Gamma \left(\frac{x - r\vartheta}{|x - r\vartheta|} \right) \left[1 - \chi \left(\frac{|x - r\vartheta|}{Ra} \right) \right] dS_\vartheta dr \\ &= \int_{\frac{a}{2}}^\infty r^{-1} \int_{|\vartheta|=1} q_N(\vartheta) \Gamma \left(\frac{x - r\vartheta}{|x - r\vartheta|} \right) \left[1 - \chi \left(\frac{|x - r\vartheta|}{Ra} \right) \right] dS_\vartheta dr \\ &= +\infty, \end{aligned}$$

since

$$\begin{aligned} \lim_{r \rightarrow +\infty} \int_{|\vartheta|=1} q_N(\vartheta) \left[1 - \chi \left(\frac{|x - r\vartheta|}{Ra} \right) \right] \Gamma \left(\frac{x - r\vartheta}{|x - r\vartheta|} \right) dS_\vartheta \\ = \int_{\vartheta \in W} q_N(-\vartheta) dS_\vartheta \geq C^{-1}. \end{aligned}$$

Thus, $f * g^a$ is not in L^∞ for any $a > 0$.

The same argument shows that $(1 - \chi)f * g^a$ is not in L^∞ for any $a > 0$. Further, rescaling by $r' = r/a$ shows that $\|\chi(f * g^a)\|_\infty \sim -\log a$ as $a \rightarrow 0$, so that neither $(1 - \chi)f$ nor χf satisfies (4.2) for $p = \infty$ with a constant independent of a . But, $\|f_1\|_\infty \leq \|\hat{f}_1\|_1 \leq (R)^n$ implies that $f_1 \in L^1(|x| \leq R)$, so that χf_1 satisfies (4.2). Similarly, $\| |x|^{n+1} f_2(x) \|_\infty \leq \|D_\xi^{n+1} \hat{f}_2\|_1 \leq C$, so that $f_2 \in L^1(|x| \geq R)$, and $(1 - \chi)f_2$ satisfies (4.2). Thus, if either f_1 or f_2 satisfied (4.2), then either $\chi(f_1 + f_2) = \chi f$ or $(1 - \chi)(f_1 + f_2) = (1 - \chi)f$ would as well, a contradiction.

To see that f is not an L^1 multiplier, let $g^a(x) = a^{-n}g(x/a)$ be a sequence of compactly supported test functions converging weakly to δ . Then $(f * g^a)$ converges pointwise to f as $a \rightarrow 0$, except at the point $x = 0$, while $\|g^a\|_1 \equiv 1$. Thus

$$\lim_{a \rightarrow 0} \int_\Omega |f * g^a| = \int_\Omega |f|$$

for any compact $\Omega \subset \mathbb{R}^n \setminus \{0\}$. On the other hand, if f were an L^1 multiplier, then there would be a constant C such that

$$(4.25) \quad C \geq \|f * g^a\|_1 \geq \int_\Omega |f * g^a| ,$$

and therefore that

$$C \geq \int_\Omega |f| ,$$

contradicting (4.22). Clearly, the same reasoning holds for $\chi(x)f(x)$ and $(1 - \chi(x))f(x)$, so that, as above, neither $f_1(x)$ nor $f_2(x)$ satisfies (4.2). $\square$

5. A preliminary decay result. We are now ready to study the decay to equilibrium of solutions of the Navier-Stokes equations with real viscosity. We begin, in this section, by rederiving an L^p , $p \geq 2$, decay result of Kawashima [8], under weaker smoothness assumptions on the initial data. The decay rate we obtain for the solution is optimal only in L^2 ; we will improve it later on.

Setting $u = \begin{bmatrix} \bar{\rho} \\ \bar{m} \end{bmatrix} = \begin{bmatrix} \rho - \rho^* \\ m - m^* \end{bmatrix}$, we can write the nonlinear system (1.1) in the form

$$(5.1) \quad u_t - Lu = \sum_k D_{x_k} Q^k; \quad u(x, 0) = u_0(x),$$

where

$$(5.2) \quad L \begin{bmatrix} \bar{\rho} \\ \bar{m} \end{bmatrix} \equiv \begin{bmatrix} -\operatorname{div}(\bar{m}) \\ -c^2 \nabla \bar{\rho} + \varepsilon \Delta \bar{m} + \eta \nabla \operatorname{div} \bar{m} \end{bmatrix},$$

$$(5.3) \quad Q^k = Q_1^k + Q_2^k = \begin{bmatrix} 0 \\ q_1^k(u) \end{bmatrix} + \begin{bmatrix} 0 \\ D_x q_2^k(u) \end{bmatrix},$$

and

$$(5.4) \quad q_i^k(u) = O(|u|^2)$$

for $|u|$ sufficiently small.

Kawashima, [8], has proved the following basic existence result, using energy methods.

Proposition 5.1. *Let $u_0 \in H^{s+\ell}(\mathbb{R}^n)$, $s = [n/2] + 1$, with $E \equiv \|u_0\|_{H^{s+\ell}}$ sufficiently small. Then, there exists a unique, global, classical solution $u \in H^{s+\ell}$ of (5.1), satisfying*

$$(5.5) \quad \left. \begin{aligned} &\|D_x^\alpha u\|_2, && 0 \leq |\alpha| \leq s + \ell \\ &\left(\int_0^\infty \|D_x^\alpha u\|_2^2 dt \right)^{1/2}, && 1 \leq |\alpha| \leq s + \ell \\ &\|D_x^\alpha u\|_\infty, && 0 \leq |\alpha| \leq \ell \end{aligned} \right\} = O(1)E,$$

The solution operator e^{LT} of the linear (left hand side) part of (5.1) can be expressed as

$$(5.6) \quad e^{Lt} u_0 = G(t) * u_0,$$

where G is the Green's function (more precisely, distribution) described in Section 3. Taking D_x^α of both sides of (5.1) and applying Duhamel's principle, we obtain the representation

$$(5.7) \quad \begin{aligned} D_x^\alpha u(t) &= G(t) * D_x^\alpha u_0 + \int_0^t \sum_k D_{x_k} G(t-s) * D_x^\alpha Q^k(s) ds \\ &= I + II, \end{aligned}$$

where we have suppressed the x arguments for convenience.

Lemma 5.2. *Let*

$$(5.8) \quad \chi(\xi) \equiv \begin{cases} 1 & |\xi| \leq R \\ 0 & |\xi| \geq R+1 \end{cases}$$

be a smooth cut off function, and decompose G as $G = G_1 + G_2$, where

$$(5.9) \quad \begin{aligned} \widehat{G}_1(\xi, t) &\equiv \chi(\xi) \widehat{G}(\xi, t), \\ \widehat{G}_2(\xi, t) &\equiv (1 - \chi(\xi)) \widehat{G}(\xi, t). \end{aligned}$$

Then:

(a) *for any $R > 0$, $D_x^\alpha G_1 \in L^p$ for all α, p , with*

$$(5.10) \quad \|D_x^\alpha G_1\|_p \leq C \begin{cases} t^{-n/2(1-1/p)-|\alpha|/2} & t \geq 1, p \geq 2, \\ 1 & t \leq 1, \text{ all } p; \end{cases}$$

(b) *for R sufficiently large, there is $b > 0$ for which*

$$(5.11) \quad \widehat{G}_2 = e^{-bt} M(\xi, t)$$

and

$$(5.12) \quad \widehat{D_{x_j} G_2} \cdot \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = e^{-bt} \left(1 + t^{-1/2}\right) \mathcal{M}(\xi, t),$$

where M and $\mathcal{M}$ are, respectively, strong ($1 \leq p \leq \infty$) and ordinary ($1 < p < \infty$) L^p multipliers, with norms bounded by C and $C(p)$ independent of time.

Proof. Let p, q denote Hölder conjugates, $1/p + 1/q = 1$.

(a) The estimates in Lemma 3.2(a), together with the fact that $D_\xi^\beta \chi(\xi)$ is supported on $\{R \leq |\xi| \leq R+1\}$, for $|\beta| > 0$, give

$$(5.13) \quad |D_\xi^\beta (\xi^\alpha \widehat{G}_1)| \leq C |\xi|^{|\alpha|} (1 + |\xi|)^{|\beta|} (1+t)^{|\beta|} e^{-\vartheta |\xi|^2 t},$$

for some $\vartheta > 0$. Taking $|\beta| = 0$, we find that $\xi^\alpha \widehat{G}_1$ is Schwartz class, so that $D_x^\alpha G_1 \in L^p$ for all p . For $p \geq 2$, the Hausdorff-Young inequality gives

$$\begin{aligned}
 (5.14) \quad \|D_x^\alpha G_1(x, t)\|_p &\leq \|\xi^\alpha \widehat{G}_1(\xi, t)\|_q \\
 &\leq \left(\int_{|\xi| \leq R} |\xi|^{|\alpha|q} e^{-\vartheta|\xi|^2 t} d\xi \right)^{1/q} \\
 &\leq C \begin{cases} t^{-n/2(1-1/p)-|\alpha|/2} & t \geq 1, \\ 1 & t \leq 1. \end{cases}
 \end{aligned}$$

For $t \leq 1$, (5.13) gives

$$(5.15) \quad |D_\xi^\beta (\xi^\alpha \widehat{G}_1)| \leq C' \begin{cases} 1 & |\xi| \leq R+1 \\ 0 & |\xi| \geq R+1. \end{cases}$$

Thus, Proposition 4.2 implies that

$$(5.16) \quad \|D_x^\alpha G_1\|_1 \leq C.$$

Interpolating between (5.14) and (5.16), we obtain

$$(5.17) \quad \|D_x^\alpha G_1\|_p \leq C, \quad t \leq 1, \text{ all } p,$$

completing the proof of (a).

(b) The estimates of Lemma 3.2(b) give

$$(5.18) \quad D_\xi^\beta \widehat{G}_2(\xi, t) \leq C \begin{cases} 0 & |\xi| \leq R, \\ |\xi|^{-|\beta|} \left(e^{-at/2} |\xi|^{-1} + e^{-\vartheta|\xi|^2 t} \right) & |\xi| \geq R. \end{cases}$$

Applying Proposition 4.1, we have

$$(5.19) \quad \widehat{G}_2(x, t) = e^{-bt} M,$$

where $b < \min\{a/2, \vartheta R^2\}$ and M is an L^p multiplier with norm bounded by C_p independent of t .

To see that M is in fact a strong L^p multiplier, we must look more closely. Setting

$$(5.20) \quad \hat{g}_2(\xi, t) = \widehat{G}_2(\xi, t) - (1 - \chi(\xi)) \sum \hat{K}_\ell \mathcal{M}_\ell,$$

where $\hat{K}_\ell$, $\mathcal{M}_\ell$ are as in (3.12), we have

$$(5.21) \quad D_\xi^\beta \hat{g}_2(\xi, t) \leq C \begin{cases} 0 & |\xi| \leq R, \\ |\xi|^{-|\beta|-1} e^{-bt} & |\xi| \geq R, |\beta| > 0. \end{cases}$$

Applying Proposition 4.2, we find that $\hat{g}_2(x, t) = e^{-bt} M(\xi, t)$, M a strong L^p multiplier. More precisely, we have

$$(5.22) \quad g_2(x, t) = e^{-at} \delta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + e^{-bt} m(x, t),$$

where $m(\cdot, t) \in L^1$, $\|m(\cdot, t)\|_1 \leq C$, and $a = c^2/\nu = c^2/(\varepsilon + \eta)$.

The remaining term, $(1 - \chi) \left(\sum \hat{K}_\ell \mathcal{M}_\ell(\xi) \right)$, has Fourier transform in L^p for all p , with

$$(5.23) \quad \left\| D_x^\alpha \widehat{\sum \hat{K}_\ell \mathcal{M}_\ell(\xi)} \right\|_1 \leq C e^{-bt} t^{-|\alpha|/2}, \quad b > 0.$$

For, in the factorization

$$(5.24) \quad (1 - \chi(\xi)) e^{-\vartheta_\ell |\xi|^2 t} \mathcal{M} \left(\frac{\xi}{|\xi|} \right) \\ = (1 - \chi(\xi)) e^{-(1/2)\vartheta_\ell |\xi|^2 t} \mathcal{M} \left(\frac{\xi}{|\xi|} \right) \times e^{-(1/2)\vartheta(\xi)^2 t},$$

the first term satisfies the hypotheses of Proposition 4.1 with constant $C'(t) \leq C e^{-(1/2)\vartheta_\ell R^2 t}$, and the second is the transform of the heat kernel. Applying Proposition 4.3, we obtain (5.23), with $b < \min_\ell \{ \frac{1}{2} \vartheta_\ell R^2 \}$. This completes the proof of (5.11).

