The Pluricomplex Green Function, Capacitative Notions, and Approximation Problems in $\mathbf{C}^n$

STEPHANIE NIVOCHÉ

ABSTRACT. We prove here the following result: If D is a bounded strictly hyperconvex domain in $\mathbf{C}^n$ containing O , then the sequence of ξ -directional analytic capacities converges to the ξ -directional harmonic capacity, quasi-everywhere in $S(O, 1)$. We deduce two results from this property. The first one is a comparison between the ξ -directional harmonic capacities associated respectively to the Green function of $\mathbf{C}^n \setminus K$ with pole at infinity and to the pluricomplex Green function of the dual open set K^* with pole at O . The second is a characterization of the harmonic capacity associated to the dual domain K^* of a regular Martineau compact K in $\mathbf{C}^n$ by the polynomial approximations of the exponential function on K .

1. Introduction. Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing O . L. Lempert [10], M. Klimek [7] and J. P. Demailly [3] proved the existence and the uniqueness of a function g_D , continuous on $\bar{D}$ with values in $[-\infty, 0]$, plurisubharmonic in D such that $g_D|_{\partial D} \equiv 0$, $(dd^c g_D)^n = 0$ on $D \setminus \{O\}$ and $g_D(z) \sim \log \|z\|$ when $z \rightarrow O$. This function exists if and only if the bounded domain D is hyperconvex. In this case, this function is called the pluricomplex Green function with logarithmic pole at O in D and it satisfies $(dd^c g_D)^n = (2\pi)^n \delta_O$ and

$$g_D(z) = \sup\{v(z) : v \in \text{PSH}(D, [-\infty, 0]), v(y) \leq \log \|y\| + O_v(1) \text{ when } y \rightarrow O\}.$$

For $\xi \in S(O, 1)$ (the unit sphere in $\mathbf{C}^n$), we define the ξ -directional harmonic capacity $C_D(\xi)$ of the domain D by:

$$-\log C_D(\xi) = \limsup_{\lambda \in \mathbf{C}, \lambda \rightarrow 0} (g_D(\lambda \xi) - \log |\lambda|).$$

$C_D(\xi)$ was introduced by P. Lelong in [9] and E. Bedford and B. A. Taylor [2] called $\varrho_D(\xi) = -\log C_D(\xi)$ the Robin function of the domain D . In Section 2, we study the convergence of sequences (g_{D_j}) and (C_{D_j}) , where (D_j) is an increasing or decreasing sequence of domains.

Let D be a domain in $\mathbf{C}^n$ containing O and p be an integer ≥ 1 . We define a compact set $\mathcal{E}_p(D)$ of $H(D)$ (the complex vector space of holomorphic functions in D , with the topology of uniform convergence on every compact set of D) and a continuous and plurisubharmonic function h_p on D by:

$$\begin{aligned}\mathcal{E}_p(D) &= \{f \in H(D) : \sup_{z \in D} |f(z)| \leq 1, D^{(\nu)} f(O) = 0 \ \forall \nu \in \mathbf{N}^n / |\nu| \leq p-1\}, \\ h_p(z) &= \sup_{f \in \mathcal{E}_p(D)} \frac{1}{p} \log |f(z)|, \ \forall z \in D.\end{aligned}$$

For every $\xi \in S(O, 1)$, let $\gamma_p(D, \xi)$ be the ξ -directional analytic capacity of order p defined by

$$\gamma_p(D, \xi) = \{\max\{|d_\xi^{(p)} f(0)/p!|^{1/p} : f \in \mathcal{E}_p(D)\}\}^{-1}.$$

h_p and γ_p were introduced in [13] for $n = 1$, and in [6] and [14] for the multidimensional case.

From a result of L. Lempert [10], it follows that on a strictly convex domain D in $\mathbf{C}^n$ with analytic boundary, $h_p = g_D$ on D and $\gamma_p(D, \xi) = C_D(\xi)$ on $S(O, 1)$, $\forall p \geq 1$. The main result of this article is the following theorem which generalizes the property above:

Theorem 1.1. *If D is a bounded strictly hyperconvex domain in $\mathbf{C}^n$ containing O , then*

$$\begin{aligned}\lim_{p \rightarrow \infty} h_p(z) &= g_D(z) \quad \forall z \in D, \\ \lim_{p \rightarrow \infty} \gamma_p(D, \xi) &= C_D(\xi) \quad \text{quasi-everywhere in } S(O, 1).\end{aligned}$$

From this theorem which is proved in Section 3, we deduce two corollaries. The first one, in Section 4, gives a comparison between several capacities. If K is a non-pluripolar compact set in $\mathbf{C}^n$ with the generalized Green function g_K , for all $\xi \in S(O, 1)$, we define $C_K(\xi)$, the ξ -directional harmonic capacity of the

compact set K by:

$$-\log C_K(\xi) = \limsup_{\lambda \in \mathbf{C}, |\lambda| \rightarrow \infty} (g_K(\lambda\xi) - \log |\lambda|).$$

The function $-\log C_K(\xi)$ is the Robin function of the compact set K [19]. We also generalize the notion of Tchbycheff constant in the multidimensional case introducing constants $\tau^*(K, \xi)$ and $\tau(K, \xi)$ and obtain the following corollary, in which we use definitions from Section 4.

Theorem 1.2. *Let K be a compact set in $\mathbf{C}^n$.*

- (1) *If its dual open set K^* is a bounded hyperconvex domain in $\mathbf{C}^{n*}$, then we have*

$$C_{K^*}(\xi) \leq \tau^*(K, \xi)^{-1} \leq \tau(K, \xi)^{-1} \leq C_K(\bar{\xi})^{-1}, \quad \forall \xi \in S(O, 1).$$

- (2) *If K is L -regular, we have $\tau(K, \xi) = C_K(\bar{\xi})$, $\forall \xi \in S(O, 1)$.*

- (3) *If in addition, K is a regular Martineau compact then we get finally that $C_{K^*}(\xi) = \tau^*(K, \xi)^{-1}$ quasi-everywhere in $S(O, 1)$.*

In the case of a compact set K in $\mathbf{C}$ endowed with a Green function with pole at infinity, we have $g_K(z) = -g_{K^*}(-1/z)$ on $\mathbf{C}$ and $C(K) = C(K^*)^{-1}$. Also according to G. Szeg (1924), we have $C(K) = \tau(K)$. In $\mathbf{C}^n$, the situation is different. If K is a closed polydisk in $\mathbf{C}^n$ with center at O , then there exist points ξ in $S(O, 1)$ satisfying $C_{K^*}(\xi) < C_K(\bar{\xi})^{-1}$.

