Remarks on the Pluricomplex Green Function

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ABSTRACT. We prove that the pluricomplex Green function g and the Azukawa pseudometric A have the product property on $G_1 \times G_2$ if G_2 is pseudoconvex. Moreover, we show that g_G and A_G are upper semicontinuous for any domain $G \subset \mathbb{C}^n$. Explicit examples are given.

Let G be a domain in $\mathbb{C}^n$. We put

$$g_G(a, z) := \sup\{u(z) : u \in \mathcal{K}_G(a)\}, \quad a, z \in G,$$

where $\mathcal{K}_G(a)$ denotes the set of all logarithmically plurisubharmonic functions $u : G \rightarrow [0, 1)$ such that there exist $M, r > 0$, with $u(z) \leq M\|z - a\|$ for all $z \in B(a, r) \subset G$ ($\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$ and $B(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$). The function g_G is called the *pluricomplex Green function for G* . It is known that $g_G(a, \cdot) \in \mathcal{K}_G(a)$ (cf. [Jar-Pfl], Lemma 4.2.3). Let A_G denote the *Azukawa pseudometric for G* given by the formula

$$A_G(a, X) := \limsup_{\lambda \nearrow 0} \frac{1}{|\lambda|} g_G(a, a + \lambda X), \quad a \in G, X \in \mathbb{C}^n.$$

Define

$$\begin{aligned} \tilde{k}_G^*(a, z) &:= \inf\{\sigma \in [0, 1) : \exists \varphi \in \mathcal{O}(E, G) : \varphi(0) = a, \varphi(\sigma) = z\}, \quad a, z \in G, \\ \kappa_G(a; X) &:= \inf\{\sigma \in [0, +\infty) : \exists \varphi \in \mathcal{O}(E, G) : \varphi(0) = a, \sigma \cdot \varphi'(0) = X\}, \\ &\quad a \in G, X \in \mathbb{C}^n, \end{aligned}$$

where E denotes the unit disc in $\mathbb{C}$. The function $\tanh^{-1}(\tilde{k}_G^*)$ is called the *Lempert function*; κ_G is the *Kobayashi–Royden pseudometric*.

Recall that the systems $(g_G)_G$ and $(\tilde{k}_G^*)_G$ are m -contractible families of functions; the systems $(A_G)_G$ and $(\kappa_G)_G$ are contractible families of pseudometrics (cf. [Jar-Pfl]).

1. The upper semicontinuity. In all papers discussing the fundamental properties of g_G and A_G the following theorem can be found (cf. [Kli 1–3], [Jar-Pfl]).

If G is pseudoconvex, then g_G and A_G are upper semicontinuous.

The proof of this result is based on the continuity of g_G for hyperconvex domains.

One might get the impression that the assumption ‘ G is pseudoconvex’ is somehow important. This is not so. We will show by elementary methods that g_G and A_G are upper semicontinuous for *arbitrary* domains G .

Theorem 1. *For an arbitrary domain $G \subset \mathbb{C}^n$ the pluricomplex Green function $g_G : G \times G \rightarrow [0, 1)$ and the Azukawa pseudometric $A_G : G \times \mathbb{C}^n \rightarrow [0, +\infty)$ are upper semicontinuous.*

Proof. First, let us recall some known facts (cf. [Jar-Pfl], Sections 4.1, 4.2; see also [Kli 2]).

- (a) If g_G is upper semicontinuous, then A_G is upper semicontinuous.
- (b) If G is bounded, then for any ball $B(a, r) \subset G$ and for any $\varepsilon > 0$ there exists $\delta \in (0, r)$ such that

$$[g_G(x, z)]^{1+\varepsilon} \leq g_G(a, z), \quad x \in B(a, \delta), z \in G \setminus B(a, r).$$

- (c) If $G_\nu \nearrow G$, then $g_{G_\nu} \searrow g_G$.
- (d) g_G is continuous at all points $(a, a) \in G \times G$.