From Lemma 3.2(b), we have, for $|\beta| \geq 0$,

$$(5.25) \quad |D_\xi^\beta \hat{g}_2(\xi) \cdot \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}| \leq C \begin{cases} 0 & |\xi| \leq R \\ |\xi|^{-|\beta|-1} e^{-bt} & |\xi| \geq R. \end{cases}$$

With Proposition 4.1, this implies that

$$(5.26) \quad \widehat{D_{x_j} g_2(x, t)} = e^{-bt} \mathcal{M}'(\xi, t),$$

with $\mathcal{M}'$ an L^p multiplier uniformly bounded in time. With (5.23), $|\alpha| = 1$, this gives (5.12), and we are done. $\square$

Remark. More precisely, we have

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \delta(x) + m(x),$$

where

$$|m(x)| \leq \begin{cases} C|x|^{-n+1} & |x| \leq R \\ C(N)|x|^{-N} & |x| > R, \end{cases}$$

for any integer N . The first fact follows by the expansion (3.12), the second from the proof of Lemma 4.2.

Remark. Example 4.4 shows that $\mathcal{M}$ is definitely **not** a strong L^p multiplier. For, the $(1, 1+k)$ entry of $\xi_j \widehat{G}_2(\xi, t)$ can be expressed more precisely as

$$e^{-bt} (1 - \chi(\xi)) \left[\frac{\xi_j \xi_k}{|\xi|^2} + O\left(\frac{1}{|\xi|}\right) \mathcal{M}' \right].$$

By Proposition 4.2, the term $O(1/|\xi|)(1 - \chi(\xi)) \mathcal{M}'$ is a strong L^p multiplier, while, by Example 4.4, $(1 - \chi(\xi))(\xi_j \xi_k / |\xi|^2)$ is not.

Theorem 5.3. Let $u_0 \in H^{s+\ell} \cap L^1(\mathbb{R}^n)$, where $s = [n/2] + 1$ and ℓ is a positive integer. If

$$(5.27) \quad E \stackrel{\text{def}}{=} \max\{\|u_0\|_{H^{s+\ell}}, \|u_0\|_1\}$$

is sufficiently small, then the solution $u(\cdot, t)$ of (5.1) satisfies

$$(5.28) \quad \|D_x^\alpha u\|_p \leq E \begin{cases} C(|\alpha|)(1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, \ell-2-|\alpha|\}} & \text{for } |\alpha| \leq \ell-2, \ 2 \leq p \leq \infty, \ \ell \geq 3 \\ C(|\alpha|, p)(1+t)^{-n/2(1-1/p)-1/2 \{\min|\alpha|, \ell-1-|\alpha|\}} & \text{for } |\alpha| \leq \ell-1, \ 2 \leq p < \infty, \end{cases}$$

$$(5.29) \quad \|D_x^\alpha (u - e^{Lt} u_0)\|_p \leq E^2 L(t) \begin{cases} C(|\alpha|)(1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, (\ell-3-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-2, \ 2 \leq p \leq \infty, \ \ell \geq 3 \\ C(|\alpha|, p)(1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, (\ell-1-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-1, \ 2 \leq p < \infty, \end{cases}$$

where e^{Lt} is as in (5.6) and $L(t) = \log(1+t)$ if $n = 2$, $L(t) = 1$ otherwise.

Proof. We first prove the estimates (5.28) and (5.29) for $3 \leq |\alpha| \leq \ell - 2$. Set

$$(5.30) \quad A(t) \stackrel{\text{def}}{=} \sup_{\substack{0 \leq s \leq t \\ 2 \leq p \leq \infty \\ |\alpha| \leq \ell - 2}} (1+s)^{n/2(1-1/p)+1/2 \min\{|\alpha|, \ell-2-|\alpha|\}} \|D_x^\alpha u(\cdot, s)\|_p,$$

$$(5.31) \quad B(t) \stackrel{\text{def}}{=} \sup_{\substack{0 \leq s \leq t \\ 2 \leq p \leq \infty \\ |\alpha| \leq \ell - 2}} L(t)^{-1} (1+s)^{n/2(1-1/p)+1/2 \min\{|\alpha|, \ell-3-|\alpha|\}+1/2} \\ \times \|D_x^\alpha (u(\cdot, s) - e^{Ls} u_0)\|_p$$

Claim. For E sufficiently small,

$$(5.32) \quad A(t) \leq C (E + A(t)^2 + A(t)^{\ell-2}), \quad t \geq 0$$

$$(5.33) \quad B(t) \leq C (E^2 + A(t)^2 + A(t)^{\ell-2}), \quad t \geq 0.$$

Assuming (5.32)–(5.33), we obtain (5.28)–(5.29) by continuous induction. For, rearranging (5.32) gives

$$(5.34) \quad A(t) \leq \frac{CE}{1 - C(A(t) + A(t)^{\ell-2})}.$$

Taking $4C^2E < \frac{1}{2}$, we obtain $A(t) \leq 2CE$ for all $t \geq 0$, using continuity of $A(t)$, (5.34), and induction. This proves (5.28) and, together with (5.33), (5.29), for $3 \leq |\alpha| \leq \ell - 2$.

Proof of Claim. Let $|\alpha| \leq \ell - 2$. By Proposition 5.1 and interpolation, we have

$$(5.35) \quad \left\{ \|D_x^\beta u\|_p, \ 2 \leq p \leq \infty \right\} \left(\int_0^t \|D_x^\beta u\|_2^2(s) ds \right)^{1/2} \leq CE \quad \text{for } |\beta| \leq |\alpha| + 2.$$

Likewise, from (5.3)–(5.4), we have, for $2 \leq p \leq \infty$,

$$(5.36) \quad \|D_x^\alpha Q\|_p \leq CE^2,$$

$$\begin{aligned}
(5.37) \quad \int_0^t \|Q\|_1 ds &\leq C \int_0^t \left(|u|^2 + |u| \|D_x u\|_1 \right) ds \\
&\leq C \int_0^t (\|u\|_2^2 + \|D_x u\|_2^2) ds \\
&\leq C \int_0^t \left(A(s)^2 (1+s)^{-n/2} + \|D_x u\|_2^2 \right) ds \\
&\leq C (E^2 + A(t)^2) \begin{cases} \log(1+t) & n=2 \\ 1 & n \geq 3, \end{cases}
\end{aligned}$$

$$\begin{aligned}
(5.38) \quad &\left. \begin{aligned} &\|D_x^\alpha Q_1\|_{2p/(p+2)} \\ &\|D_x^\beta Q_2\|_{2p/(p+2)}, \quad |\beta| = |\alpha| - 1 \end{aligned} \right\} \\
&\leq C \sup_{\sum |\gamma_j| = |\alpha|} \|D_x^{\gamma_1} u\|_p \|D_x^{\gamma_2} u\|_2 \prod_{j \geq 3} \|D_x^{\gamma_j} u\|_\infty \\
&\leq C \left(A(t)^2 + A(t)^{|\alpha|} \right) (1+t)^{-n/2(1-1/p) - 1/2 \min\{|\alpha|, \ell-2-|\alpha|\} - n/4},
\end{aligned}$$

and the critical estimate,

$$\begin{aligned}
(5.39) \quad &\|D_x D_x^\alpha Q_j\|_p \leq \\
&\leq C \sup_{\sum |\gamma_j| = |\alpha| + j} \|D_x^{\gamma_1} u\|_p \prod_{j \geq 2} \|D_x^{\gamma_j} u\|_\infty \\
&\leq A(t) (E + A(t)^2 + A(t)^{|\alpha|}) (1+t)^{-n/2(1-1/p) - 1/2 \min\{|\alpha|, \ell-3-|\alpha|\} - 1/2}, \\
&\quad |\alpha| \leq \ell - 2.
\end{aligned}$$

(We use the condition $\ell \geq 3$ precisely in (5.39), to bound the term $\|D_x D_x^\alpha u\|_p \|D_x u\|_\infty$ occurring for $|\alpha| = \ell - 3$.)

Let I and II be as in (5.7) and $2 \leq p \leq \infty$. Applying Lemma 5.2, we have

$$\begin{aligned}
(5.40) \quad &\|I\|_p = \|D^\alpha (G * u_0)\|_p \\
&\leq \|D_x^\alpha G_1 * u_0\|_p + \|G_2 * D_x^\alpha u_0\|_p \\
&\leq \|D_x^\alpha G_1\|_p \|u_0\|_1 + e^{-bt} \|D_x^\alpha u_0\|_p \\
&\leq CE(1+t)^{-n/2(1-1/p) - |\alpha|/2}.
\end{aligned}$$

Likewise, combining (5.36)–(5.39) with Lemma 5.2, we have for $|\alpha| > 0$ and $|\beta| = |\alpha| - 1$,

$$\begin{aligned}
(5.41) \quad \|II\|_p &= \left\| \int_0^t D_x G(t-s) * D_x^\alpha Q(s) ds \right\|_p \\
&\leq \int_0^{t/2} \|D_x^\alpha D_x G_1 * Q\|_p + \int_{t/2}^t \|D_x G_1 * D_x^\alpha Q_1\|_p \\
&\quad + \int_{t/2}^t \|D_x^2 G_1 * D_x^\beta Q_2\|_p + \int_0^t \|G_2 * D_x D_x^\alpha Q\|_p \\
&\leq \int_0^{t/2} \|D_x^\alpha D_x G_1\|_p \|Q\|_1 + \int_{t/2}^t \|D_x G_1\|_2 \|D_x^\alpha Q_1\|_{2p/(p+2)} \\
&\quad + \int_{t/2}^t \|D_x^2 G_1\|_2 \|D_x^\beta Q_2\|_{2p/(p+2)} + \int_0^t e^{-b(t-s)} \|D_x D_x^\alpha Q\|_p \\
&\leq C(1+t)^{-n/2(1-1/p)-|\alpha|/2-1/2} \int_0^{t/2} \|Q(s)\|_1 ds \\
&\quad + C(A(t)^2 + A(t)^{|\alpha|}) (1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, \ell-2-|\alpha|\}-n/4} \\
&\quad \quad \quad \times \int_{t/2}^t (1+(t-s))^{-n/4-1/2} ds \\
&\quad + CA(t) (E + A(t) + A(t)^{|\alpha|}) \int_0^t e^{-b(t-s)} \\
&\quad \quad \quad \times (1+s)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}-1/2} ds.
\end{aligned}$$

In the case $|\alpha| = 0$, we can replace $\int_{t/2}^t \|D_x^2 G_1 * D_x^\beta Q_2\|_p$ with $\int_{t/2}^t \|D_x G_1 * D_x^\alpha Q_2\|_p$ in the second line of (5.41) and obtain the same result. We therefore have

$$\begin{aligned}
(5.42) \quad \|II\|_p &\leq C(E^2 + A(t)^2 + A(t)^{|\alpha|}) \\
&\quad \times (1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}-1/2} \begin{cases} \log t, & n = 2 \\ 1, & n \geq 3. \end{cases}
\end{aligned}$$

Combining (5.40)–(5.42) with

$$\begin{aligned}
(5.43) \quad D_x^\alpha u &= I + II, \\
D_x^\alpha (u - e^{Lt} u_0) &= II,
\end{aligned}$$

we obtain (5.32)–(5.33), and we are done.

Finally, the case that $|\alpha| = \ell - 1$ or $|\alpha| \leq 2$ follows by a similar argument, with

$$(5.44) \quad A(t) \stackrel{\text{def}}{=} \sup\{(1+s)^{n/2(1-1/p)+1/2 \min\{|\alpha|, \ell-1-|\alpha|\}} \|D_x^\alpha u(\cdot, s)\|_p\},$$

$$(5.45) \quad B(t) \stackrel{\text{def}}{=} \sup\{(1+s)^{n/2(1-1/p)+1/2 \min\{|\alpha|, \ell-1-|\alpha|\}} \|D_x^\alpha (u - e^{Lt} u_0)\|_p\},$$

by substituting

$$\int D_x G_2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} * D_x^\alpha Q$$

for

$$\int G_2 * D_x D_x^\alpha Q$$

in (5.42), and replacing (5.39) by the bound

$$(5.46) \quad \|D_x^\alpha Q\|_p \leq A(t) \left(E + A(t)^2 + A(t)^{|\alpha|} \right) \\ \times (1+t)^{-n/2(1-1/p)-1/2 \min\{|\alpha|, \ell-1-|\alpha|\}-1/2}.$$

Since $D_x G_2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is only an L^p multiplier for $2 \leq p < \infty$, the bound $C(|\alpha|)$ must be replaced by $C(|\alpha|, p)$. $\square$

Remark. The results of Theorem 5.3 can be established in L^2 without resort to Proposition 4.1 and Lemma 5.2, since $\mathcal{M}$ is trivially an L^2 multiplier. With enough $([n/2] + 1)$ extra smoothness on the initial data, the results for L^p , $p \geq 2$ then follow by Sobolev bounds. However, the decomposition of the Green's function obtained in Lemma 5.2 via Proposition 4.1 allows us to obtain the results for $p > 2$ with far less smoothness assumed on the initial data. Further, Lemma 5.2 will be essential in our analysis, in Sections 7 and 8, of the case $p < 2$.