The second corollary, which is proved in Section 5, characterizes the ξ -directional harmonic capacity of the dual domain K^* of a regular Martineau compact K , in terms of polynomial approximations of the exponential function on K . This corollary generalizes a result of G. M. Henkin and A. Shananin [5]. For $N \in \mathbf{N}$, we denote by $\mathcal{P}_N$ the complex vector space of holomorphic polynomials of degree less or equal to N . If K is a compact set in $\mathbf{C}^n$ and f is a continuous function on K , then $E_N(K, f) = \inf\{\sup\{|f(z) - p(z)| : z \in K\} : p \in \mathcal{P}_N\}$. If we denote by $\mathcal{C}(K)$ the Banach space of continuous functions on K with the uniform norm on K , then $E_N(K, f)$ is the distance in $\mathcal{C}(K)$ between the function f and the closed subspace $\mathcal{P}_N$.

Theorem 1.3. *If K is a regular Martineau compact set in $\mathbf{C}^n$ with dual domain K^* , then for every complex number λ and quasi-everywhere in $S(O, 1)$*

$$\lim_{N \rightarrow \infty} N \cdot E_N^{1/N}(K, e^{\lambda\xi \cdot z}) = e|\lambda|C_{K^*}(\xi)^{-1}.$$

The main results of this article were announced in [14].

Notations. On $\mathbf{C}^n$, we only use the hermitian norm: $\|z\| = (\sum_{j=1}^n |z_j|^2)^{1/2}$ and we note $z \cdot z' = \sum_{j=1}^n z_j z'_j$ if $z = (z_1, \dots, z_n)$ and $z' = (z'_1, \dots, z'_n) \in \mathbf{C}^n$. Let R be a positive real number. We will denote by $D(0, R)$ the open disk with center at 0 and radius R in $\mathbf{C}$ and by $B(O, R)$ the open ball with center at O and radius R in $\mathbf{C}^n$. If $R = (R_1, \dots, R_n) \in (\mathbf{R}^+)^n$ and $z \in \mathbf{C}^n$, we will denote by $P(z, R)$ the open polydisk with center at z and multiradius R . If $f \in \mathcal{C}^\infty(D)$ (where D is an open set in $\mathbf{C}^n$), then for every $\xi \in S(O, 1)$ and all $p \in \mathbf{N}$,

$$d_\xi^{(p)} f(0) = \frac{\partial^p (f(\lambda \xi))}{\partial \lambda^p} \Big|_{\lambda=0}$$

is the ξ -directional derivative of order p of the function f at O . We will use the abbreviation psh for plurisubharmonic; $\text{PSH}(D)$ is the set of psh functions on the open set D and $H(D)$ is the set of holomorphic functions on D .

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2. Pluricomplex Green function and harmonic capacity. A domain D in $\mathbf{C}^n$ is called hyperconvex if there exists a continuous psh exhaustion function $\varrho : D \rightarrow]-\infty, 0[$. This definition has been introduced by J. L. Stehlé. In what follows, we will often need the notion of strictly hyperconvex domain defined as below.

Definition 2.1. A bounded domain D in $\mathbf{C}^n$ is said to be strictly hyperconvex if there exists a bounded domain Ω and a function $\varrho \in \mathcal{C}(\Omega,]-\infty, 1[) \cap \text{PSH}(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$, ϱ is exhaustive for Ω and for all real number $c \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected.

The two following properties study the convergence of sequences (g_{D_j}) and (C_{D_j}) , where (D_j) is an increasing or decreasing sequence of hyperconvex domains.

Proposition 2.2. ([3], [2]) Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing O . If (D_j) is an increasing sequence of hyperconvex domains in $\mathbf{C}^n$ containing O such that $\bigcup_j D_j = D$, then the sequence (g_{D_j}) converges pointwise on D to g_D and the sequence $(C_{D_j}(\xi))$ converges on $S(O, 1)$ to $C_D(\xi)$.

Proposition 2.3. Let D be a bounded strictly hyperconvex domain in $\mathbf{C}^n$ containing O (as in Definition 2.1). If $(D_j)_{j \geq 1}$ is the decreasing sequence of bounded hyperconvex domains D_j , defined by $D_j = \{z \in \Omega : \varrho(z) < 1/j\}$ for

every $j \geq 1$, then:

(1) the sequence $(g_{D_j})_{j \geq 1}$ converges uniformly to g_D on $\bar{D}$

(2) $\lim_{j \rightarrow \infty} C_{D_j}(\xi) = C_D(\xi)$, for all ξ in $S(O, 1)$.

Proof. First we verify that $\lim_j g_{D_j}(z) = 0 = g_D(z)$, $\forall z \in bD$. Next, for every $j \geq 1$ let $c_j = \inf_{z \in bD} g_{D_j}(z)$. Since g_{D_j} is a negative, continuous and real valued function on the compact set bD , $c_j \in]-\infty, 0[$. The sequence (g_{D_j}) increases and converges pointwise to the restriction of the function g_D on the boundary of D , i.e., to 0. Then according to Dini's theorem, the sequence (g_{D_j}) converges uniformly to 0 on bD and $\lim_j c_j = 0$. For every $j \geq 1$, consider g_j the function defined on D_j by

$$g_j(z) = \begin{cases} g_{D_j}(z) & \text{if } z \in D_j \setminus D, \\ \max\{g_{D_j}(z), g_D(z) + c_j\} & \text{if } z \in \bar{D}. \end{cases}$$

g_j is a negative continuous psh function on D_j and it has a logarithmic pole at O . Consequently $g_j \leq g_{D_j}$ on D_j and in particular, $g_D + c_j \leq g_{D_j}$ on $\bar{D}$. Also $D \subset D_j$ and $g_{D_j} \leq g_D$. Thus we get the two following statements:

$$g_{D_j} \leq g_D \leq g_{D_j} - c_j, \quad \text{on } \bar{D},$$

$$\text{and } C_{D_j}(\xi) \geq C_D(\xi) \geq C_{D_j}(\xi)e^{c_j}, \quad \forall \xi \in S(O, 1);$$

and the proposition is proved. $\square$

3. Analytic capacities of a domain in $\mathbf{C}^n$.

Lemma 3.1. *Let D be a domain in $\mathbf{C}^n$ containing O . Then:*

- (1) $\forall p \geq 1$, h_p is a continuous and psh function on D , with values at $[-\infty, 0[$.
- (2) $h_p(z)$ is equivalent to $\log \|z\|$ when z tends to O .
- (3) $h_p(\lambda\xi) - \log |\lambda|$ converges to $-\log \gamma_p(D, \xi)$ when λ ($\in \mathbf{C}^*$) tends to 0, $\forall \xi \in S(O, 1)$.
- (4) The sequences (h_p) and $(\gamma_p(D, \xi))$ satisfy: $\forall (p, q) \in (\mathbf{N}^*)^2$

$$(3) \quad (p+q)h_{p+q} \geq ph_p + qh_q, \quad \text{on } D,$$

$$(4) \quad \gamma_{p+q}(D, \xi)^{(p+q)} \leq \gamma_p(D, \xi)^p \cdot \gamma_q(D, \xi)^q, \quad \text{on } S(O, 1),$$

and they converge $\forall z \in D$ and $\forall \xi \in S(O, 1)$, respectively.