By virtue of (a) and (c), we can restrict ourselves to the case where G is bounded. Because of (d), it suffices to prove that g_G is upper semicontinuous at any point $(a, b) \in G \times G$ with $a \neq b$. Fix $r > 0$ such that $b \notin \bar{B}(a, r) \subset G$. Suppose that $g_G(a, b) < \alpha < \beta$ and let $\varepsilon > 0$ be such that $\alpha < \beta^{1+\varepsilon}$. Take $\delta \in (0, r)$ as in (b). Since the function g_G is upper semicontinuous (recall that $g_G(a, \cdot) \in \mathcal{K}_G(a)$), there exists $\eta > 0$ such that $g_G(a, z) < \alpha$, $z \in B(b, \eta) \subset G \setminus B(a, r)$. Finally,

$$g_G(x, z) \leq [g_G(a, z)]^{1/(1+\varepsilon)} < \alpha^{1/(1+\varepsilon)} < \beta, \quad x \in B(a, \delta), z \in B(b, \eta),$$

which completes the proof. $\square$

Remarks.

- (a) Observe that the same argument shows that if G is bounded and if $g_G(a, \cdot)$ is continuous at b , then g_G is continuous at (a, b) . The implication is not true if G is unbounded (cf. [Jar-Pfl], Proposition 4.4.2).
- (b) Using the description of g_G proposed by Poletsky in [Pol], one can give another proof of Theorem 1.
- (c) The fact that A_G is upper semicontinuous permits us to define the integrated form of A_G and next to apply the general theory to study this object (see [Jar-Pfl], Section 4.3, for more details).

2. The product property. Let $G_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, be arbitrary domains. We say that g , respectively A , has the *product property* on $G_1 \times G_2$ if

$$(1) \quad g_{G_1 \times G_2} = \max\{g_{G_1}, g_{G_2}\},$$

respectively,

$$(2) \quad A_{G_1 \times G_2} = \max\{A_{G_1}, A_{G_2}\}.$$

It is known that (2) is a consequence of (1) (cf. [Jar-Pfl], Remark 9.4(a)). Moreover, *the product properties hold if G_j , $j = 1, 2$, are pseudoconvex*; the proof is based on the boundary behavior of g_G for hyperconvex domains, the Monge–Ampère equation, and the domination principle (cf. [Jar-Pfl], Theorem 9.8). We would like to point out that in general there is no simple relation between g_G and $g_{\tilde{G}}$, where $\tilde{G}$ is the envelope of holomorphy of G (cp. Section 3).

In this note we will prove (Theorem 2(a)) that the product property is also true on $G_1 \times G_2$ if G_2 is pseudoconvex (without assuming G_1 to be pseudoconvex). The more precise formulation of our result is as follows.

Theorem 2. *Let $G_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, be domains.*

- (a) *If G_2 is pseudoconvex, then g and A have the product properties on $G_1 \times G_2$.*
- (b) *Assume that for $b, w_0 \in G_2$ we have*

$$g_{G_2}(b, w_0) = \tilde{k}_{G_2}^*(b, w_0).$$

Then,

$$g_{G_1 \times G_2}((a, b), (z, w_0)) = \max\{g_{G_1}(a, z), g_{G_2}(b, w_0)\}, \quad a, z \in G_1.$$

- (c) *Assume that for $b \in G_2$, $Y_0 \in \mathbb{C}^{n_2}$ we have*

$$A_{G_2}(b; Y_0) = \kappa_{G_2}(b; Y_0).$$

Then,

$$A_{G_1 \times G_2}((a, b); (X, Y_0)) = \max\{A_{G_1}(a; X), A_{G_2}(b; Y_0)\}, \quad a \in G_1, X \in \mathbb{C}^{n_1}.$$

To prove Theorem 2, it is sufficient to verify the following lemma.

Lemma 3. *The product properties hold on $G \times E$ for arbitrary domain $G \subset \mathbb{C}^n$.*

Proof that Lemma 3 $\Rightarrow$ Theorem 2. For (a): We already know that it suffices to prove the product property for g .

Let $(G_{2,\nu})_{\nu \in \mathbb{N}}$ be a sequence of bounded hyperconvex domains such that $G_{2,\nu} \nearrow G_2$. Then $g_{G_{2,\nu}} \searrow g_{G_2}$ and $g_{G_1 \times G_{2,\nu}} \searrow g_{G_1 \times G_2}$, and therefore, we assume that G_2 is bounded hyperconvex. By virtue of Lemma 3 we only have to show that

$$(3) \quad g_{G_1 \times G_2}((a,b), (z,w)) \leq g_{G_1 \times E}((a,0), (z, g_{G_2}(b,w)))$$

for all $(a,b), (z,w) \in G_1 \times G_2$.