6. Effective artificial viscosity and L^p decay to diffusion waves, $p \geq 2$. We continue the study of the large-time behavior of the solution $u(\cdot, t)$ of the nonlinear Navier-Stokes system (1.1) in L^p for $p \in [2, \infty]$. As we saw in Theorem 5.3, $u(\cdot, t)$ is well-approximated for large time by $G(\cdot, t) * u_0$, where G is the Green's matrix for the linearized Navier-Stokes system (1.4). While this result clarifies in a significant way the asymptotic behavior of u , it raises the issue of the “shape” of $G(\cdot, t) * u_0$ itself for large time. Observe that the linearized system (1.4) is not strictly parabolic, nor does the symbol associated with its first-order convection terms commute with that of its second order diffusion terms. As we shall see in Theorem 6.2 below, however, the matrices B_{jk} occurring in (1.4) (in the notation of §2) may be replaced by artificial viscosity matrices $\tilde{B}_{jk}$, in such a way that the resulting system does become strictly parabolic, the symbols $\xi_j A_j$ and $\xi_j \xi_k \tilde{B}_{jk}$ do commute, and, most important, the solution $u(\cdot, t)$ of the nonlinear Navier-Stokes system (1.1) is just as well approximated by $\tilde{G}(\cdot, t) * u_0$ as by $G(\cdot, t) * u_0$, in the sense of Theorem 5.3. The aforementioned commutativity implies that $\tilde{G}(\cdot, t)$ is simply the convolution of two fundamental solutions—one corresponding to the hyperbolic part (linearized Euler equations),

and the other to an explicit parabolic operator canonically associated to the Navier-Stokes system. These results enable us to greatly refine the description given in Theorem 5.3 of the asymptotic behavior of u . In addition, the strict parabolicity of the artificial viscosity system enables us to show that, when u_0 has a first moment, the corresponding solution $u(\cdot, t)$ to the nonlinear Navier-Stokes system is asymptotic to the convecting diffusion wave $\tilde{G}(\cdot, t)U$, where U is the total mass of u_0 . These results, stated in Theorems 6.7 and 6.8 below, represent the culmination of our analysis of solutions of the Navier-Stokes system in L^p for $p \in [2, \infty]$.

We begin with a brief discussion of artificial viscosity in a rather general setting. Thus consider the following pde for an unknown vector function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^m$:

$$(6.1) \quad u_t + A_j u_{x_j} = B_{jk} u_{x_j x_k},$$

where A_j and B_{jk} are constant, real $m \times m$ matrices. If the differential operator in (6.1) satisfies the Hadamard-Petrovsky condition, then there will be a Green's matrix $G \in C^\infty([0, \infty); \mathcal{S}'(\mathbb{R}^n))$: the columns of G are solutions of (6.1), and $G(0) = \delta I$. In addition, if (6.1) is dissipative in a suitable sense, then, as we proved in Section 5 for the linearized Navier-Stokes system and in Section 2 for more general linear systems,

$$(6.2) \quad \|G(\cdot, t) * u_0\|_{L^2} \leq C(u_0) t^{-n/4}$$

for u_0 in a fairly broad class. We seek to replace the system (6.1) by a system which enjoys the special properties discussed above, but without altering the asymptotic character of its solutions in an essential way.

Definition. Let G be the Green's matrix for (6.1) as described above, and assume that there is a symmetric, positive definite matrix P such PA_j is symmetric for each j , as in Section 2. An effective artificial viscosity for (6.1) is then a measurable mapping $\xi \mapsto \tilde{B}(\xi)$ from $\mathbb{R}^n$ into the set of $m \times m$ matrices such that

$$(6.3) \quad \tilde{B}(\xi) = |\xi|^2 \tilde{B} \left(\frac{\xi}{|\xi|} \right), \quad \xi \in \mathbb{R}^n;$$

$$(6.4) \quad \tilde{B}(\xi) \text{ and } \xi_j A_j \text{ commute, } \xi \in \mathbb{R}^n;$$

there is a positive constant ε such that

$$(6.5) \quad \langle P\tilde{B}(\xi)w, w \rangle_{\mathbb{R}^m} \geq \varepsilon |\xi|^2 |w|^2,$$

$$\xi \in \mathbb{R}^n, w \in \mathbb{R}^m;$$

$$(6.6) \quad t^{n/4} \| [G(\cdot, t) - \tilde{G}(\cdot, t)] * u_0 \|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty$$

for all $u_0 \in L^1 \cap L^2$, where $\tilde{G}(\cdot, t) = \exp(-i\xi_j A_j t) \exp(-\tilde{B}(\xi)t)$.

Though we do not prove it here, there exists an effective artificial viscosity for any hyperbolic-parabolic system which is stable in the sense of Kawashima [8]. We shall demonstrate the existence of an effective artificial viscosity for the Navier-Stokes system by explicit computation; the procedure in the general case is similar (see (6.9) below). Uniqueness is quite straightforward:

Proposition 6.1. *Assume that there is a symmetric, positive definite matrix P such that PA_j is symmetric for each j , and let $\tilde{B}_1$ and $\tilde{B}_2$ be effective artificial viscosities for the system (6.1). Then $\tilde{B}_1 = \tilde{B}_2$.*

Proof. Fix $\bar{u} \in \mathbb{R}^m$ and define $u_0 \in L^1 \cap L^2$ by $\hat{u}_0(\xi) = e^{-|\xi|^2} \bar{u}$. Letting $\tilde{G}_1$ and $\tilde{G}_2$ be the corresponding Green's matrices $\hat{\tilde{G}}_\ell(\xi, t) = \exp[(-i\xi_j A_j - \tilde{B}_\ell(\xi))t]$, we then find from (6.3), (6.4), and the symmetrizability that

$$\begin{aligned} & t^{n/2} \| [G_2(\cdot, t) - G_1(\cdot, t)] * u_0 \|_2^2 \\ &= t^{n/2} \int |e^{-i\xi_j A_j t} [e^{-\tilde{B}_2(\xi)t} - e^{-\tilde{B}_1(\xi)t}] \bar{u}|^2 e^{-2|\xi|^2} d\xi \\ &\geq C^{-1} \int |[e^{-\tilde{B}_2(y)} - e^{-\tilde{B}_1(y)}] \bar{u}|^2 e^{-2|y|^2/t} dy \\ &\rightarrow C^{-1} \int |[e^{-\tilde{B}_2(y)} - e^{-\tilde{B}_1(y)}] \bar{u}|^2 dy \end{aligned}$$

as $t \rightarrow \infty$. This last integral must be zero, by (6.6), so that $e^{-\tilde{B}_2(y)} = e^{-\tilde{B}_1(y)}$ for almost all y . (6.3) then shows that $e^{-\tilde{B}_2(\xi)t} = e^{-\tilde{B}_1(\xi)t}$ for all $t > 0$ and almost all ξ . Differentiating with respect to t , we then conclude that $\tilde{B}_2 = \tilde{B}_1$ a.e. $\square$

Next we show that an effective artificial viscosity does indeed exist for the linearized Navier-Stokes system (1.4). Taking Fourier transforms in (1.4), we find that, in the notation of (6.1),

$$(6.7) \quad i\xi_j A_j = \begin{bmatrix} 0 & i\xi^t \\ ic^2\xi & 0 \end{bmatrix}$$

and

$$(6.8) \quad B(\xi) \equiv \xi_j \xi_k B_{jk} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon|\xi|^2 I + \eta\xi\xi^t \end{bmatrix}.$$

The symmetrizer P is evidently given by

$$P = \begin{bmatrix} c^2 & 0 \\ 0 & I \end{bmatrix},$$

and the matrix $\xi_j A_j$ may be diagonalized: there is an invertible matrix $R(\xi)$ and a diagonal matrix $\Lambda(\xi)$ such that

$$R^{-1}(\xi)(\xi_j A_j)R(\xi) = \Lambda(\xi).$$

We define $\tilde{B}(\xi)$ to be the diagonal of $B(\xi)$ in the $R(\xi)$ coordinates:

$$(6.9) \quad R^{-1}(\xi)\tilde{B}(\xi)R(\xi) = \text{diag} [R^{-1}(\xi)B(\xi)R(\xi)].$$

The symbols $\xi_j A_j$ and $\tilde{B}(\xi)$ therefore commute because $R^{-1}(\xi_j A_j)R$ and $R^{-1}\tilde{B}R$ are diagonal matrices, and (6.3) holds because B is homogeneous of degree two, and R of degree zero, in ξ .

To check the other two requirements, (6.5) and (6.6), we exploit the rotational symmetry to compute $\tilde{B}$ explicitly. Thus fix $\xi \in \mathbb{R}^n - \{0\}$ and choose an orthogonal matrix $V(\xi)$ such that $V^t \xi = |\xi|e_1$, where $e_1^t = (1, 0, \dots, 0)$. Setting

$$(6.10) \quad Q(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & V(\xi) \end{bmatrix}$$

we then compute that

$$Q^t(i\xi_j A_j)Q = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$W = \begin{bmatrix} 0 & i|\xi| \\ ic^2|\xi| & 0 \end{bmatrix}.$$

Now,

$$(6.11) \quad H^{-1}WH = \begin{bmatrix} -ic|\xi| & 0 \\ 0 & ic|\xi| \end{bmatrix},$$

where

$$H = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix},$$

so that

$$(6.12) \quad Q^t \exp(-i\xi_j A_j t) Q = \left[\begin{array}{cc|c} \cos c|\xi|t & -ic^{-1} \sin c|\xi|t & 0 \\ -ic \sin c|\xi|t & \cos c|\xi|t & \\ \hline 0 & & I \end{array} \right].$$

Evidently

$$(6.13) \quad R(\xi) = Q(\xi) \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix},$$

whose norm is bounded above and below away from zero, uniformly in ξ (this is also a consequence of the symmetrizability; see Theorem 2.1a, for example). Combining (6.8), (6.10), and (6.13), we then find that

$$R^{-1}BR = \left[\begin{array}{cc|c} \frac{1}{2}\nu|\xi|^2 & -\frac{1}{2}\nu|\xi|^2 & 0 \\ -\frac{1}{2}\nu|\xi|^2 & \frac{1}{2}\nu|\xi|^2 & \\ \hline 0 & & \varepsilon|\xi|^2 I \end{array} \right],$$

where $\nu = \varepsilon + \eta$. A straightforward computation based on (6.9) then shows finally that

$$(6.14) \quad \tilde{B}(\xi) = \begin{bmatrix} \frac{1}{2}(\varepsilon + \eta)|\xi|^2 & 0 \\ 0 & \varepsilon|\xi|^2 I + \frac{1}{2}(\eta - \varepsilon)\xi\xi^t \end{bmatrix}.$$

The homogeneity (6.3) (which we have already checked) and the strict parabolicity (6.5) are now obvious. The approximation property (6.6) will be verified below. Observe first, however, that replacing $B(\xi)$ by $\tilde{B}(\xi)$ corresponds to replacing the linearized Navier-Stokes system (1.4) by the artificial viscosity system

$$(6.15) \quad \begin{cases} \rho_t + \operatorname{div} m = \frac{1}{2}(\varepsilon + \eta)\Delta\rho \\ m_t^j + c^2\rho_{x_j} = \varepsilon\Delta m^j + \frac{1}{2}(\eta - \varepsilon)\operatorname{div} m_{x_j}. \end{cases}$$

That is, (6.15) is the linear system whose Green's matrix $\tilde{G}$ is given by

$$(6.16) \quad \hat{\tilde{G}}(\xi, t) = \exp \left[-(i\xi_j A_j + \tilde{B}(\xi))t \right].$$

Once (6.6) has been established for $\tilde{G}$, then, we will have proved that (6.15) is the canonical artificial viscosity system associated to the Navier-Stokes equations (1.1): (6.15) is strictly parabolic, the symbols of its first-order convection terms and second-order diffusion terms commute, and solutions of the original system (1.1) are asymptotic to those of (6.15) in the sense of (6.6). Actually, a much stronger asymptotic approximation result holds; we make this part of the following theorem, which summarizes these results:

Theorem 6.2. *The unique effective artificial viscosity for the linearized Navier-Stokes system (1.4) is the mapping $\xi \mapsto \tilde{B}(\xi)$ given by (6.14). Moreover, if G and $\tilde{G}$ are the respective Green's matrices for (1.4) and (6.15), given by (3.1) and (6.16), then for each nonnegative integer k there are positive constants C and b such that, for $p \in [2, \infty]$, $u_0 \in L^1 \cap W^{k,p}$, $t \geq 1$, and $|\alpha| \leq k$,*

$$(6.17) \quad \begin{aligned} & \|D_x^\alpha [G(\cdot, t) - \tilde{G}(\cdot, t)] * u_0\|_p \\ & \leq t^{n/2(1/p-1)-|\alpha|/2} C \left[t^{-1/2} \|u_0\|_1 + e^{-bt} \|u_0\|_{W^{k,p}} \right]. \end{aligned}$$