Proof. For every $f \in \mathcal{E}_p(D)$, $(\log |f|)/p$ is a continuous function on D , so h_p is lower semicontinuous on D . By using Montel's Theorem and the fact that $\mathcal{E}_p(D)$ is a compact set in $H(D)$, it follows that h_p is also upper semicontinuous on D . The plurisubharmonicity of h_p is a consequence of its definition.

If $B(O, r) \subset D \subset B(O, R)$, let us show that $\log(\|z\|/R) \leq h_p(z) \leq \log(\|z\|/r)$ on D . If $z \in D \setminus \{O\}$, then $z = \|z\|\xi_0$, where $\xi_0 = z/\|z\| \in S(O, 1)$. The function $f(w) = (w \cdot \bar{\xi}_0)^p/R^p$ belongs to $\mathcal{E}_p(D)$ and $h_p(z) \geq \log(\|z\|/R)$. By Schwarz's Lemma for the ball $B(O, r) \subset D$, we obtain $|f(z)| \leq \|z\|^p/r^p$, $\forall z \in B(O, r)$, and $\forall f \in \mathcal{E}_p(D)$. Thus $h_p(z) \leq \log(\|z\|/r)$ on D .

For ξ fixed in $S(O, 1)$, by using again Montel's theorem, we prove property (iii). Let us prove the inequality (3) and the pointwise convergence on D of the sequence (h_p) . For each function $f \in \mathcal{E}_p(D)$ and $g \in \mathcal{E}_q(D)$, the function $h = fg$ is in $\mathcal{E}_{p+q}(D)$. Consequently, $(p+q)h_{p+q} \geq \sup\{\log |fg| : f \in \mathcal{E}_p(D), g \in \mathcal{E}_q(D)\} = ph_p + qh_q$. Let z be fixed in $D \setminus \{O\}$ (for $z = O$, $h_p(O) = -\infty$ $\forall p \geq 1$ and then $\lim_p h_p(O) = \sup_p h_p(O) = -\infty$). Denote $L(z) = \limsup_p h_p(z)$ and $\ell(z) = \liminf_p h_p(z)$. $\forall \varepsilon > 0$, $\exists p \geq 1$ such that $h_p(z) \geq L(z) - \varepsilon$. $\forall q \geq p$, $q = \alpha p + s$ with $\alpha \geq 1$ and $0 \leq s \leq p - 1$. Then $qh_q(z) = (\alpha p + s)h_{\alpha p + s}(z) \geq \alpha ph_{\alpha p}(z) + sh_s(z) \geq \alpha ph_p(z) + sh_s(z) \geq \alpha p(L(z) - \varepsilon) + sh_s(z)$ and we obtain that $\ell(z) = L(z)$. The inequality (4) follows from the inequality (3) and from the property (iii). The convergence of the sequence $(\gamma_p(D, \xi))$ is proved as the convergence of (h_p) . $\square$

Remark 3.2. From Lemma 3.1, we deduce two inequalities: $\forall (p, \alpha) \in (\mathbf{N}^*)^2$,

$$(3') \quad h_{\alpha p} \geq h_p \quad \text{on } D,$$

$$(4') \quad \gamma_{\alpha p}(D, \xi) \leq \gamma_p(D, \xi), \quad \forall \xi \in S(O, 1).$$

In the following property, we give a first comparison of functions h_p and g_D and of capacities $\gamma_p(D, \xi)$ and $C_D(\xi)$.

Lemma 3.3. If D is a bounded hyperconvex domain in $\mathbf{C}^n$ containing O , then for every $p \in \mathbf{N}^*$:

$$(5) \quad h_p \leq g_D, \quad \text{on } D, \text{ and}$$

$$(6) \quad \gamma_p(D, \xi) \geq C_D(\xi), \quad \text{for each } \xi \in S(O, 1).$$

Proof. For every function $f \in \mathcal{E}_p(D)$, $\log |f| \in \text{PSH}(D)$ and is negative on D . According to Lemma 3.1, we have $(\log |f(z)|)/p \leq \log \|z\| - \log r$ on D , if $B(O, r) \subset D$. Therefore, from the maximality of g_D , we obtain the inequality $(\log |f(z)|)/p \leq g_D(z)$ $\forall z \in D$, which gives us (5). The inequality (6) is a consequence of (5) and of Lemma 3.1 (iii). $\square$

Proposition 3.4. *Let D be a bounded strictly convex open set in $\mathbf{C}^n$, with analytic boundary and containing O . Then for all $p \in \mathbf{N}^*$ we have*

$$\begin{aligned} g_D(z) &= h_p, & \forall z \in D, \\ \gamma_p(D, \xi) &= C_D(\xi), & \forall \xi \in S(O, 1). \end{aligned}$$

Proof. Identity of functions g_D and h_1 was proved by L. Lempert in [10]. We deduce the equality between $C_D(\xi)$ and $\gamma_1(D, \xi)$, $\forall \xi \in S(O, 1)$. According to the inequalities (3'), (4'), (5) and (6), we get $\gamma_p(D, \xi) = C_D(\xi)$ on $S(O, 1)$ and $h_p = g_D$ on D , $\forall p \geq 1$. $\square$

Now, we are going to prove the convergence of the sequence (h_p) to g_D , using an idea of J. P. Demailly (see [4]). Let us first suppose that D is a bounded hyperconvex domain in $\mathbf{C}^n$ containing O . For any $p \in \mathbf{N}^*$, let $\mathcal{H}_D(pg_D) = \{f \in H(D) : \int_D |f(z)|^2 e^{-2pg_D(z)} d\mu(z) < \infty\}$ be the Hilbert space provided with the scalar product

$$(f, g)_{2,p} = \int_D f(z) \overline{g(z)} e^{-2pg_D(z)} d\mu(z)$$

(the norm is denoted by $\|\cdot\|_{2,p}$) and let g_p be the function defined on D by $g_p(z) = (\log(\sum_l |\sigma_\ell(z)|^2)) / (2p)$, where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_D(pg_D)$.

Just by using the fact that $\log(\|z\|/R) \leq g_D(z) \leq \log(\|z\|/r)$ on D , if $B(O, r) \subset D \subset B(O, R)$, we prove the following lemma.