Fix $(a,b) \in G_1 \times G_2$. Recall (Lemma 3) that

$$g_{G_1 \times E}((a,0), (z,\lambda)) = \max\{g_{G_1}(a,z), |\lambda|\}, \quad (z,\lambda) \in G_1 \times E.$$

In particular, for each $z \in G_1$, the function $g_{G_1 \times E}((a,0), (z, \cdot))$ is continuous on E . Consequently, inequality (3) follows from:

$$(4) \quad g_{G_1 \times G_2}((a,b), (z,w)) \leq g_{G_1 \times E}((a,0), (z,\lambda))$$

for all $(z,w,\lambda) \in G_1 \times G_2 \times E$ with $g_{G_2}(b,w) < |\lambda|$.

Observe that for fixed $(w,\lambda) \in G_1 \times E$ both sides of (4) are plurisubharmonic on G_1 . Thus, it is enough to show that there exists a subset $S \subset G_1$ of zero Lebesgue measure such that

$$(5) \quad g_{G_1 \times G_2}((a,b), (z,w)) \leq g_{G_1 \times E}((a,0), (z,\lambda))$$

for all $(z,w,\lambda) \in (G_1 \setminus S) \times G_2 \times E$ with $g_{G_2}(b,w) < |\lambda|$.

Let

$$\begin{aligned} \varphi(w) &:= \log g_{G_2}(b,w), & w &\in G_2, \\ V(z,w) &:= \log g_{G_1 \times G_2}((a,b), (z,w)), & (z,w) &\in G_1 \times G_2, \\ u(z,\zeta) &:= \sup\{V(z,w) : (z,w) \in G_1 \times G_2, \varphi(w) < \operatorname{Re} \zeta\}, & (z,\zeta) &\in G_1 \times H, \\ u_0(z,\lambda) &:= \exp(u(z, \log |\lambda|)), & (z,\lambda) &\in G_1 \times E_*, \end{aligned}$$

where $H := \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta < 0\}$ and $E_* := E \setminus \{0\}$.

Note that:

- φ is a continuous plurisubharmonic function with $\{w \in G_2 : \varphi(w) < t\} \in G_2$ for negative t , and $(dd^c)^{n_2}\varphi \equiv 0$ in $G_2 \setminus \{0\}$,
- V is plurisubharmonic,
- $u(z, \zeta) = u(z, \operatorname{Re} \zeta)$, $(z, \zeta) \in G_1 \times H$.

To get (5) it suffices to prove that

$$(6) \quad u_0(z, \lambda) \leq g_{G_1 \times E}((a, 0), (z, \lambda)), \quad (z, \lambda) \in (G_1 \setminus S) \times E_*.$$

By [Dem], Corollary 6.6 (b), for each $z \in G_1$, the function $u(z, \cdot)$ is continuous on H . Hence, inequality (6) is equivalent to:

$$(7) \quad u_0(z, \lambda) \leq g_{G_1 \times E}((a, 0), (z, \lambda)) \quad \text{for a.a. } (z, \lambda) \in G_1 \times E_*.$$

Let $M > 0$, $r \in (0, 1)$ be such that

$$g_{G_1 \times G_2}((a, b), (z, w)) \leq M(\|z - a\| + \|w - b\|) \\ \text{for all } (z, w) \in B(a, r) \times B(b, r) \subset G_1 \times G_2.$$

Choose $R \geq 1$ such that $G_2 \subset B(b, R)$. Then,

$$g_{G_2}(b, w) \geq \frac{\|w - b\|}{R}, \quad w \in G_2,$$

and, consequently,

$$(8) \quad u_0(z, \lambda) \leq MR(\|z - a\| + |\lambda|), \quad z \in B(a, r), 0 < |\lambda| < \frac{r}{R}.$$

We claim that the function u is upper semicontinuous on $G_1 \times H$. Let $G_1 \times H \ni (z_\nu, \zeta_\nu) \rightarrow (z_0, \zeta_0) \in G_1 \times H$ and let $w_\nu \in G_2$ be such that $\varphi(w_\nu) \leq \operatorname{Re} \zeta_\nu$, $u(z_\nu, \zeta_\nu) = V(z_\nu, w_\nu)$, $\nu \in \mathbb{N}$. We may assume that $w_\nu \rightarrow w_0 \in G_2$ with $\varphi(w_0) \leq \operatorname{Re} \zeta_0$. Hence,

$$\limsup_{\nu \rightarrow \infty} u(z_\nu, \zeta_\nu) = \limsup_{\nu \rightarrow \infty} V(z_\nu, w_\nu) \leq V(z_0, w_0) \leq u(z_0, \zeta)$$

whenever $\operatorname{Re} \zeta_0 < \operatorname{Re} \zeta < 0$.