Proof. We define a smooth cut-off function $\chi = \chi(|\xi|)$ such that $\chi = 1$ for $|\xi| \leq R$ and $\chi = 0$ for $|\xi| \geq R+1$, where R is as in Theorem 3.2b. We then take $\hat{G}_1 = \chi \hat{G}$ and $G_2 = G - G_1$, as in Section 5, so that by Lemma 5.2b,

$$\begin{aligned} & \|D_x^\alpha [G(\cdot, t) - \tilde{G}(\cdot, t)] * u_0\|_p \\ & \leq \|G_2(\cdot, t) * D_x^\alpha u_0\|_p + C \| |\xi|^{|\alpha|} [\hat{G}_1(\cdot, t) - \hat{\tilde{G}}(\cdot, t)] \hat{u}_0 \|_q \\ & \leq C e^{-bt} \|u_0\|_{W^{k,p}} + C \|\hat{u}_0\|_\infty \| |\xi|^{|\alpha|} [\chi \hat{G}(\cdot, t) - \hat{\tilde{G}}(\cdot, t)] \|_q, \end{aligned}$$

where q is the Hölder conjugate of p . However, from Theorem 3.2a and (2.8),

$$|\widehat{G}(\xi, t)| \leq Ce^{-\vartheta|\xi|^2 t}, \quad |\xi| \leq R$$

and

$$|\hat{\tilde{G}}(\xi, t)| \leq Ce^{-\vartheta|\xi|^2 t}, \quad \xi \in \mathbb{R}^n,$$

for a positive constant ϑ . Thus, for $r > 0$,

$$\left[\int_{|\xi| \geq r} |\xi|^{|\alpha|q} \left(|\chi \widehat{G}|^q + |\hat{\tilde{G}}|^q \right) d\xi \right]^{1/q} \leq C(k)e^{-bt},$$

and

$$(6.18) \quad \begin{aligned} \|D_x^\alpha [G(\cdot, t) - \tilde{G}(\cdot, t)] * u_0\|_p &\leq Ce^{-bt} (\|u_0\|_{W^{k,p}} + \|u_0\|_1) + C\|u_0\|_1 \\ &\times \left[\int_{|\xi| \leq r} |\xi|^{|\alpha|q} |\widehat{G}(\xi, t) - \hat{\tilde{G}}(\xi, t)|^q d\xi \right]^{1/q}. \end{aligned}$$

It is therefore the low frequencies $|\xi| \leq r$ that are important here, and for these it is most convenient to work in the coordinates given by the matrix $Q(\xi)$ in (6.10). We therefore compute from (6.10), (6.12), and (6.14) that

$$\begin{aligned} Q^t \hat{\tilde{G}} Q &= (Q^t e^{-i\xi_j A_j t} Q) (Q^t e^{-\tilde{B}(\xi)t} Q) \\ &= \begin{bmatrix} \tilde{S} & 0 \\ 0 & e^{-\varepsilon|\xi|^2 t} I \end{bmatrix} \end{aligned}$$

where

$$(6.19) \quad \tilde{S} = e^{-\nu|\xi|^2 t/2} \begin{bmatrix} \cos c|\xi|t & -ic^{-1} \sin c|\xi|t \\ -ic \sin c|\xi|t & \cos c|\xi|t \end{bmatrix},$$

and again $\nu = \varepsilon + \eta$. On the other hand, if we write

$$\widehat{G} = \left[\begin{array}{c|c} g_{11} & -ig_{12}\xi^t \\ \hline -ic^2 g_{12}\xi & e^{-\varepsilon|\xi|^2 t} I + (g_{22} - e^{-\varepsilon|\xi|^2 t}) \frac{\xi \xi^t}{|\xi|^2} \end{array} \right],$$

as in Lemma 3.1, we find that

$$Q^t \widehat{G} Q = \begin{bmatrix} S & 0 \\ 0 & e^{-\varepsilon|\xi|^2 t} I \end{bmatrix},$$

where

$$S = \begin{bmatrix} g_{11} & -ig_{12}|\xi| \\ -ic^2g_{12}|\xi| & g_{22} \end{bmatrix}.$$

We shall show that, for r small, there are positive constants C and b such that

$$(6.20) \quad |S(\xi, t) - \tilde{S}(\xi, t)| \leq C|\xi|e^{-\vartheta|\xi|^2 t}, \quad 0 \leq |\xi| \leq r.$$

Since Q is orthogonal, it will then follow that

$$(6.21) \quad \left[\int_{|\xi| \leq r} |\xi|^{|\alpha|q} |\widehat{G}(\xi, t) - \hat{\tilde{G}}(\xi, t)|^q d\xi \right]^{1/q} \leq Ct^{n/2(1/p-1)-|\alpha|/2-1/2},$$

which, together with (6.18), establishes the desired decay result (6.17).

(6.20) holds in a much more general setting, and can be derived by matrix perturbation techniques, as in Kawashima [8]. We shall check (6.20) directly, and for the upper left component only; the verification for the others is similar. First, we have from the definitions of $\lambda_{\pm}(\xi)$ in Lemma 3.1 that, for small $|\xi|$,

$$\lambda_{\pm}(\xi) = -\frac{1}{2}\nu|\xi|^2 \pm ic|\xi|(1+E),$$

where E is real and $|E| \leq C|\xi|^2$. The upper left entry of S is then

$$\begin{aligned} g_{11} &= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \\ &= e^{-\nu|\xi|^2 t/2} \cos[c|\xi|t(1+E)] + O(|\xi|e^{-\nu|\xi|^2 t/2}) \\ &= e^{-\nu|\xi|^2 t/2} [\cos c|\xi|t + O(|\xi|^3 t)] + O(|\xi|e^{-\nu|\xi|^2 t/2}) \\ &= e^{-\nu|\xi|^2 t/2} \cos c|\xi|t + O(|\xi|e^{-\nu|\xi|^2 t/4}). \end{aligned}$$

Comparison with (6.19) then shows that $S^{11} - \tilde{S}^{11}$ is bounded as required in (6.20). $\square$

Observe that the effective artificial viscosity system (6.15) satisfies the hypotheses of Theorem 2.3, so that, by the remarks made at the end of Section 2, the decay rate $\|D_x^\alpha \tilde{G}(\cdot, t) * u_0\|_2 \sim t^{-(n/4)-|\alpha|/2}$ may indeed be attained. The same is therefore true also for solutions of the nonlinear Navier-Stokes system, for if $u(\cdot, t)$ is such a solution with initial data u_0 , then

$$(6.22) \quad u(\cdot, t) = \tilde{G}(\cdot, t) * u_0 + [G(\cdot, t) - \tilde{G}(\cdot, t)] * u_0 + [u(\cdot, t) - G(\cdot, t) * u_0].$$

Theorems 6.2 and 5.3 show that the α -derivatives of the rightmost two terms here decay more rapidly than $t^{-n/4-|\alpha|/2}$. Consequently, if u_0 is chosen so that $\|D_x^\alpha \tilde{G}(\cdot, t) * u_0\|_2 \geq C^{-1} t^{-n/4-|\alpha|/2}$, then the same lower bound will hold for the corresponding solution $u(\cdot, t)$.

We now assemble the conclusions of Theorems 2.3, 5.3, and 6.2 to give an asymptotic approximation result for solutions $u(\cdot, t)$ of the nonlinear Navier-Stokes system (4.2): for large time, $u(\cdot, t)$ is well-approximated by the convolution $\tilde{G}(\cdot, t) * u_0$, where $\tilde{G}$ is the Green's matrix for the linear artificial viscosity system (6.15), as well as by the solution $\tilde{u}(\cdot, t)$ of the nonlinear artificial viscosity system (1.3), which consists of the nonlinear Euler equations together with the artificial viscosity of (6.15).

Theorem 6.3. *Let $s = [n/2] + 1$, ℓ positive integer, and*

$$E = \max\{\|u_0\|_{H^{s+\ell}}, \|u_0\|_{L^1}\}.$$

Let u be the solution of the nonlinear Navier-Stokes system (1.1), $\tilde{G}$ the Green's matrix (6.16) for the linear artificial viscosity system (6.15), and $\tilde{u}$ the solution of the nonlinear artificial viscosity system (1.3). Then if E is sufficiently small,

$$(6.23) \quad \|D_x^\alpha [u(\cdot, t) - \tilde{G}(\cdot, t) * u_0]\|_p, \|D_x^\alpha [u(\cdot, t) - \tilde{u}(\cdot, t)]\|_p \\ \leq C(|\alpha|)EL(t)(1+t)^{n/2(1/p-1)-|\alpha|/2-1/2} \\ + E^2L(t) \begin{cases} C(|\alpha|)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-3-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-2, 2 \leq p \leq \infty, \ell \geq 3 \\ C(|\alpha|, p)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-1-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-1, 2 \leq p < \infty. \end{cases}$$

Here $L(t) = \log(1+t)$ when $n = 2$, and $L(t) = 1$ for $n \geq 3$.

Proof. The estimate (6.23) for $u - \tilde{G} * u_0$ follows directly from the bound (5.29) for $u - G * u_0$ and the bound (6.17) for $G * u_0 - \tilde{G} * u_0$. To prove the estimates for $u - \tilde{u}$, we recall the result (2.28) of Theorem 2.3, which applies to the system (1.3) to give a similar decay estimate for $\tilde{G} * u_0 - \tilde{u}$. Combining this estimate with the bound above for $u - \tilde{G} * u_0$, we then obtain the required bound (6.23) for $u - \tilde{u}$. $\square$

We are now in a position to refine our description of the asymptotic behavior of the solution $\begin{bmatrix} \rho \\ m \end{bmatrix}$ of the nonlinear Navier-Stokes system (1.1) in L^p for $p \in$

$[2, \infty]$. The final results, stated in Theorems 6.7 and 6.8 below, are obtained from a detailed analysis of the Green's matrix $\tilde{G}$, together with the decay estimate (6.23) for the difference $u - \tilde{G} * u_0$.

To analyze $\tilde{G}$, we first observe that, by virtue of the commutativity (6.4), we may write

$$\hat{\tilde{G}}(\xi, t) = \exp(-i\xi_j A_j t) \exp\left(-\tilde{B}(\xi)t\right),$$

where $i\xi_j A_j$ and $\tilde{B}$ are given in (6.7) and (6.14). As in Section 2, we define $\hat{W} = (2\pi)^{-n/2} \exp(-i\xi_j A_j t)$ and $\hat{G}^B = \exp(-\tilde{B}(\xi)t)$. W and G^B are then the respective Green's matrices for the hyperbolic system (1.11) and the strictly parabolic system (1.10). Elementary computations based on (6.7) and (6.14) then enable us to identify $\tilde{G} = G^B * W$ explicitly:

Proposition 6.4. *The Green's matrix $\tilde{G}$ for the artificial viscosity system (6.15) is given by*

$$(6.24) \quad \tilde{G} = \begin{bmatrix} w_t * K_{(\varepsilon+\eta)/2} & -K_\varepsilon * \nabla w^t + (K_\varepsilon - K_{(\varepsilon+\eta)/2}) * R * \nabla w^t \\ -c^2 K_{(\varepsilon+\eta)/2} * \nabla w & K_\varepsilon * (\delta I - R) + K_{(\varepsilon+\eta)/2} * w_t * R \end{bmatrix},$$

where, for $\vartheta > 0$, K_ϑ is the heat kernel

$$K_\vartheta(x, t) = (4\pi\vartheta t)^{-n/2} \exp\left(-\frac{|x|^2}{4\vartheta t}\right);$$

$w : [0, \infty) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is the fundamental solution for the wave operator:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \\ w(0) = 0 \\ w_t(0) = \delta \end{cases};$$

and R is the matrix of tempered distributions defined by $\widehat{R^{jk}}(\xi) = (2\pi)^{-n/2} \cdot (\xi_j \xi_k / |\xi|^2)$.