Lemma 3.5. *Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing O .*

- (1) *If $p \geq n$, every function f in $\mathcal{H}_D(pg_D)$ satisfies $D^{(\nu)}f(O) = 0 \ \forall \nu \in \mathbf{N}^n / |\nu| \leq p - n$.*
- (2) *If $p < n$, then for every ν there is a function f in $\mathcal{H}_D(pg_D)$ such that $D^{(\nu)}f(O) \neq 0$.*

Lemma 3.6. *If D is a bounded hyperconvex domain in $\mathbf{C}^n$ containing O , then $\forall p \in \mathbf{N}^*$*

$$g_p(z) = \sup_{f \in B(O, 1)_{2,p}} \frac{1}{p} \log |f(z)| \quad \text{on } D,$$

where $B(O, 1)_{2,p}$ is the unit ball in $\mathcal{H}_D(pg_D)$. Moreover, g_p is a continuous psh function on D .

Proof. $\forall w \in D$, let ϕ_w be the evaluation linear form $f \mapsto f(w)$ on $\mathcal{H}_D(pg_D)$. We verify that its norm $\|\phi_w\|$ is equal to $(\sum_l |\sigma_\ell(w)|^2)^{1/2}$. Since g_D is negative on D , the L^2 -topology in the space $\mathcal{H}_D(pg_D)$ is stronger than the topology of

uniform convergence on every compact set in D . Thus the series $\sum_l |\sigma_l|^2$ converges uniformly on every compact set in D and is real analytic and continuous on D . Hence g_p is a continuous and psh function on D . $\square$

Proposition 3.7. *Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing O . Then there exist a constant $c_1 > 0$, depending only on n and the diameter of D and a constant $c_2 > 0$, such that*

$$(7) \quad g_p(z) \geq g_D(z) - \frac{c_1}{p},$$

$$(8) \quad g_p(z) \leq \sup_{w \in B(z, r)} g_D(w) + \frac{1}{p}(\log c_2 - n \log r)$$

for all z in D and all real $r > 0$ such that $r < \text{dist}(z, bD)$. In particular, the sequence (g_p) converges pointwise to the function g_D on D .

Proof. We are going to use the following result of T. Ohsawa (Corollary 2 in [15]):

Let Y be a pure dimensional closed complex submanifold of $\mathbf{C}^n$, let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and φ a psh function on D . Then, for any holomorphic function f on $D \cap Y$ with $\int_{D \cap Y} |f|^2 e^{-\varphi} d\mu_Y < +\infty$, there exists holomorphic extension F to D such that $\int_D |F|^2 e^{-\varphi} d\mu \leq A \int_{D \cap Y} |f|^2 e^{-\varphi} d\mu_Y$. The constant A depends on Y and $\sup\{\|z\| : z \in D\}$, but does not depend either on f or on φ .

This result applied to the 0-dimensional subvariety $Y = \{z\}$ (z is any point in D) and to the function $\varphi = 2pg_D$ (p is any integer ≥ 1) shows that for any $a \in \mathbf{C}$ there exists a function f in $H(D)$ such that $f(z) = a$ and

$$\int_D \frac{|f(w)|^2 e^{-2pg_D(w)}}{(1 + \|w - z\|^2)^{n+1}} d\mu(w) \leq c_n |f(z)|^2 e^{-2pg_D(z)},$$

where c_n (see the proof of the Corollary 2 in [15]) is a constant which only depends on the dimension n . As $1 + \|w - z\|^2 \leq 1 + \text{diam}(D)^2$, $\forall (z, w) \in D^2$, we obtain:

$$\int_D |f(w)|^2 e^{-2pg_D(w)} d\mu(w) \leq (\text{diam}(D)^2 + 1)^{n+1} c_n e^{-2pg_D(z)} |a|^2.$$

If we choose $a \in \mathbf{C}$ such that the right-hand side of the previous inequality is equal to 1, then letting $c_1 = (\log c_n + (n+1) \log(1 + \text{diam}(D)^2))/2$ and according to Lemma 3.6, we get the inequality (7).

Let us prove now inequality (8). Take z and r as above. Let f be a function in $B(O, 1)_{2,p}$. The mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned} |f(z)|^2 &\leq \frac{c_2^2}{r^{2n}} \int_{B(z,r)} |f(w)|^2 d\mu(w) \\ &\leq \frac{c_2^2}{r^{2n}} \left(\int_{B(z,r)} |f(w)|^2 e^{-2pg_D(w)} d\mu(w) \right) C(z), \end{aligned}$$

where $C(z) = \exp(2p \sup\{g_D(w) : w \in B(z, r)\})$ and $c_2 = \sqrt{n!/\pi^n}$. Using again Lemma 3.6, we are done. $\square$

Now we suppose that D is a bounded strictly hyperconvex domain in $\mathbf{C}^n$ containing O .

Lemma 3.8. $\forall j \geq 1$ and $\forall p \geq 1$ we have the following inequality

$$(9) \quad \frac{p+n-1}{p} g_{p+n-1, D_j} - \frac{\log(c_2 \delta(j)^{-n})}{p} \leq h_p \text{ on } D,$$

where

$$\begin{aligned} B(0, 1)_{2,p+n-1, D_j} &= \left\{ f \in H(D_j) : \int_{D_j} |f|^2 e^{-2(p+n-1)g_{D_j}} d\mu \leq 1 \right\}, \\ \delta(j) &= \text{dist}(\bar{D}, bD_j), \text{ and} \\ g_{p+n-1, D_j} &= \sup \left\{ \frac{(\log |f|)}{(p+n-1)} : f \in B(0, 1)_{2,p+n-1, D_j} \right\}. \end{aligned}$$

Proof. Let $f \in B(0, 1)_{2,p+n-1, D_j}$. There exists $z \in bD$ such that $|f(z)| = \sup\{|f(w)| : w \in \bar{D}\}$. By the mean value inequality applied to the psh function $|f|^2$ on the ball $B(z, \delta(j)) \subset D_j$, we get

$$|f(z)|^2 \leq \int_{B(z, \delta(j))} |f(w)|^2 d\mu(w) c_2^2 / \delta(j)^{2n}.$$

As g_{D_j} is negative on D_j , we deduce that

$$\begin{aligned} |f(z)|^2 &\leq \frac{c_2^2}{\delta(j)^{2n}} \int_{B(z, \delta(j))} |f(w)|^2 e^{-2(p+n-1)g_{D_j}(w)} d\mu(w) \\ &\leq \frac{c_2^2}{\delta(j)^{2n}} \|f\|_{2,p+n-1, D_j}^2. \end{aligned}$$

From Lemmas 3.5 and 3.6 and the inequality above, we obtain the required inequality.