Then continuity of $u(z_0, \cdot)$ gives the required result.

By Theorem 6.11 in [Dem] the function u is weakly plurisubharmonic on $G_1 \times H$, i.e., there exists a plurisubharmonic function $\tilde{u}$ on $G_1 \times H$ such that $\tilde{u} = u$ a.e. on $G_1 \times H$. Since u is upper semicontinuous, $\tilde{u} \leq u$ on $G_1 \times H$.

Observe that $\tilde{u}(z, \zeta) = \tilde{u}(z, \operatorname{Re} \zeta)$, $(z, \zeta) \in G_1 \times H$. For fix $\zeta_0 \in H$. Then the two plurisubharmonic functions

$$(z, \zeta) \rightarrow \tilde{u}(z, \zeta_0 + \zeta) \quad \text{and} \quad (z, \zeta) \rightarrow \tilde{u}(z, \operatorname{Re} \zeta_0 + \zeta)$$

coincide a.a. on $G_1 \times \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta + \zeta_0) < 0\}$. Therefore, they coincide everywhere.

Define $\tilde{u}_1(z, \lambda) := \tilde{u}(z, \log \lambda)$, $(z, \lambda) \in G_1 \times E_*$. It is clear that $\tilde{u}_1$ is well-defined, plurisubharmonic, and $\tilde{u}_1 < 0$. We extend $\tilde{u}_1$ plurisubharmonically on the whole $G_1 \times E$ setting,

$$\tilde{u}_1(z, 0) := \limsup_{G_1 \times E_* \ni (z', \lambda) \rightarrow (z, 0)} \tilde{u}_1(z', \lambda), \quad z \in G_1.$$

Define $\tilde{u}_0 := \exp(\tilde{u}_1)$. Then $\tilde{u}_0 : G_1 \times E \rightarrow [0, 1]$ is a log-plurisubharmonic function and $\tilde{u}_0 \leq u_0$ on $G_1 \times E_*$. In particular, by (8) we have

$$\tilde{u}_0(z, \lambda) \leq MR(\|z - a\| + |\lambda|), \quad z \in B(a, r), 0 < |\lambda| < \frac{r}{R}.$$

Since both sides are plurisubharmonic on $G_1 \times E$, the inequality above holds for all $z \in B(a, r)$ and $|\lambda| < r/R$.

Thus $\tilde{u}_0 \in \mathcal{K}_{G_1 \times E}((a, 0))$ and therefore, $\tilde{u}_0 \leq g_{G_1 \times E}((a, 0), \cdot)$. Since $\tilde{u}_0 = u_0$ a.e. on $G_1 \times E_*$, we end up with (7), so the proof of (a) is complete.

For part (b): Fix $a, z \in G_1$. Choose a sequence of holomorphic mappings $\varphi_\nu : E \rightarrow G_2$ with $\varphi_\nu(0) = b$ and $\varphi_\nu(\sigma_\nu) = w_0$ for some $\sigma_\nu \in [0, 1]$ such that $\sigma_\nu \searrow \tilde{k}_{G_2}^*(b, w_0) = g_{G_2}(b, w_0)$. Using contractibility and Lemma 3 gives

$$\begin{aligned} g_{G_1 \times G_2}((a, b), (z, w_0)) &\leq g_{G_1 \times E}((a, 0), (z, \sigma_\nu)) \\ &= \max\{g_{G_1}(a, z), \sigma_\nu\} \searrow \max\{g_{G_1}(a, z), g_{G_2}(b, w_0)\}. \end{aligned}$$

Recall that the remaining inequality is always satisfied.

The proof of (c) is analogous. $\square$

Lemma 3 is a consequence of the following lemma.

Lemma 4. *Let $G \subset \mathbb{C}^n$ be a bounded domain. Then, for $a, z \in G$, $\lambda \in E$ such that $g_G(a, z) = |\lambda|$ we have:*

$$(9) \quad g_{G \times E}((a, 0), (z, \lambda)) \leq |\lambda|.$$

Proof that Lemma 4 $\Rightarrow$ Lemma 3. It suffices to consider only the case where G is bounded and to show that for arbitrary $a, z_0 \in G$ with $a \neq z_0$ we have

$$(10) \quad g_{G \times E}((a, 0), (z_0, \lambda)) \leq \max\{g_G(a, z_0), |\lambda|\}, \quad \lambda \in E.$$