Thus each entry of $\tilde{G}$ involves the convolution of a heat kernel with either w_t or ∇w , with the exception of the term $K_\varepsilon * (\delta I - R)$ in the 2-2 entry. This term deserves special attention because, as we shall see, it is the term which dominates for large time. The following result, which is well known, explains the physical meaning of the convolution operator $(\delta I - R)*$:

Proposition 6.5. *Let $m_0 \in [(L^2 \cap L^1)(\mathbb{R}^n)]^n$ be a given vector field. Then there exists a unique vector field $m_{0f} \in (L^2)^n$ and a unique scalar function $g \in L^2 + L^\infty$, such that $m_0 = m_{0f} + \nabla g$ and $\operatorname{div} m_{0f} = 0$. In fact, $m_{0f} = (\delta I - R) * m_0$, where R is as in Proposition 6.4.*

Thus $(\delta I - R) *$ is the operator which projects a given vector field onto its divergence-free part. Letting $\tilde{G}^{22}$ be the lower-right entry of the matrix $\tilde{G}$ in (6.24), and shifting by the constant field m^* , we may therefore write

$$\tilde{G}^{22} * (m_0 - m^*) = K_\varepsilon * (m_{0f} - m^*) + K_{(\varepsilon+\eta)/2} * w_t * \nabla g.$$

The first term here can decay no more rapidly than at the rate of the heat kernel K_ε , but the second term will decay faster, owing to the appearance of the factor w_t , which, as we shall see, enhances the overall decay rate in L^p for $p > 2$. These observations therefore suggest that we decompose $\tilde{G}$ as follows:

$$(6.25) \quad \tilde{G} = \tilde{G}_f + \tilde{G}_\perp, \text{ where } \tilde{G}_f = \begin{bmatrix} 0 & 0 \\ 0 & K_\varepsilon * (\delta I - R) \end{bmatrix}.$$

Now, in [6] we derive explicit, pointwise bounds in terms of x and t for the various entries of $\tilde{G}_f$ and $\tilde{G}_\perp$. These bounds are proved in a sequence of rather involved estimates for the convolution integrals appearing in the explicit representation (6.24) for $\tilde{G}$. We shall describe these bounds for a representative case at the end of this section, after establishing in Theorems 6.7 and 6.8 their relevance for solutions of the nonlinear Navier-Stokes system. For our present purposes, it suffices to simply quote the resulting L^p decay rates for $\tilde{G}_f$ and $\tilde{G}_\perp$:

Proposition 6.6. *Let $\tilde{G}$ be the Green's matrix (6.24) for the artificial viscosity system (6.15), and let $\tilde{G}_f$ and $\tilde{G}_\perp$ be as above in (6.25). Then:*

$$(6.26) \quad \|D_x^\alpha \tilde{G}_f(\cdot, t)\|_{L^p} \leq C(|\alpha|) \begin{cases} \log t, & p = 1 \text{ and } \alpha = 0, \\ t^{n/2(1/p-1)-|\alpha|/2}, & |\alpha| \geq 0, p \in [1, \infty], (p, \alpha) \neq (1, 0); \end{cases}$$

$$(6.27) \quad \|D_x^\alpha \tilde{G}_\perp(\cdot, t)\|_{L^p} \leq C(|\alpha|) L_\alpha(t) t^{n/2(1/p-1)-|\alpha|/2} \cdot t^{(n-1)/4(2/p-1)}, \quad |\alpha| \geq 0, p \in [1, \infty],$$

where $L_\alpha(t) = \log(1+t)$ when $n = 2$ and $\alpha = 0$, and is otherwise identically one.

Thus $D_x^\alpha \tilde{G}_f$ decays at the rate $t^{n/2(1/p-1)-|\alpha|/2}$, which is precisely that of a heat kernel. On the other hand, $D_x^\alpha \tilde{G}_\perp$ decays at the heat kernel rate times the factor $t^{(n-1)/4(2/p-1)}$. This latter factor is neutral when $p = 2$, increases the overall decay rate for $p > 2$, but retards the decay for $p < 2$ (in fact, the bound in (6.27) may actually grow, rather than decay, when p is close to 1 and $|\alpha|$ is small).

We now combine the estimates of Proposition 6.6, together with the bound for $u - \tilde{G} * u_0$ in Theorem 6.3, to give a complete asymptotic description of solutions of the nonlinear Navier-Stokes system (1.1). As in Section 1, we let $\begin{bmatrix} \rho \\ m \end{bmatrix}$ be the solution of (1.1) with initial data $\begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix}$ close to the constant state $\begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$, and we take $u = \begin{bmatrix} \rho - \rho^* \\ m - m^* \end{bmatrix}$ and $u_0 = \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix}$. We decompose u as follows:

$$\begin{aligned} (6.28) \quad u &= \begin{bmatrix} \rho - \rho^* \\ m - m^* \end{bmatrix} = \tilde{G}_f * u_0 + \tilde{G}_\perp * u_0 + (u - \tilde{G} * u_0) \\ &= \begin{bmatrix} 0 \\ K_\varepsilon * (\delta I - R) * (m_0 - m^*) \end{bmatrix} + \tilde{G}_\perp * u_0 + (u - \tilde{G} * u_0). \end{aligned}$$

We then have the following result, which gives our most complete description of the asymptotic behavior of u in L^p for $p \in [2, \infty]$:

Theorem 6.7. *Let $s = [n/2] + 1$ and ℓ a positive integer. Assume that initial data $\begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix} = u_0 + \begin{bmatrix} \rho^* \\ m^* \end{bmatrix}$ is given for which*

$$E = \max\{\|u_0\|_{H^{s+\ell}}, \|u_0\|_{L^1}\}$$

is sufficiently small, and let $\begin{bmatrix} \rho \\ m \end{bmatrix} = u + \begin{bmatrix} \rho^ \\ m^* \end{bmatrix}$ be the corresponding solution of the nonlinear Navier-Stokes system (1.1). Then the following decay rates apply to the three summands in the decomposition (6.28) of u :*

$$\begin{aligned} (6.29) \quad &\|D_x^\alpha(\tilde{G}_f * u_0)(\cdot, t)\|_{L^p} \\ &\leq C(|\alpha|)\|u_0\|_{L^1} \begin{cases} \log t, & p = 1 \text{ and } \alpha = 0, \\ t^{n/2(1/p-1)-|\alpha|/2}, & |\alpha| \geq 0, p \in [1, \infty], (p, \alpha) \neq (1, 0); \end{cases} \end{aligned}$$

$$(6.30) \quad \|D_x^\alpha(\tilde{G}_\perp * u_0)(\cdot, t)\|_{L^p} \\ \leq C(|\alpha|)L_\alpha(t)\|u_0\|_{L^1}t^{n/2(1/p-1)-|\alpha|/2}t^{(n-1)/4(2/p-1)}, \quad |\alpha| \geq 0, \quad p \in [1, \infty],$$

where $L_\alpha = \log(1+t)$ when $n = 2$ and $\alpha = 0$, and is otherwise identically 1; and

$$(6.31) \quad \|D_x^\alpha(u - \tilde{G} * u_0)(\cdot, t)\|_{L^p} \\ \leq E^2 L(t) \begin{cases} C(|\alpha|)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-3-|\alpha|)\}-1/2} \\ \quad \text{for } |\alpha| \leq \ell-2, \quad 2 \leq p \leq \infty, \quad \ell \geq 3 \\ C(|\alpha|, p)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-1-|\alpha|)\}-1/2} \\ \quad \text{for } |\alpha| \leq \ell-1, \quad 2 \leq p < \infty \end{cases}$$

where $L(t) = \log(1+t)$ when $n = 2$, and is identically 1 when $n \geq 3$.

Thus, given sufficient initial regularity, the overall decay rate for $D_x^\alpha u$ is $t^{n/2(1/p-1)-|\alpha|/2}$ in L^p for $p \in [2, \infty]$, which is exactly that of a heat kernel. However, for $p > 2$, (6.30) and (6.31) show that the difference

$$u - \tilde{G}_f * u_0 = \begin{bmatrix} \rho - \rho^* \\ m - m^* \end{bmatrix} - \begin{bmatrix} 0 \\ K_\varepsilon * (\delta I - R) * (m_0 - m^*) \end{bmatrix} = \begin{bmatrix} \rho - \rho^* \\ m - K_\varepsilon * m_{0f} \end{bmatrix}$$

decays more rapidly, by a factor of t^{-q} , where

$$q = \min \left\{ \frac{1}{2}, (n-1)(1-2/p)/4 \right\}.$$

It is therefore the first term on the right side of (6.28) which dominates for large time (in L^p , $p \in (2, \infty]$). Stated differently, the solution $\begin{bmatrix} \rho(\cdot, t) \\ m(\cdot, t) \end{bmatrix}$ is asymptotic to the flow $\begin{bmatrix} \rho^* \\ K_\varepsilon * m_{0f} \end{bmatrix}$, which is incompressible with divergence-free velocity field. We have therefore arrived at the result that *all smooth, small amplitude solutions of the nonlinear Navier-Stokes system (1.1) are asymptotically incompressible*.

There is an additional point to be made concerning the decomposition (6.25) of $\tilde{G}$ and the asymptotic description of u above. First, we may decompose the Green's matrix W for the hyperbolic system (1.11) in the same way as in (6.25),

$$W = W_f + W_\perp, \quad \text{where } W_f = W * \begin{bmatrix} 0 & 0 \\ 0 & \delta I - R \end{bmatrix},$$

so that $\tilde{G}_f = G^B * W_f$ and $\tilde{G}_\perp = G^B * W_\perp$. Now observe that the linearized Euler equations (1.11) have nontrivial steady-state solutions, namely, those flows $\begin{bmatrix} \rho - \rho^* \\ m - m^* \end{bmatrix}$ for which $\rho = \rho^*$ and $\operatorname{div} m = 0$. These steady-state solutions are thus nontrivial fixed points of the operator W_f , so that, in general, $W_f * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix}$ cannot decay. We therefore expect that the first term in the expansion (6.28)

$$\tilde{G}_f * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix} = G^B * \left(W_f * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix} \right)$$

will decay at most at the rate of a heat kernel; and this is precisely the result given in (6.29). The second term in the expansion (6.28) may be written

$$\begin{aligned} \tilde{G}_\perp * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix} &= G^B * W_\perp * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix} \\ &= G^B * W * \begin{bmatrix} \rho_0 - \rho^* \\ m_{0f} - m^* \end{bmatrix}. \end{aligned}$$

One readily checks that, because $m_0 - m_{0f}$ is the gradient of a scalar function, both components of $W * \begin{bmatrix} \rho_0 - \rho^* \\ m_{0f} - m^* \end{bmatrix}$ satisfy the standard wave equation. Since solutions of the wave equation decay in L^p for $p > 2$ (see Strauss [19]), we can expect that the entire term $G^B * W_\perp * \begin{bmatrix} \rho_0 - \rho^* \\ m_0 - m^* \end{bmatrix}$ will decay at a rate which is faster than that determined separately by either G^B or by the wave operator; and this is precisely the result (6.30).

Observe that the situation reverses itself entirely when $p < 2$; the upper bound in (6.27) for $\tilde{G}_\perp$ may actually grow, rather than decay. We shall discuss L^p rates of decay for $p < 2$ at length in Sections 7 and 8.

The asymptotic description of the solution can be further refined when the initial data u_0 has a first moment: in this case the corresponding solution u is asymptotically the Green's matrix $\tilde{G}$ itself, multiplied by the total mass vector of u_0 :

Theorem 6.8. *Let s , ℓ , and E be as in Theorem 6.7, with E sufficiently small, and assume that the initial data u_0 satisfies $(1 + |x|)u_0(x) \in L^1$. Define*

$$U = \int_{\mathbb{R}^n} u_0(x) dx,$$

and let u be the solution of the nonlinear Navier-Stokes system (1.1) with initial data u_0 . Then:

$$(6.32) \quad \|D_x^\alpha [u(\cdot, t) - \tilde{G}(\cdot, t)U]\|_p \leq (E + \| |x|u_0(x) \|_{L^1}) \begin{cases} C(|\alpha|)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-3-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-2, \ 2 \leq p \leq \infty, \ \ell \geq 3 \\ C(|\alpha|, p)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, (\ell-1-|\alpha|)\}-1/2} & \text{for } |\alpha| \leq \ell-1, \ 2 \leq p < \infty. \end{cases}$$

Proof. In light of (6.23), we have only to prove (6.32) with u replaced by $\tilde{G} * u_0$. Letting superscripts denote components, we have that

$$D_x^\alpha [(\tilde{G}(\cdot, t) * u_0)^j(x) - (\tilde{G}(x, t)U)^j] = - \int_0^1 \int D_x^\alpha \tilde{G}_{x_\ell}^{jk}(x - sy, t) y_\ell u_0^k(y) dy ds,$$

so that

$$\begin{aligned} & |D_x^\alpha (\tilde{G} * u_0 - \tilde{G}U)(x, t)|^p \\ & \leq \left(\int_0^1 \int |\nabla D_x^\alpha \tilde{G}(x - sy, t)|^p |y| |u_0(y)| dy ds \right) \left(\int |y| |u_0(y)| dy \right)^{p/q}, \end{aligned}$$

where q is the Hölder conjugate of p . Then

$$\begin{aligned} \|D_x^\alpha [\tilde{G} * u_0 - \tilde{G}U]\|_p^p & \leq C \|\nabla D_x^\alpha \tilde{G}(\cdot, t)\|_p^p \left(\int |y| |u_0(y)| dy \right)^{1+p/q} \\ & \leq C \left(t^{n/2(1/p-1)-|\alpha|/2-1/2} \| |x|u_0(x) \|_1 \right)^p \end{aligned}$$

by (2.6) of Theorem 2.1. □

The result above gives the precise sense in which the asymptotic solution is a diffusion wave, and in fact defines the correct notion of diffusion wave in the present context. Recall the decomposition $\tilde{G} = W * G^B$, where G^B and W are the Green's matrices respectively for the strictly parabolic system (1.10) and the hyperbolic system (1.11). The asymptotic state in (6.32) may therefore be written

$$(6.33) \quad \tilde{G}(\cdot, t)U = W(t) * [\tilde{G}^B(\cdot, t)U] = W(t) * \Theta(\cdot, t),$$

where $\Theta(\cdot, t)$ is the solution of the parabolic system (1.10) with initial data δU . In light of the estimates of Theorem 2.1 for the Green's matrix $\tilde{G}^B(\cdot, t)$, we may

regard Θ as a diffusion about the origin, so that $W * \Theta$ is a diffusion convected by the fundamental solution of the linearized Euler equations; that is, $W * \Theta$ is a “diffusion wave.” (6.32) and (6.33) therefore show that, asymptotically, the solution $u(\cdot, t)$ of the nonlinear Navier-Stokes system is a diffusion wave whose total mass U is the total mass of the initial data.