Finally, letting p and j go to infinity in the inequalities (5) and (9) and by Proposition 3.7, we derive that the sequence (h_p) converges pointwise to g_D on D . $\square$

Now let us prove the convergence of the sequence $(\gamma_p(D, \xi))$ to $C_D(\xi)$. First, we remark that the sequence $(h_{p^\alpha})_{\alpha \geq 1}$ increases ($\forall p > 1$), according to the inequality (3') and converges pointwise on D to the function g_D .

Let us study the sequence of the Monge-Ampère operators $((dd^c h_m)^n)_{m \geq 1}$ on D . By Property 3.1 (ii), $h_m(z)$ is equivalent to $\log \|z\|$ when z tends to O . Thus, according to the Theorem 3.9 of [3], we deduce that $(dd^c h_m)^n(\{O\}) = (2\pi)^n \forall m \geq 1$. Let us denote by Ω the open set $D \setminus \{O\}$. The increasing sequence $(h_{p^\alpha})_{\alpha \geq 1}$ is contained in $\text{PSH}(D) \cap L_{\text{loc}}^\infty(\Omega)$ and converges pointwise on Ω to the function $g_D \in \text{PSH}(D) \cap L_{\text{loc}}^\infty(\Omega)$. From Theorem 7.4 of [1], we deduce that the sequence of the Radon measures $((dd^c h_{p^\alpha})^n)_{\alpha \geq 1}$ converges, in the weak sense of measures on Ω , to the measure identically to zero, since $(dd^c g_D)^n = 0$ on Ω . So the sequence of the Radon measures $((dd^c h_{p^\alpha})^n)_{\alpha \geq 1}$ converges to the measure $(2\pi)^n \delta_0 = (dd^c g_D)^n$, on D .

By Theorem 6.6 of [2], we get that the sequence $(\gamma_{p^\alpha}(D, \xi))_{\alpha \geq 1}$ decreases to $C_D(\xi)$ quasi-everywhere in $S(O, 1)$ (i.e., except on a polar set in $S(O, 1)$, regarded as $\mathbf{CP}^{n-1}$), since

$$\lim_{\alpha \rightarrow +\infty} \int_D \log(1 + \|z\|) (dd^c h_{p^\alpha})^n = \int_D \log(1 + \|z\|) (dd^c g_D)^n = 0.$$

Finally, since the sequence $(\gamma_m(D, \xi))_m$ converges $\forall \xi \in S(O, 1)$, we are done.

4. Comparison of capacities in $\mathbf{C}^n$. Let K be a compact set in $\mathbf{C}^n$ and V_K be the extremal function on $\mathbf{C}^n$ associated with K ([17], [21]) and defined by

$$V_K(z) = \sup\{v(z) : v \in L(\mathbf{C}^n) \text{ and } v|_K \leq 0\},$$

where $L(\mathbf{C}^n) = \{v \in \text{PSH}(\mathbf{C}^n) : v(z) \leq \log(1 + \|z\|) + O_v(1)\}$. Denote by g_K the semicontinuous regularization V_K^* , called the generalized Green function for the compact set K ([21]). J. Siciak ([17]) proved that the function g_K belongs to the set $L(\mathbf{C}^n)$ if and only if K is a non-pluripolar compact set. From now on, we will assume this. Then we define $C(K)$, the harmonic capacity associated with the compact set K , by

$$-\log C(K) = \gamma(K) = \limsup_{\|z\| \rightarrow \infty} (g_K(z) - \log \|z\|),$$

where $\gamma(K)$ is the Robin constant of the set K .

Let α be a positive integer and ξ be in $S(O, 1)$. Denote by $\mathcal{P}_\alpha$ the space of all holomorphic polynomials of degree less or equal to α in $\mathbf{C}^n$ and by $\mathcal{P}_\alpha^*(\xi)$ the subspace of all polynomials p such that $p(z) = \sum_{|\sigma| \leq \alpha-1} c_\sigma z^\sigma + c_\alpha (z \cdot \xi)^\alpha$, where c_σ ($\sigma \in \mathbf{N}^n$) and c_α are complex numbers. For all polynomial p in $\mathcal{P}_\alpha$, $\hat{p}$ is its homogeneous part of degree α . We define

$$\begin{aligned} \mathcal{P}_{\alpha,u} &= \{p \in \mathcal{P}_\alpha : \|\hat{p}\|_{S(O,1)} \geq 1\} & M_\alpha(K) &= \inf_{p \in \mathcal{P}_{\alpha,u}} \|p\|_K, \\ \mathcal{P}_{\alpha,u}(\xi) &= \{p \in \mathcal{P}_\alpha : |\hat{p}(\xi)| \geq 1\} & m_\alpha(K, \xi) &= \inf_{p \in \mathcal{P}_{\alpha,u}(\xi)} \|p\|_K, \\ \mathcal{P}_{\alpha,u}^*(\xi) &= \{p \in \mathcal{P}_\alpha^*(\xi) : |c_\alpha| \geq 1\} & m_\alpha^*(K, \xi) &= \inf_{p \in \mathcal{P}_{\alpha,u}^*(\xi)} \|p\|_K. \end{aligned}$$

For any compact set K in $\mathbf{C}^n$, the sequence $(M_\alpha(K)^{1/\alpha})$ converges to a constant $\tau(K)$, called the Tchbycheff constant of the compact set K (see [21], [18] and [11]). In the same manner we first prove that, $\forall \xi \in S(O, 1)$ and $\forall (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $m_{\alpha_1+\alpha_2}(K, \xi) \leq m_{\alpha_1}(K, \xi)m_{\alpha_2}(K, \xi)$ and $m_{\alpha_1+\alpha_2}^*(K, \xi) \leq m_{\alpha_1}^*(K, \xi)m_{\alpha_2}^*(K, \xi)$. Then we deduce that the sequences $(m_\alpha(K, \xi)^{1/\alpha})$ and $(m_\alpha^*(K, \xi)^{1/\alpha})$ are convergent and we respectively denote by $\tau(K, \xi)$ and $\tau^*(K, \xi)$ their limit. $\tau(K, \xi)$ and $\tau^*(K, \xi)$ are ξ -directional Tchbycheff constants associated to K .

At first let us compare Tchbycheff constants with harmonic capacities of a compact set in $\mathbf{C}^n$. For this we need the following result, which has been proved by V. P. Zaharyuta in the case of a L-regular compact set (i.e. a compact set K in $\mathbf{C}^n$ for which the function V_K is continuous), and then by J. Siciak in the general case.