If $|\lambda| \leq g_G(a, z_0)$, then (10) follows from (9) and the maximum principle for subharmonic functions. If $g_G(a, z_0) < |\lambda| < 1$, then (10) is a consequence of (9) and the maximum principle applied to the subharmonic function

$$\lambda \rightarrow \log g_{G \times E}((a, 0), (z_0, \lambda)) - \log |\lambda|. \quad \square$$

Finally, Lemma 4 follows directly from the fact (Lemma 5) that the function

$$G \ni z \rightarrow g_{G \times E}((a, 0), (z, g_G(a, z))) \in [0, 1]$$

belongs to $\mathcal{K}_G(a)$.

Lemma 5. *Let $G \subset \mathbb{C}^n$ be a domain and let $a \in G$. Then the function u given by the formula*

$$u(z) := g_{G \times E}((a, 0), (z, g_G(a, z))), \quad z \in G,$$

is of the class $\mathcal{K}_G(a)$.

Proof.

STEP 1. For any point $z \in G$ the subharmonic function $g_{G \times E}((a, 0), (z, \cdot))$ is invariant under rotations, i.e.,

$$g_{G \times E}((a, 0), (z, \lambda)) = g_{G \times E}((a, 0), (z, |\lambda|)), \quad \lambda \in E.$$

Hence, the function

$$[0, 1) \ni t \rightarrow g_{G \times E}((a, 0), (z, t))$$

is increasing and continuous (cf. [Kli 2]).

STEP 2. The monotonicity from Step 1 (together with the fact that the functions $g_{G \times E}((a, 0), (\cdot, \cdot))$ and $g_G(a, \cdot)$ are upper semicontinuous) implies that u is upper semicontinuous on G . Moreover, since $g_{G \times E}((a, 0), (\cdot, \cdot)) \in \mathcal{K}_{G \times E}((a, 0))$ and $g_G(a, \cdot) \in \mathcal{K}_G(a)$, we see that there exist $M, r > 0$ with $u(z) \leq M\|z - a\|$ for all $z \in B(a, r) \subset G$.

It remains to prove that u is logarithmically plurisubharmonic.

STEP 3. Fix $z_0 \in G$ and $B(z_0, r) \Subset G$. Choose a sequence $(h_\nu)_{\nu=1}^\infty$ of negative continuous plurisubharmonic functions on $B(z_0, r)$ such that $h_\nu \searrow \log g_G(a, \cdot)$ on $B(a, r)$. Then, $\tilde{h}_\nu := \exp(h_\nu) \searrow g_G(a, \cdot)$. Put

$$v_\nu(z) := \log g_{G \times E}((a, 0), (z, \tilde{h}_\nu(z))), \quad z \in B(z_0, r).$$

In view of Step 1, we get that $v_\nu \searrow \log u$ on $B(z_0, r)$.

Thus, it remains to show that all the functions v_ν are plurisubharmonic near z_0 .

Fix ν . Applying the theorem of Bremermann (cf. [Sib]), we find sequences $(f_j)_{j=1}^\infty \subset \mathcal{O}(B(z_0, r/2))$, $(c_j)_{j=1}^\infty \subset (0, +\infty)$ ($f_j \not\equiv 0$, $j \in \mathbb{N}$) such that

$$h_\nu = \sup_{j \in \mathbb{N}} \{c_j \log |f_j|\} \quad \text{i.e.,} \quad \tilde{h}_\nu = \sup_{j \in \mathbb{N}} \{|f_j|^{c_j}\} < 1 \quad \text{on } B\left(z_0, \frac{r}{2}\right).$$

Hence, by virtue of Step 1, we have

$$v_\nu(z) = \sup_{j \in \mathbb{N}} \{ \log g_{G \times E}((a, 0), (z, |f_j(z)|^{c_j})) \}, \quad z \in B\left(z_0, \frac{r}{2}\right).$$