We conclude this section with a brief description and discussion of our pointwise bounds for $\tilde{G}$ in the representative case $n = 3$. The following is a special case of the results of [6]:

Proposition 6.9. *Let $\tilde{G}$ be the Green’s matrix of Proposition 6.6, and let $n = 3$. Then for all x and t ,*

$$(6.34) \quad |\tilde{G}_f(x, t)| \leq C \min\{t^{-3/2}, |x|^{-3}\}$$

and

$$(6.35) \quad |\tilde{G}_\perp(x, t)| \leq Ct^{-2} \exp\left(-\frac{(|x| - ct)^2}{Ct}\right).$$

The bound (6.34) for $\tilde{G}_f = G^B * W_f$ results from a straightforward estimate for the convolution $K_\varepsilon * (\delta I - R)$ appearing in (6.25). There is no hyperbolic effect in this term, since, as we saw above, the operator $W_f *$ has nontrivial fixed points. The bound for $\tilde{G}_\perp = G^B * W_\perp$ is considerably more interesting: the dissipative hyperbolic operator $W_\perp$ interacts with the three-dimensional diffusion $G^B \sim t^{-3/2} e^{-|x|^2/Ct}$ by convecting the bulk of its mass along the light cone $\{x : |x| = ct\}$, and by increasing its pointwise decay rate by a factor of $t^{-1/2}$. The L^p decay rates for $\tilde{G}_f$ and $\tilde{G}_\perp$ given in Proposition 6.6 then result simply by integrating the pointwise bounds (6.34) and (6.35). Complete technical details are given in [6].

7. Pointwise estimates of Green’s matrices. Before studying behavior in L^p , $p \leq 2$, we address the general problem of converting frequency domain estimates on Green’s matrices into pointwise estimates in the spatial domain.

General, symmetrizable, hyperbolic-strictly parabolic systems. We first consider systems

$$(7.1) \quad u_t + A_j u_{x_j} - B_{jk} u_{x_j x_k} = 0$$

for which there exists a symmetric matrix $P > 0$ such that PA_j is symmetric, for all j , and the operator $PB_{jk} u_{x_j x_k}$ is strictly parabolic. By the coordinate

change $v = P^{1/2}u$, we can assume without loss of generality that the A_j are symmetric, and that $B_{jk}u_{x_j, x_k}$ is strictly parabolic, i.e.

$$(7.2) \quad \sum_{j,k} B_{jk} \xi_j \xi_k \geq \vartheta |\xi|^2, \quad \vartheta > 0.$$

The Green's matrix $G(x, t)$ of (7.1) has transform

$$(7.3) \quad \widehat{G}(\xi, t) = e^{(-i\xi_j A_j - B_{jk} \xi_j \xi_k)t}.$$

Clearly, $\widehat{G}$ is an entire function of $\xi \in \mathbb{C}^n$.

Lemma 7.1. *Let C and D be real $m \times m$ matrices. Then,*

$$|e^{(iC+D)t}| \leq e^{\bar{\sigma}(i((C-C^*)/2) + ((D+D^*)/2))t},$$

where $|\cdot|$ is the standard operator norm over $\ell^2(\mathbb{R}^n)$ and $\bar{\sigma}(M)$ denotes the largest eigenvalue of M .

Proof. The vector function $v(t) \stackrel{\text{def}}{=} e^{(iC+D)t}v_0$ satisfies the ODE

$$(7.4) \quad \frac{d}{dt}v = (iC + D)v; \quad v(0) = v_0.$$

From the energy estimate

$$(7.5) \quad \begin{aligned} \frac{d}{dt}|v|^2 &= \left\langle v, \frac{d}{dt}v \right\rangle + \left\langle \frac{d}{dt}v, v \right\rangle \\ &= \langle v, [i(C - C^*) + (D + D^*)]v \rangle \\ &\leq \bar{\sigma}(i(C - C^*) + (D + D^*))|v|^2, \end{aligned}$$

we obtain

$$(7.6) \quad |v(t)|^2 \leq e^{\bar{\sigma}(i((C-C^*)/2) + ((D+D^*)/2))t} |v_0|^2,$$

or

$$|e^{(iC+D)t}v_0| \leq e^{\bar{\sigma}(i((C-C^*)/2) + ((D+D^*)/2))t} |v_0|.$$

Since v_0 is arbitrary, the result follows. $\square$

Lemma 7.2. *For any $\xi, \eta \in \mathbb{R}^n$,*

$$(7.7) \quad |\widehat{G}(\xi + i\eta, t)| \leq e^{(-(\vartheta/2)|\xi|^2 + \Lambda(\eta/|\eta|)|\eta| + \mu|\eta|^2)t},$$

where $\widehat{G}$ is as in (7.3), $\vartheta > 0$ is as in (7.2),

$$(7.8) \quad \Lambda\left(\frac{\eta}{|\eta|}\right) = \frac{|A(\eta)|}{|\eta|} \stackrel{\text{def}}{=} \frac{|\sum_j A_j \eta_j|}{|\eta|},$$

and μ is a positive constant.

Proof. Writing

$$(7.9) \quad -i(\xi + i\eta)_j A_j - (\xi + i\eta)_j (\xi + i\eta)_k B_{jk} = iC + D,$$

we have

$$(7.10) \quad C = (-\xi_j A_j - (\xi_j \eta_k + \eta_j \xi_k) B_{jk})$$

and

$$(7.11) \quad D = (\eta_j A_j - \xi_j \xi_k B_{jk} + \eta_j \eta_k B_{jk}).$$

By symmetry of A_j ,

$$(7.12) \quad \bar{\sigma} \left(\frac{C - C^*}{2} \right) \leq K|\eta| |\xi| \leq \frac{K}{\varepsilon} |\eta|^2 + K\varepsilon |\xi|^2,$$

$\varepsilon > 0$ arbitrary. By (7.2), (7.8),

$$(7.13) \quad \bar{\sigma} \left(\frac{D + D^*}{2} \right) \leq \Lambda \left(\frac{\eta}{|\eta|} \right) |\eta| + K|\eta|^2 - \vartheta |\xi|^2.$$

Thus, choosing ε sufficiently small, we have

$$(7.14) \quad \bar{\sigma} \left(i \left(\frac{C - C^*}{2} \right) + \left(\frac{D + D^*}{2} \right) \right) < \left(-\frac{\vartheta}{2} |\xi|^2 + \Lambda \left(\frac{\eta}{|\eta|} \right) |\eta| + \mu |\eta|^2 \right),$$

for $\mu = K(1 + 1/\varepsilon)$. The result now follows from Lemma 7.1. $\square$

Estimate (7.7) gives us pointwise bounds on $G(x, t)$, via the following weak version of the Paley-Wiener Theorem.

Proposition 7.3. *Let $\hat{f}(\xi)$ be an entire function of ξ satisfying*

$$(7.15) \quad |\hat{f}(\xi + i\eta)| \leq C_1(1 + |\xi|k)^{-N} e^{\mu_1(\eta/|\eta|)|\eta| + \mu_2|\eta|^2}$$

for all $\xi, \eta \in \mathbb{R}^n$, for some $N > n$, where μ_1 is a function from S^n to $\mathbb{R}$ and μ_2 is a positive constant. Then:

$$(7.16) \quad |f(x)| \leq \left(\frac{C}{k^n}\right) e^{-[t(|x| - \mu_1(x/|x|))^+]^2/(4\mu_2)},$$

where $(r)^+ \stackrel{\text{def}}{=} (r + |r|)/2$, and C is a constant independent of k, μ_j .

Proof. By the Fourier inversion formula, we have

$$(7.17) \quad f(x) = (2\pi)^{-n/2} \int e^{ix \cdot \xi} \hat{f}(\xi) d\xi.$$

Since $\hat{f}$ is entire and rapidly decaying in ξ , we can move the surface of integration to $\xi' = \xi + i\eta$, where

$$(7.18) \quad \eta = \alpha \frac{x}{\sqrt{\mu_2}}; \quad \alpha = \frac{(1 - \mu_1/|x|)^+}{2\sqrt{\mu_2}},$$

and $\mu_1 = \mu_1(\eta/|\eta|) = \mu_1(x/|x|)$. We obtain

$$(7.19) \quad |f(x)| = \left| \int e^{ix \cdot (\xi + i\eta)} \hat{f}(\xi + i\eta) d\xi \right| \\ \leq e^{(\mu_1|\eta| + \mu_2|\eta|^2 - \eta \cdot x)} \int C_1(1 + |\xi|k)^{-N} d\xi \\ = e^{|\eta|\sqrt{\mu_2} - ((x - (\mu_1\eta)/|\eta|)/(2\sqrt{\mu_2}))|^2} e^{-\left|(x - \mu_1(\eta/|\eta|))/(2\sqrt{\mu_2})\right|^2} \frac{C}{k^n}.$$

For $\alpha = 0$, we have $\eta = 0$, and (7.16) follows from the second line of (7.19). For $\alpha > 0$, we have $\eta/|\eta| = x/|x|$, and the third line of (7.19) becomes

$$(7.20) \quad \frac{C}{k^n} e^{-((|x| - \mu_1)/(2\sqrt{\mu_2}))^2},$$

giving the result. □

Corollary 7.4. *The Green's matrix $G(x, t)$ of a symmetrizable, hyperbolic-strictly parabolic system, (7.1), satisfies*

$$(7.21) \quad |D_x^\alpha G(x, t)| \leq C(|\alpha|) t^{-n/2-|\alpha|/2} e^{-(|x|-\Lambda(x/|x|)t)^{+2}/(4\mu t)},$$

for some $\mu > 0$, where $\Lambda(x/|x|)$ is as in (7.8), i.e., the maximum characteristic speed in the direction $x/|x|$ of the associated hyperbolic system,

$$(7.22) \quad u_t + A_j u_{x_j} = 0.$$

Proof. By Lemma 7.2, the entire function $\widehat{D_x^\alpha G} = (i\xi)^\alpha \widehat{G}$ satisfies the decay estimate

$$(7.23) \quad \begin{aligned} |\widehat{D_x^\alpha G}(\xi + i\eta)| &\leq C_1 |\xi + i\eta|^{|\alpha|} (1 + |\xi| t^{1/2})^{-N} e^{(\Lambda((\eta/|\eta|))|\eta| + \mu|\eta|^2)t} \\ &\leq C t^{-|\alpha|/2} (1 + |\xi| t^{1/2})^{|\alpha| - N} e^{(\Lambda(\eta/|\eta|)|\eta| + C\mu|\eta|^2)t}. \end{aligned}$$

Applying Lemma 7.3, with $k = t^{1/2}$, $\mu_1 = \Lambda t$, and $\mu_2 = C\mu t$, and $N > n + |\alpha|$, we obtain the result. $\square$

According to the Corollary, $G(x, t)$ falls off like a heat kernel outside the cone of propagation, $\{x : |x| \leq \Lambda(x/|x|)t\}$ of system (7.1). This is the most we can say without specific information about the propagation of (7.1), or its interaction with the parabolic operator.

However, in the case of an artificial viscosity system, as described in Section 2, we have the decomposition, due to commutation of $-i\xi_j A_j$ and $\xi_j \xi_k B_{jk}$, given by

$$(7.24) \quad G = W * G^B,$$

where W is the Green's matrix for (7.23) and G^B is the Green's matrix for the associated parabolic system,

$$(7.25) \quad u_t = B_{jk} u_{x_j x_k}.$$

In this case, we can apply Lemma 7.2 and Proposition 7.3 to G^B , to obtain

$$(7.26) \quad |D_x^\alpha G^B(x, t)| \leq C(|\alpha|) t^{-n/2-|\alpha|/2} e^{-|x|^2/(4\mu t)}.$$

Together with (7.24), (7.26) gives a fairly sharp pointwise description of $G(x, t)$. However, note (cf. calculations, Section 5) that precise decay information depends strongly on the details of the hyperbolic solution operator W .

Linearized Navier-Stokes. In the case of nonstrictly parabolic systems with real viscosity, such as the linearized Navier-Stokes equations, we don not have estimate (7.7), and the argument above breaks down. However, we can easily establish a weaker pointwise bound sufficient for our purposes, using the ξ -derivative bounds of Lemma 3.2 and standard, real variable estimates.

Proposition 7.5. *Let $G(x, t)$ be the Green's matrix of the linearized Navier-Stokes equations (1.4), and $G_1(x, t)$ be its smooth part, as defined in (5.9). Then*

$$(7.27) \quad |D_x^\alpha G_1(x, t)| \leq C_N t^{-n/2-|\alpha|/2} \left(\frac{|x|}{t} \right)^{-N} \quad \text{for } t \geq 1,$$

for any N, x . Moreover, as a distribution on $\mathbb{R}^n - \{0\}$, $G(\cdot, t)$ is a function satisfying

$$(7.28) \quad |G(x, t)| \leq C_N \begin{cases} t^{-n/2} \left(\frac{|x|}{t} \right)^{-N} & \text{for } t \geq 1, \\ |x|^{-N} & \text{for } t \leq 1, \end{cases}$$

for any $N > n$.