Proposition 4.1. ([22], [17], [18]) *Let K be a compact set in $\mathbf{C}^n$.*

$$\sup_{\alpha \geq 1} \left\{ \sup \left\{ \frac{1}{\alpha} \log |p(z)| : p \in \mathcal{P}_\alpha, \|p\|_K \leq 1 \right\} \right\} = V_K(z), \quad \text{on } \mathbf{C}^n.$$

Proposition 4.2. *Let K be a compact set in $\mathbf{C}^n$.*

- (1) $C(K) = \tau(K)$.
- (2) *If K is L-regular, $\tau(K, \xi) = C_K(\bar{\xi})$, on $S(O, 1)$. Moreover, we have $\tau(K, \xi) \leq \tau^*(K, \xi)$ on $S(O, 1)$.*

Proof. The equality $C(K) = \tau(K)$ results from [16], [23] and [8]. Next, let us prove property (ii), by using properties of the orthogonal polynomials and Proposition 4.1.

Let us remark that $m_\alpha(K, \xi)^{-1} = \sup\{|\hat{p}(\xi)| : p \in \mathcal{P}_\alpha, \|p\|_K \leq 1\}$. Since $(\log |p|)/\alpha \leq g_K$ on $\mathbf{C}^n$, for all $p \in \mathcal{P}_\alpha$ such that $\|p\|_K \leq 1$, we deduce that $\tau(K, \xi) \geq C_K(\bar{\xi})$.

Moreover, there exists a sequence (B_k) of orthonormal polynomials in the Hilbert space $L^2(K)$, which satisfies Theorem 2 in [23]. Then $\forall \varepsilon > 0$, there exists

a strictly increasing sequence $(k_j)_j$ of integers such that $|\hat{B}_{k_j}(\bar{\xi})| \geq e^{(\gamma_K(\bar{\xi})-\varepsilon)d_j} \forall j$, where d_j is the degree of the polynomial B_{k_j} . According to Bernstein-Markov's inequality, there exists a constant $c_\varepsilon > 0$ such that $\|B_{k_j}\|_K \leq c_\varepsilon(1+\varepsilon)^{d_j} \forall j$. If we note $P_j = c_\varepsilon^{-1}(1+\varepsilon)^{-d_j}B_{k_j}$, then $\|P_j\|_K \leq 1$. We deduce that $m_{d_j}(K, \xi)^{-1} \geq |\hat{P}_j(\bar{\xi})| \geq c_\varepsilon^{-1}(1+\varepsilon)^{-d_j}e^{(\gamma_K(\bar{\xi})-\varepsilon)d_j}$ and $m_{d_j}(K, \xi)^{1/d_j} \leq c_\varepsilon^{1/d_j}(1+\varepsilon)e^{-\gamma_K(\bar{\xi})+\varepsilon}$. Finally, letting j go to infinity and ε go to 0 in the inequality above, we obtain that $\tau(K, \xi) \leq C_K(\xi)$. $\square$

Remark 4.3. We can prove that $C(K) = \inf\{C_K(\xi) : \xi \in S(O, 1)\}$ and $\tau(K) = \inf\{\tau(K, \xi) : \xi \in S(O, 1)\}$. Therefore we can derive (i) from (ii) in the previous proposition, in the L -regular case.

Let us recall some information about analytic functionals [12]. Let K be a compact set in the n -dimensional complex projective space $\mathbf{CP}^n$. We denote by K^* its dual open set in $(\mathbf{CP}^n)^*$, defined by $K^* = \{\tau \in (\mathbf{CP}^n)^* : \langle z, \tau \rangle = \sum_{j=0}^n z_j \tau_j \neq 0, \forall z \in K\}$, where $[z_0 : \dots : z_n]$ and $[\tau_0 : \dots : \tau_n]$ are the homogeneous coordinates of $z \in \mathbf{CP}^n$ and $\tau \in (\mathbf{CP}^n)^*$, respectively. Geometrically K^* corresponds to the set of all complex hyperplanes in $\mathbf{CP}^n$ not intersecting K . A compact set K in $\mathbf{CP}^n$ is called linearly convex if through every point z of the complement of K , it passes at least one complex hyperplane not intersecting K .

If U is an open set in $\mathbf{CP}^n$, we denote by $H(U)$ the space of holomorphic functions in U , homogeneous of degree 0. It is a Fréchet space endowed with the topology of the uniform convergence on every compact set in U . If K is a compact set in $\mathbf{CP}^n$, we note $H(K)$ the space of holomorphic functions in an open neighbourhood of K . The topology on $H(K)$ is defined by the inductive limit $H(K) = \varinjlim H(U)$, where U is any open neighbourhood of K . We denote by $H'(K)$ (respectively $H'(U)$) the topological dual space of $H(K)$ (respectively $H(U)$). An element in $H'(K)$ is called an analytic functional on K .

Let K be a compact set in $\mathbf{CP}^n$ non-intersecting the hyperplane at infinity $\{z \in \mathbf{CP}^n : z_0 = 0\}$. We define the Fantappiè transform $\mathcal{F}\mu$ of an analytic functional $\mu \in H'(K)$, by

$$\mathcal{F}\mu(\tau) = \mu_z(\tau_0 z_0 / \langle z, \tau \rangle) \quad \forall \tau \in K^*.$$

This function is analytic on K^* and is equal to zero on the hyperplane at infinity in $(\mathbf{CP}^n)^*$ $\{\tau \in (\mathbf{CP}^n)^* : \tau_0 = 0\}$. The mapping $\mathcal{F} : H'(K) \rightarrow H_0(K^*)$, where $H_0(K^*)$ is the closed subspace of $H(K^*)$ consisting of the functions equal to zero on the hyperplane at infinity, is linear and continuous.

Let K be a compact set in $\mathbf{C}^n$. We define the Fourier-Laplace transform $\hat{\mu}$ of an analytic functional $\mu \in H'(K)$ by

$$\hat{\mu}(p) = \mu(e^{ip \cdot z}) \quad \forall p \in \mathbf{C}^n.$$

($\hat{\mu}$ is an entire function.)

In what follows, we will consider only compact sets K in $\mathbf{CP}^n$ containing the point $P_0 = [1 : 0 : \dots : 0]$ and included in $\mathbf{CP}^n \setminus \{z_0 = 0\} \sim \mathbf{C}^n$. Therefore, K^* is included in $(\mathbf{CP}^n)^* \setminus \{\tau_0 = 0\} \sim \mathbf{C}^{n*}$ and contains the point $P_0^* = [1 : 0 : \dots : 0]$. We write $K^* = \{\tau \in \mathbf{C}^{n*} : 1 + \tau \cdot z \neq 0, \forall z \in K\}$, where $\tau \cdot z = \sum_{j=1}^n \tau_j z_j$. Under these conditions, the Fantappiè transform of an analytic functional μ defined on K , is the function $\mathcal{F}\mu$ of n complex variables defined by $\mathcal{F}\mu(\tau_1, \dots, \tau_n) = \mu_z((1 + \tau \cdot z)^{-1})$.