Since v_ν is an upper semicontinuous function, it suffices to show that all the upper semicontinuous functions

$$B\left(z_0, \frac{r}{2}\right) \ni z \xrightarrow{\psi_j} \log g_{G \times E}((a, 0), (z, |f_j(z)|^{c_j}))$$

are plurisubharmonic. In view of Step 1, we see that ψ_j is plurisubharmonic on $B(z_0, r/2) \setminus Z_j$, where $Z_j := \{z \in B(z_0, r/2) : f_j(z) = 0\}$. Since the set Z_j is analytic, ψ_j can be continued as a plurisubharmonic function $\tilde{\psi}_j$ on the whole $B(z_0, r/2)$. We only need to prove that $\tilde{\psi}_j = \psi_j$ on Z_j . Take a point $z^* \in Z_j$. Using the plurisubharmonicity of the function $z \rightarrow \log g_{G \times E}((a, 0), (z, 0))$, it follows (again by Step 1) that

$$\begin{aligned} \psi_j(z^*) &= \limsup_{Z_j \ni z \rightarrow z^*} \log g_{G \times E}((a, 0), (z, 0)) \\ &\leq \limsup_{Z_j \ni z \rightarrow z^*} \log g_{G \times E}((a, 0), (z, |f_j(z)|^{c_j})) \\ &= \limsup_{Z_j \ni z \rightarrow z^*} \psi_j(z) \\ &= \tilde{\psi}_j(z^*) \leq \psi_j(z^*). \end{aligned} \quad \square$$

Remark.

- (a) We would like to point out that the proof of Theorem 2(a) given here is independent of Theorem 9.8 in [Jar-Pfl].
- (b) A. Edigarian informed us that, using the description of g_G from [Pol], one can give another proof of Theorem 2(b).

Remark. Having Theorem 2, we think that the product property for g (and A) is true for *arbitrary* domains.

3. An example. In [Jar-Pfl], Example 4.2.8, and in [Kli 3], explicit formulas for g_G for certain Reinhardt domains are given. The following more general example contains, in particular, these results.

Example. Let $0 \leq a_1 < a_2 \leq 1$, $0 \leq b_1 < b_2 \leq 1$, $Q := [a_1, a_2] \times [b_1, b_2]$ be such that $(0, 0) \notin Q$. Put

$$G := \{(z_1, z_2) \in E \times E : (|z_1|, |z_2|) \notin Q\}.$$

There are two cases which surprisingly give two different types of Green functions.

CASE 1. $\overset{\circ}{Q} \cap \{(t, t) : t \in [0, 1]\} = \emptyset$. Then,

$$g_G((0, 0), (z_1, z_2)) = \max\{|z_1|, |z_2|\}.$$

CASE 2. $\overset{\circ}{Q} \cap \{(t, t) : t \in [0, 1]\} \neq \emptyset$. Then,

$$\begin{aligned} & g_G((0, 0), (z_1, z_2)) \\ &= \max \left\{ |z_1|, |z_2|, \left(e^{\log(b_2/a_1) \log(a_2/b_1)} |z_1|^{\log(a_2/b_1)} |z_2|^{\log(b_2/a_1)} \right)^{1/\log(a_2 b_2/a_1 b_1)} \right\} \\ & \hspace{15em} \text{if } |z_1| < a_1 \text{ or } |z_2| < b_1; \\ & g_G((0, 0), (z_1, z_2)) \\ &= \max \left\{ |z_1|, |z_2|, \left(|z_1|^{\log b_2 \log(b_1/a_2)} |z_2|^{\log a_2 \log(a_1/b_2)} \right)^{1/(\log a_1 \log b_1 - \log a_2 \log b_2)} \right\} \\ & \hspace{15em} \text{if } |z_1| > a_2 \text{ or } |z_2| > b_2. \end{aligned}$$

For $a_1 = 0$ (resp. $b_1 = 0$), the formulas should be understood as limits.

The proof of the formulas above is based on the Schwarz Lemma and the Hadamard Three Circles Theorem for subharmonic functions.

REFERENCES

- [Dem] J.-P. DEMAILLY, *Measures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N. S.) **19** (1985), 1–125.
- [Jar-Pfl] M. JARNICKI & P. PFLUG, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Expositions in Mathematics 9, Walter de Gruyter, 1993.
- [Kli 1] M. KLIMEK, *Infinitesimal pseudometrics and the Schwarz Lemma*, Proc. Amer. Math. Soc. **105** (1989), 134–140.
- [Kli 2] ———, *Pluripotential Theory*, Oxford University Press, 1991.
- [Kli 3] ———, *Invariant pluricomplex Green functions*, Banach Center Publications (to appear).
- [Pol] E. A. POLETSKY, *Holomorphic currents*, Indiana Univ. Math. J. **42** (1993), 85–144.
- [Sib] N. SIBONY, *Prolongement de fonctions holomorphes bornées et metrique de Carathéodory*, Invent. Math. **29** (1975), 205–230.

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