Proof. By Hausdorff-Young,

$$(7.29) \quad \begin{aligned} |x^\beta G(x, t)| &\leq \|\widehat{x^\beta G(\cdot, t)}\|_1 \\ &= \|D_\xi^\beta \widehat{G}(\cdot, t)\|_1, \end{aligned}$$

while

$$(7.30) \quad \begin{aligned} |x^\beta D_x^\alpha G_1(x, t)| &\leq \|\widehat{x^\beta D_x^\alpha G_1(\cdot, t)}\|_1 \\ &= \|D_\xi^\beta (\xi^\alpha \widehat{G}_1(\xi, t))\|_1. \end{aligned}$$

Thus, it is sufficient to show that, for $|\beta| = N$,

$$(7.31) \quad \|D_\xi^\beta \widehat{G}(\xi, t)\|_1 \leq C_N \begin{cases} t^{N-(n/2)} & t \geq 1 \\ 1 & t \leq 1, \end{cases}$$

when $N > n$, and

$$(7.32) \quad \|D_\xi^\beta (\xi^\alpha \widehat{G}_1(\xi, t))\|_1 \leq C_N t^{N-(n/2)-|\alpha|/2} \quad \text{for } t \geq 1.$$

But, (7.31) - (7.32) follow easily from the ξ -derivative bounds given in Lemma 3.2. $\square$

At the expense of further effort, we can in fact recover a close approximation of (7.21), by interpolating between the real and complex analytic arguments. We present this result for completeness, though it is not needed for our later analysis.

Lemma 7.6. *Let $r^2 = (\xi + i\eta) \cdot (\xi + i\eta) = (|\xi|^2 - |\eta|^2) + i(\xi \cdot \eta)$. Then, there exists $R > 0$ such that:*

(i) *For $|r^2| < R^2$,*

$$(7.33) \quad |D_{\xi+i\eta}^\alpha \widehat{G}(\xi + i\eta, t)| \leq C(1 + |\xi| + |\eta|)^{|\alpha|} (1+t)^{|\alpha|} e^{(-\vartheta|\xi|^2 + c|\eta| + \mu|\eta|^2)t},$$

and

(ii) *For $|r^2| \geq R^2$,*

$$(7.34) \quad |D_{\xi+i\eta}^\alpha \widehat{G}(\xi + i\eta, t)| \leq C \left(\frac{|\xi| + |\eta|}{|r|} \right)^{|\alpha|} (e^{-at/2} |r|^{-|\alpha|-1} + e^{-\vartheta r^2 t} |r|^{-|\alpha|}),$$

where $a, \vartheta > 0$.

Proof. Note that $r^2 = |\xi|^2$ in the case $|\eta| = 0$, in agreement with the notation of Section 3. This has the consequence that expressions (3.1)–(3.2), and (3.28)–(3.29) for $\widehat{G}(\xi, t)$ remain valid for $\widehat{G}(\xi + i\eta, t)$ if we everywhere replace ξ by $\xi + i\eta$ and $|\xi|^2$ by r^2 . For, it is easily checked that these expressions are analytic functions of r^2 : (3.28)–(3.29) by construction, and (3.1) by the observation that terms such as $(e^{\lambda_+ t} - e^{\lambda_- t})/(\lambda_+ - \lambda_-)$ are invariant under permutation of λ_- and λ_+ and bounded at the branch point $\lambda_- = \lambda_+$, hence analytic despite the branch singularity in $\lambda_\pm(r^2)$. As entire expressions agreeing with $\widehat{G}$ on $|\eta| = 0$, they must be valid also for arbitrary η . The same result can equally well be obtained by rewording the derivations of (3.1), (3.28), (3.29) in terms of r^2 . With this observation, it is easy to extend the bounds of Theorem 3.2 to the complex plane. For example, the series expansion (3.12) at $|\xi| = \infty$ is valid, more generally, around $|r^2| = \infty$, by the same derivation; in one space dimension, this reduces to the fact that $\widehat{G}(\xi + i\eta, t)$ is holomorphic at $|\xi + i\eta| = \infty$ (cf. [16]). Likewise, (3.15) carries over in modified form, with the observation that, for $|r^2| \leq R^2$,

$$\operatorname{Re} \lambda_\mp(r^2) \leq -\vartheta|\xi|^2 + \mu|\eta|^2 + c|\eta|,$$

which follows by direct calculation from (3.2). Carrying through the analysis of Theorem 3.2, line by line, with r^2 substituted for $|\xi|^2$, and noting that $\partial(r^2)/\partial(\xi + i\eta) = 2(\xi + i\eta)$, we obtain the claimed bounds. $\square$

Proposition 7.7. *For $|x| > 0$,*

$$|D_x^\beta G(x, t)| \leq C(1 + t^{-2n-2|\beta|-1}) t^{-n/2-|\beta|/2} \left(\frac{|x|}{t} \right)^{-n-|\beta|} e^{-(|x|-ct)^2/4\mu t}.$$

Proof. First, consider the case $|\beta| = 0$. Taking $|\alpha| = n$ and combining (7.33)- (7.34), we have

$$\begin{aligned} |x^\alpha \widehat{G}(\xi + i\eta, t)| &= |D_\xi^\alpha \widehat{G}| \\ &\leq C \left(\frac{|\xi| + |\eta|}{R + |r|} \right)^{2n+1} (1 + |\xi|)^{-1-n} e^{-at/2} + Ct^n e^{(-\vartheta|\xi|^2 + \mu|\eta|^2 + c|\eta|)t}. \end{aligned}$$

With

$$\begin{aligned} \left| \frac{|\xi| + |\eta|}{R + |r|} \right| &\leq C \left(\frac{|r^2| + 2|\eta|^2}{R^2 + |r^2|} \right)^{1/2} \\ &\leq C(1 + |\eta|) \\ &\leq C(1 + t)e^{\mu|\eta|^2 t}, \end{aligned}$$

this gives

$$|x^\alpha \widehat{G}(\xi + i\eta, t)| \leq C(1 + t^{-2n-1})(1 + t)^n (1 + |\xi|t)^{-n-1} e^{\mu|\eta|^2 t + c|\eta|t},$$

for $|\alpha| = n$. Applying Proposition 7.3, we have the result. The case $|\beta| > 0$ follows similarly, using $|\alpha| = n + |\beta|$. $\square$

Corollary 7.8. For $|x| > 0$,

$$\begin{aligned} |D_x^\beta (G - \tilde{G})(x, t)| \\ \leq C(1 + t^{-2n-2|\beta|-1}) t^{-n/2-|\beta|/2-1/2} \left(\frac{|x|}{t} \right)^{-n-|\beta|} e^{-(|x|-ct)^2/(4\mu t)}. \end{aligned}$$

Proof. This follows by the argument of Proposition 7.7, after the observation that, for $|r^2| < R^2$,

$$\begin{aligned} (7.35) \quad |D_{\xi+i\eta}^\alpha (\hat{G} - \hat{\tilde{G}})(\xi + i\eta, t)| \\ \leq C|r|(1 + |\xi| + |\eta|)^{|\alpha|} (1 + t)^{|\alpha|} e^{(-\vartheta|\xi|^2 + c|\eta| + \mu|\eta|^2)t} \end{aligned}$$

The bound (7.35) follows by expanding $\hat{\tilde{G}}$ as in (3.1), by the technique used to expand $\hat{G}$, and comparing directly. $\square$

Remark. Proposition 7.7 and Corollary 7.8 imply that, for any $\sigma > 0$,

$$|D_x^\beta G(x, t)| \leq C(|\beta|) t^{-n/2-|\beta|/2} e^{-(|x|-\Lambda(x/|x|)t)^2/4\mu t^{1+\sigma}},$$

and

$$|D_x^\beta (G - \tilde{G})(x, t)| \leq C(|\beta|) t^{-n/2-|\beta|/2-1/2} e^{-(|x|-\Lambda(x/|x|)t)^2/4\mu t^{1+\sigma}},$$

for $|x| > 0$. In dimension $n = 1$, $\sigma = 0$ can be attained by taking into account cancellation in terms of order $|\xi|^{-1}$, rather than differentiating with respect to ξ . $\square$

8. Decay to diffusion waves in L^p , $p < 2$. Using the pointwise bounds derived in Section 7, we can now establish the decay and asymptotic behavior of solutions of (1.1) in L^p for $p < 2$.

Decompose the Green's matrix $G(x, t)$ of the linearized Navier-Stokes equations, (1.4), as

$$(8.1) \quad G = \tilde{G} + (G_1 - \tilde{G}_1) + (G_2 - \tilde{G}_2),$$

where $\tilde{G}(x, t)$ is the Green's function of the associated, effective artificial viscosity system, (1.7), and $\hat{G}_j, \tilde{G}_j$ are as in (5.9), with R sufficiently large. By Lemma 5.2 applied to G and Proposition 4.3 applied to $\tilde{G}$, the term $(G_2 - \tilde{G}_2)$ is negligible, satisfying

$$(8.2) \quad (G_2 - \tilde{G}_2) = e^{-bt} \widehat{M(\xi, t)}$$

and

$$(8.3) \quad D_{x_j}(G_2 - \tilde{G}_2) \cdot \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = e^{-bt} \left(1 + t^{-1/2}\right) \widehat{\mathcal{M}(\xi, t)},$$

where $b > 0$ and M and $\mathcal{M}$ are strong and ordinary L^p multipliers, respectively.

Lemma 8.1. *For all $p \leq 2$, $\sigma > 0$,*

$$(8.4) \quad \|D_x^\alpha(G_1 - \tilde{G}_1)\|_p \leq C(|\alpha|, \sigma)(1+t)^{n/2(1/p-1)-|\alpha|/2+(n/4)(2/p-1)-1/2+\sigma}.$$

Proof. For $t \leq 1$, the result follows from (5.10) in Lemma 5.2. Thus, we may assume $t \geq 1$. From (6.21), we have

$$(8.5) \quad \|D_x^\alpha(G_1 - \tilde{G}_1)\|_2 \leq Ct^{-n/4-|\alpha|/2-1/2},$$

while from (7.27) and (7.21) we have

$$(8.6) \quad |D_x^\alpha(G_1 - \tilde{G}_1)(x, t)| \leq C_N t^{-n/2-|\alpha|/2} \left(\frac{|x|}{t}\right)^{-N}, \quad \text{for } t \geq 1,$$

for any $N > n$.

Fixing t , let

$$(8.7) \quad \Omega^j \stackrel{\text{def}}{=} \left\{ x : j \leq \frac{|x|}{t} \leq j+1 \right\},$$

and denote $\|f\|_{L^p(\Omega^j)}$ by $\|f\|_{p,\Omega^j}$. By Hölder's inequality, we have

$$(8.8) \quad \|f\|_{p,\Omega^j} \leq \|f\|_{p',\Omega^j} |\Omega^j|^{1/p-1/p'}, \text{ for } p \leq p'.$$

A second application of Hölder's inequality gives

$$(8.9) \quad \begin{aligned} & \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{2+\sigma,\Omega^j} \\ & \leq \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{2,\Omega^j}^{2/(2+\sigma)} \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{\infty,\Omega^j}^{\sigma/(2+\sigma)} \\ & \leq \left(C t^{-n/4-|\alpha|/2-1/2} \right)^{2/(2+\sigma)} \left(C_N t^{-n/2-|\alpha|/2} j^{-N} \right)^{\sigma/(2+\sigma)}, \end{aligned}$$

for any $\sigma > 0$, $j \geq 1$. Applying (8.8) and (8.6), we have, for $j \geq 1$, $p \leq 2$, and σ small (depending on n),

$$(8.10) \quad \begin{aligned} & \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{p,\Omega^j} \\ & \leq \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{2+\sigma,\Omega^j} |\Omega^j|^{1/p-1/(2+\sigma)} \\ & \leq C \left(t^{-n/4-|\alpha|/2-1/2+\sigma_2} \right) j^{-(N\sigma/(2+\sigma))+1} ((jt)^{n-1}t)^{1/p-1/2} \\ & \leq C \left(j^{(n-1)/2-(N\sigma/(2+\sigma))+1} \right) (t^{\sigma_2}) \left(t^{-n/2(1-1/p)-|\alpha|/2+(n/4)(2/p-1)-1/2} \right), \end{aligned}$$

where $\sigma_2(\sigma) > 0$ goes to zero as $\sigma \rightarrow 0$, while

$$(8.11) \quad \begin{aligned} & \|D_x^\alpha(G_1 - \tilde{G}_1)\|_{p,\Omega^0} \leq \|D_x^\alpha(G_1 - \tilde{G}_1)\|_2 |\Omega^0|^{1/p-1/2} \\ & \leq C t^{-n/2(1-1/p)-|\alpha|/2+n/4(2/p-1)-1/2}. \end{aligned}$$