Definition 4.4. Let K be a linearly convex compact set in $\mathbf{CP}^n$ non-intersecting the hyperplane at infinity. We say that K is a Martineau compact if the Fantappiè transform $\mathcal{F}$ is an isomorphism from $H'(K)$ to $H_0(K^*)$.

Martineau's main result in [12] is: Each convex compact set in $\mathbf{C}^n$ is a Martineau compact. For $n = 1$, all compact sets in $\mathbf{C}$ are Martineau compacta, according to C. L. da Silva, A. Grothendieck and G. Köthe.

Definition 4.5. A linearly convex compact set K in $\mathbf{C}^n$ is a regular Martineau compact if it has a bounded hyperconvex dual domain K^* in $\mathbf{C}^{n*}$ and if it satisfies the following property: there exists an increasing sequence (K_j) of Martineau compacta in $\mathbf{C}^n$ such that $\bigcup_j K_j = \overset{\circ}{K}$. In addition, we assume that (K_j^*) is a decreasing sequence of bounded hyperconvex domains in $\mathbf{C}^{n*}$, satisfying properties (1) and (2), with $\bar{K}^* = \bigcap_j K_j^*$.

Now let us compare a Tchétcheff constant of a compact set and the harmonic capacity of its dual domain. For this, we need to introduce a new analytic capacity associated to the dual domain. Let K be a compact set in $\mathbf{C}^n$ containing O . If μ is an analytic functional on K , then we note $\|\mu\|_0 = \sup\{|\mu(h)| : h \in H(K), \|h\|_K \leq 1\}$ ($\|\mu\|_0$ can be infinite). Moreover, we verify that $\forall \tau \in K^*$ and $\forall \sigma \in \mathbf{N}^n$, $D^{(\sigma)}\mathcal{F}\mu(\tau) = (-1)^{|\sigma|}|\sigma|!\mu(z^\sigma/(1 + z \cdot \tau)^{|\sigma|+1})$. We deduce that $D^{(\sigma)}\mathcal{F}\mu(O) = (-1)^{|\sigma|}|\sigma|!\mu(z^\sigma)$, $\forall \sigma \in \mathbf{N}^n$ and $d_\xi^{(\alpha)}(\mathcal{F}\mu)(0)/\alpha! = (-1)^\alpha \mu((z \cdot \xi)^\alpha)$, $\forall \alpha \in \mathbf{N}$. Let us define for all integer $\alpha \geq 1$, the subspace $\mathcal{F}_\alpha(K)$ of $H'(K)$ by $\mathcal{F}_\alpha(K) = \{\mu \in H'(K) : \|\mu\|_0 \leq 1, \mu(z^\nu) = 0 \forall \nu \in \mathbf{N}^n/|\nu| \leq \alpha - 1\}$. We denote by $\tilde{\mathcal{E}}_\alpha(K^*)$ its image in $H(K^*)$ by the Fantappiè transform $\mathcal{F}$. Then we define the function $\tilde{h}_\alpha$ on K^* and the ξ -directional analytic capacity of order α $\tilde{\gamma}_\alpha(D, \xi)$ on $S(O, 1)$ by:

$$\begin{aligned} \tilde{h}_\alpha &= \sup_{\mu \in \mathcal{F}_\alpha(K)} \frac{1}{\alpha} \log |\mathcal{F}\mu|, \\ \tilde{\gamma}_\alpha(K^*, \xi) &= \left\{ \sup_{\mu \in \mathcal{F}_\alpha(K)} |\mu((\xi \cdot z)^\alpha)|^{1/\alpha} \right\}^{-1} \\ &= \left\{ \sup_{\mu \in \mathcal{F}_\alpha(K)} \left| \frac{1}{\alpha!} d_\xi^{(\alpha)}(\mathcal{F}\mu)(0) \right|^{1/\alpha} \right\}^{-1}. \end{aligned}$$

Property 4.6. *Let K be a compact set in $\mathbf{C}^n$ containing O . For all $\xi \in S(O, 1)$, the sequence of the ξ -directional analytic capacities $(\tilde{\gamma}_\alpha(K^*, \xi))$ converges to $\tau^*(K, \xi)^{-1}$.*

Proof. We are going to express $\tilde{\gamma}_\alpha(K^*, \xi)$ through $m_\alpha^*(K, \xi)$. Let $\mu \in \mathcal{F}_\alpha(K)$ and Q be any polynomial in $\mathcal{P}_\alpha^*(\xi)$ such that $Q(z) = \sum_{|\nu| \leq \alpha-1} c_\nu z^\nu + c_\alpha (z \cdot \xi)^\alpha$. Since $\mu(z^\sigma) = 0 \ \forall \sigma \in \mathbf{N}^n / |\sigma| \leq \alpha - 1$, we get that $\mu(Q) = c_\alpha \mu((z \cdot \xi)^\alpha)$. Thus, $\sup\{|\mu(Q)| : Q \in \mathcal{P}_\alpha^*(\xi), \|Q\|_K \leq 1\} = \tilde{m}_\alpha^*(K, \xi) |\mu((z \cdot \xi)^\alpha)|$. We denote by T the restriction of the analytic functional μ to the subspace $\mathcal{P}_\alpha^*(\xi)$. By Hahn-Banach's Theorem, there exists a continuous linear form μ' on $H_u(K)$, the space of holomorphic functions on a neighbourhood of K endowed with the topology of the uniform convergence on K , such that its restriction to $\mathcal{P}_\alpha^*(\xi)$ is T and such that its norm $\|\mu'\|_0$ is equal to $\tilde{m}_\alpha^*(K, \xi) |\mu((z \cdot \xi)^\alpha)|$. Since the topology of the uniform convergence on K is coarser than the inductive topology in $H(K)$, μ' is an analytic functional on K .