Choosing N so large that $(n-1)/2 - N\sigma/(2+\sigma) + 1 < -1$, and summing over j , we have the result. $\square$

Theorem 8.2. *Let $u_0 \in H^{s+\ell} \cap L^1(\mathbb{R}^n)$, for $s \geq [n/2] + 1$ and ℓ a positive integer. For*

$$E = \max \left\{ \|u_0\|_{H^{s+\ell}}, \|u_0\|_{W^{\ell-1,1}} \right\}$$

sufficiently small, the solution u of (4.1) satisfies

$$(8.12) \quad \|D_x^\alpha u\|_p \leq E \begin{cases} C(|\alpha|)L_\alpha(t)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,\ell-2-|\alpha|\}+((n-1)/4)(2/p-1)} & \text{for } |\alpha| \leq \ell-2, 1 \leq p \leq 2, \ell \geq 3, \\ C(|\alpha|,p)L_\alpha(t)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,\ell-1-|\alpha|\}+((n-1)/4)(2/p-1)} & \text{for } |\alpha| \leq \ell-1, 1 < p \leq 2, \end{cases}$$

and, for any $\sigma > 0$,

$$(8.13) \quad \|D_x^\alpha(u - \tilde{u})\|_p, \|D_x^\alpha(u - G * u_0)\|_p, \|D_x^\alpha(u - \tilde{G} * u_0)\|_p \leq E \begin{cases} C(|\alpha|, \sigma)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,\ell-3-|\alpha|\}+(n/4)(2/p-1)-1/2+\sigma} & \text{for } |\alpha| \leq \ell-2, 1 \leq p \leq 2, \ell \geq 3, \\ C(|\alpha|, p, \sigma)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,(\ell-1-|\alpha|)\}+(n/4)(2/p-1)-1/2+\sigma} & \text{for } |\alpha| \leq \ell-1, 1 < p \leq 2, \end{cases}$$

where $\tilde{u}$ is the solution of the associated nonlinear artificial viscosity system (1.5). Here, $L_\alpha(t) = \log(1+t)$ when $n = 2$ and $\alpha = 0$ and is otherwise one, and $L(t) = \log(1+t)$ when $n = 2$ and is otherwise one. If, further, $(1+|x|)u_0(x) \in L^1$, then

$$(8.14) \quad \|D_x^\alpha(u - \tilde{G}U)\|_p \leq (E + \|xu_0\|_1) \begin{cases} C(|\alpha|, \sigma)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,\ell-3-|\alpha|\}+(n/4)(2/p-1)-1/2+\sigma} & \text{for } |\alpha| \leq \ell-2, 1 \leq p \leq 2, \ell \geq 3, \\ C(|\alpha|, p, \sigma)(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|,\ell-1-|\alpha|\}+(n/4)(2/p-1)-1/2+\sigma} & \text{for } |\alpha| \leq \ell-1, 1 < p \leq 2, \end{cases}$$

where $U = \int_{\mathbb{R}^n} u_0 dx$.

Proof. Let $|\alpha| \leq \ell-2$, $\ell \geq 3$. By Proposition 6.6, we have

$$(8.15) \quad \begin{aligned} & \|D_x^\alpha \tilde{G} * u_0\|_p \\ & \leq \min\{\|D_x^\alpha \tilde{G}\|_p \|u_0\|_1, \|\tilde{G}\|_1 \|D_x^\alpha u_0\|_p\} \\ & \leq EC(|\alpha|)L_\alpha(t)(1+t)^{n/2(1/p-1)-|\alpha|/2+((n-1)/4)(2/p-1)}. \end{aligned}$$

Likewise, (8.4) and (8.2) give, respectively,

$$\begin{aligned}
 (8.16) \quad & \|D_x^\alpha(G_1 - \tilde{G}_1) * u_0\|_p \\
 & \leq \|D_x^\alpha(G_1 - \tilde{G}_1)\|_p \|u_0\|_1 \\
 & \leq C(|\alpha|, \sigma) E(1+t)^{n/2(1/p-1)-|\alpha|/2+(n/4)(2/p-1)-1/2+\sigma},
 \end{aligned}$$

and

$$\begin{aligned}
 (8.17) \quad & \|D_x^\alpha(G_2 - \tilde{G}_2) * u_0\|_p \leq C e^{-bt} \|D_x^\alpha u_0\|_p \\
 & \leq C E e^{-bt}.
 \end{aligned}$$

Following (5.7), write

$$\begin{aligned}
 (8.18) \quad & D_x^\alpha u(t) = G(t) * D_x^\alpha u_0 + \int_0^t \sum_k D_{x_k} G(t-s) * D_x^\alpha Q^k(s) ds \\
 & = I + II.
 \end{aligned}$$

We can immediately bound the linear term I by

$$\begin{aligned}
 (8.19) \quad & \|I\|_p \leq \|D_x^\alpha \tilde{G} * u_0\|_p + \|D_x^\alpha(G_1 - \tilde{G}_1) * u_0\|_p + \|D_x^\alpha(G_2 - \tilde{G}_2) * u_0\|_p \\
 & \leq E C L_\alpha(t) (1+t)^{n/2(1/p-1)-|\alpha|/2+((n-1)/4)(2/p-1)},
 \end{aligned}$$

using (8.15)–(8.17) and the fact that $\frac{1}{4}(2/p-1) > -\frac{1}{2}$.

To bound the nonlinear term II , we observe that the source term Q , being quadratic, can be estimated using the decay rates for L^p , $p \geq 2$, already known from Theorem 5.3. For example, as in (5.36),

$$\begin{aligned}
 (8.20) \quad & \int_0^t \|Q\|_1 ds \leq C \int_0^t \| |u|^2 + |u| |D_x u| \|_1 ds \\
 & \leq C \int_0^t (\|u\|_2^2 + \|D_x u\|_2^2) ds \\
 & \leq C(|\alpha|) E^2 L(t).
 \end{aligned}$$

Likewise, for Q_1 and Q_2 as defined in (5.3) we have, similarly to (5.38) and (5.39),

$$\begin{aligned}
 (8.21) \quad & \left. \begin{aligned} & \|D_x^\alpha Q_1\|_1 \\ & \|D_x^\beta Q_2\|_1, |\beta| = |\alpha| - 1 \end{aligned} \right\} \\
 & \leq C \sup_{\sum |\gamma_j| = |\alpha|} \|D_x^{\gamma_1} u\|_2 \|D_x^{\gamma_2} u\|_2 \prod_{j \geq 3} \|D_x^{\gamma_j} u\|_\infty \\
 & \leq C(|\alpha|) E^2 (1+t)^{-n/2-1/2 \min\{|\alpha|, \ell-2-|\alpha|\}}
 \end{aligned}$$

and

$$\begin{aligned}
(8.22) \quad \|D_x D_x^\alpha Q_j\|_p &\leq C \sup_{\sum |\gamma_j| = |\alpha| + j} \|D_x^{\gamma_1} u\|_{2p/(2-p)} \|D_x^{\gamma_2} u\|_2 \prod_{j \geq 3} \|D_x^{\gamma_j} u\|_\infty \\
&\leq C(|\alpha|) E^2 (1+t)^{n/2(1/p-1)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}-1/2}, \\
&\quad |\alpha| \leq \ell - 2.
\end{aligned}$$

Thus, similarly to (5.41), (5.42), we have, for $|\alpha| > 0$, $|\beta| = |\alpha| - 1$,

$$\begin{aligned}
(8.23) \quad &\|II\|_p \\
&\leq \int_0^{t/2} \|D_x^\alpha D_x \tilde{G} * Q\|_p + \int_{t/2}^t \|D_x \tilde{G} * D_x^\alpha Q_1\|_p + \int_{t/2}^t \|D_x^2 \tilde{G} * D_x^\beta Q_2\|_p \\
&\quad + \int_0^{t/2} \|D_x^\alpha D_x (G_1 - \tilde{G}_1) * Q\|_p + \int_{t/2}^t \|D_x (G_1 - \tilde{G}_1) * D_x^\alpha Q_1\|_p \\
&\quad + \int_{t/2}^t \|D_x^2 (G_1 - \tilde{G}_1) * D_x^\beta Q_2\|_p + \int_0^t \|(G_2 - \tilde{G}_2) * D_x D_x^\alpha Q\|_p \\
&\leq \int_0^{t/2} \|D_x^\alpha D_x \tilde{G}\|_p \|Q\|_1 + \int_{t/2}^t \|D_x \tilde{G}\|_p \|D_x^\alpha Q_1\|_1 + \int_{t/2}^t \|D_x^2 \tilde{G}\|_p \|D_x^\beta Q_2\|_1 \\
&\quad + \int_0^{t/2} \|D_x^\alpha D_x (G_1 - \tilde{G}_1)\|_p \|Q\|_1 + \int_{t/2}^t \|D_x (G_1 - \tilde{G}_1)\|_p \|D_x^\alpha Q_1\|_1 \\
&\quad + \int_{t/2}^t \|D_x^2 (G_1 - \tilde{G}_1)\|_p \|D_x^\beta Q_2\|_1 + \int_0^t e^{-b(t-s)} \|D_x D_x^\alpha Q\|_p \\
&\leq C(|\alpha|) (1+t)^{n/2(1/p-1)-|\alpha|/2+((n-1)/4)(2/p-1)-1/2} \int_0^{t/2} \|Q(s)\|_1 ds \\
&\quad + C(|\alpha|) E^2 (1+t)^{-n/2-1/2 \min\{|\alpha|, \ell-2-|\alpha|\}} \int_{t/2}^t (1+(t-s))^{n/2(1/p-1)-1/2} ds \\
&\quad + C(|\alpha|) E^2 \int_0^t e^{-b(t-s)} (1+s)^{n/2(1/p-1)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}-1/2} ds, \\
&\leq C(|\alpha|) E^2 L(t) (1+t)^{n/2(1/p-1)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}+((n-1)/4)(2/p-1)-1/2}.
\end{aligned}$$

In the case $|\alpha| = 0$, we can replace $\int_{t/2}^t \|D_x^2 \tilde{G} * D_x^\beta Q_2\|_p$ with $\int_{t/2}^t \|D_x \tilde{G} * D_x^\alpha Q_2\|_p$ in the first line of (8.23) and obtain the same result.

From (8.18), (8.19) and (8.23), we obtain (8.12) and

$$\begin{aligned}
(8.24) \quad &\|D_x^\alpha (u - G * u_0)\|_p \\
&\leq C(|\alpha|) E^2 L(t) (1+t)^{n/2(1/p-1)-1/2 \min\{|\alpha|, \ell-3-|\alpha|\}+((n-1)/4)(2/p-1)-1/2}.
\end{aligned}$$

Equations (8.16) and (8.17) give

$$(8.25) \quad \begin{aligned} & \|D_x^\alpha(G - \tilde{G}) * u_0\|_p \\ & \leq C(|\alpha|, \sigma) E(1+t)^{n/2(1/p-1)-|\alpha|/2+(n/4)(2/p-1)-1/2+\sigma}. \end{aligned}$$

Triangulating with (8.24), we find that

$$(8.26) \quad \begin{aligned} & \|D_x^\alpha(u - \tilde{G} * u_0)\|_p \\ & \leq C(|\alpha|, \sigma) E(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, \ell-3-|\alpha|\}+(n/4)(2/p-1)-1/2+\sigma}, \end{aligned}$$

as well.

Similar analysis applied to the artificial viscosity system (1.7), gives

$$(8.27) \quad \begin{aligned} & \|D_x^\alpha(\tilde{u} - \tilde{G} * u_0)\|_p \\ & \leq C(|\alpha|) E^2 L(t) (1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, \ell-3-|\alpha|\}+((n-1)/4)(2/p-1)-1/2}, \end{aligned}$$

just as in (8.24). Triangulating with (8.26), we obtain

$$(8.28) \quad \begin{aligned} & \|D_x^\alpha(u - \tilde{u})\|_p \\ & \leq C(|\alpha|, \sigma) E(1+t)^{n/2(1/p-1)-1/2\min\{|\alpha|, \ell-3-|\alpha|\}+(n/4)(2/p-1)-1/2+\sigma}, \end{aligned}$$

completing the proof of (8.13).

Finally, the proof of Theorem 6.8 shows that

$$\begin{aligned} \|D_x^\alpha(\tilde{G} * u_0 - \tilde{G}U)\|_p & \leq \|D_x^\alpha D_x \tilde{G}\|_p \|xu_0\|_1, \\ & \leq C(1+t)^{n/2(1/p-1)-|\alpha|/2+((n-1)/4)(2/p-1)-1/2} \|xu_0\|_1. \end{aligned}$$

Triangulating with (8.26) gives (8.14), completing the proof in the case $3 \leq \ell$, $|\alpha| \leq \ell - 2$. The case that $|\alpha| = \ell - 1$ or $\ell \leq 2$ follows similarly. $\square$

Theorem 8.2 shows that the convected diffusion wave $\tilde{G} = W * G^{\tilde{B}}$ dominates the asymptotic behavior of u , also in L^p , $p < 2$.

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