We have $\mu(Q) = \mu'(Q) \ \forall Q \in \mathcal{P}_\alpha^*(\xi)$, thus $\mu(z^\sigma) = \mu'(z^\sigma) = 0 \ \forall \sigma \in \mathbf{N}^n / |\sigma| \leq \alpha - 1$ and $|\mu((z \cdot \xi)^\alpha)| = |\mu'((z \cdot \xi)^\alpha)| = \|\mu'\|_0 \tilde{m}_\alpha^*(K, \xi)^{-1}$. Consequently,

$$\begin{aligned} \sup\{|\mu'((z \cdot \xi)^\alpha)| : \mu' \in \mathcal{F}_\alpha(K)\} &= \sup\{|\mu((z \cdot \xi)^\alpha)| : \mu \in \mathcal{F}_\alpha(K, \xi)\} \\ &= \tilde{m}_\alpha^*(K, \xi)^{-1}, \end{aligned}$$

where $\mathcal{F}_\alpha(K, \xi) = \{\mu \in H'(K), \mu(z^\sigma) = 0 \ \forall \sigma \in \mathbf{N}^n / |\sigma| \leq \alpha - 1, \sup\{|\mu(Q)| : Q \in \mathcal{P}_\alpha^*(\xi), \|Q\|_K \leq 1\} \leq 1\}$ and $\tilde{\gamma}_\alpha(K, \xi) = m_\alpha^*(K, \xi)^{-1/\alpha}$. $\square$

In the next Proposition 4.7, we are going to compare the limit of this sequence $(\tilde{\gamma}_\alpha(K^*, \xi))$ with the ξ -directional harmonic capacity $C_{K^*}(\xi)$ of the domain K^* , when K^* is a bounded hyperconvex domain in $\mathbf{C}^{n*}$ containing O . Then, Theorem 2 will be a direct consequence of Propositions 4.2 and 4.7 and Property 4.6.

Proposition 4.7. *Let K be a compact set in $\mathbf{C}^n$ such that its dual open set K^* is a bounded hyperconvex domain in $\mathbf{C}^{n*}$. Then, we have:*

$$\begin{aligned} \text{(i)} \quad & \limsup_{\alpha \rightarrow \infty} \tilde{h}_\alpha \leq g_{K^*} \quad \text{on } K^*, \\ & \lim_{\alpha \rightarrow \infty} \tilde{\gamma}_\alpha(K^*, \xi) \geq C_{K^*}(\xi) \quad \forall \xi \in S(O, 1). \end{aligned}$$

(ii) *Moreover if K is a regular Martineau compact, we get*

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \tilde{h}_\alpha &= g_{K^*}, \quad \text{on } K^*, \\ \lim_{\alpha \rightarrow \infty} \tilde{\gamma}_\alpha(K^*, \xi) &= C_{K^*}(\xi), \quad \text{quasi-everywhere in } S(O, 1). \end{aligned}$$

Proof. First let us note for any $\delta > 0$, $K_\delta^* = \{p \in K^* : g_{K^*}(p) < -\delta\}$ and $K_\delta = (K_\delta^*)^*$. K_δ^* is a hyperconvex domain in $\mathbf{C}^{n*}$ containing O , relatively compact in K^* and with pluricomplex Green function $g_{K^*} + \delta$ with logarithmic pole at O . K_δ is compact in $\mathbf{C}^n$ and contains K . For $\delta = 0$, $K_0^* = K^*$ and $K_0 = K$, if K is linearly convex. In addition, we have a family of semi-norms defined on $H'(K)$ by $\|\mu\|_\delta = \sup\{|\mu(h)| : h \in H(K_\delta), \|h\|_{K_\delta} \leq 1\}$, where δ is a nonnegative real number and $\|h\|_{K_\delta} = \sup\{|h(z)| : z \in K_\delta\}$. For ε and $\delta \in \mathbf{R}$ such that $0 \leq \varepsilon < \delta$, let $\text{dist}(K_\varepsilon, K_\delta^*) = \inf\{|1 + z \cdot p| : z \in K_\varepsilon, p \in K_\delta^*\}$, which is > 0 .

Let K be a compact set as in (i). By the definition of the Fantappi transform $\mathcal{F}$, we have for any $\mu \in H'(K)$, $|\mathcal{F}\mu(p)| \leq \|\mu\|_\varepsilon / \text{dist}(K_\varepsilon, K_\delta^*)$ on K_δ^* , $\forall (\varepsilon, \delta) \in \mathbf{R}^2 / O \leq \varepsilon < \delta$. Thus $\tilde{\mathcal{E}}_\alpha(K^*) \subset (\text{dist}(K_\varepsilon, K_\delta^*))^{-1} \mathcal{E}_\alpha(K_\delta^*)$ and, due to the inequality (5), we get

$$(*) \quad \tilde{h}_\alpha \leq -\frac{1}{\alpha} \log(\text{dist}(K, K_\delta^*)) + g_{K^*} + \delta, \quad \text{on } K_\delta^*.$$

Consequently, letting α go to infinity and δ go to 0, we get $\limsup_\alpha \tilde{h}_\alpha \leq g_{K^*}$ on K^* .

Since $\mathcal{F}_\alpha(K)$ is compact in $H'(K)$ and $\mathcal{F}$ is continuous, $\tilde{\mathcal{E}}_\alpha(K^*)$ is a compact set in $H(K^*)$ and $\lim_{\lambda \rightarrow 0} (\tilde{h}_\alpha(\lambda\xi) - \log|\lambda|) = -\log \tilde{\gamma}_\alpha(K^*, \xi)$, $\forall \xi \in S(O, 1)$. From inequality (*), we deduce that

$$\tilde{\gamma}_\alpha(K^*, \xi) \geq \text{dist}(K, K_\delta^*)^{1/\alpha} C_{K^*}(\xi) e^{-\delta}, \quad \forall \delta > 0.$$

Now let us assume that the compact set K is a regular Martineau compact. For all j , the Fantappi transform $\mathcal{F}$ is an isomorphism from $H'(K_j)$ to $H(K_j^*)$. For all $\alpha \geq 1$, denote by $H'_\alpha(K_j)$ the compact set in $H'(K_j)$ defined by $H'_\alpha(K_j) = \mathcal{F}^{-1}(\mathcal{E}_\alpha(K_j^*))$ ($\mathcal{F}^{-1}$ is continuous and $\mathcal{E}_\alpha(K_j^*)$ is compact in $H(K_j^*)$). Since $K_j \mathbf{b} K$, there exists a constant $c_\alpha(j) > 0$ such that $\sup_{\mu \in H'_\alpha(K_j)} \|\mu\|_0 = c_\alpha(j) < +\infty$ and $H'_\alpha(K_j) \subset c_\alpha(j) \mathcal{F}_\alpha(K)$. In addition, $H'_{\alpha+1}(K_j) \subset H'_\alpha(K_j)$ and $c_{\alpha+1}(j) \leq c_\alpha(j)$; and, in the same way, $H'_\alpha(K_j) \subset H'_\alpha(K_{j+1})$ and $c_\alpha(j) \leq c_\alpha(j+1)$. Since $\mathcal{F}$ is surjective, we have $\mathcal{E}_\alpha(K_j^*) \subset c_\alpha(j) \tilde{\mathcal{E}}_\alpha(K^*)$ and $h_{\alpha,j} \leq (\log c_\alpha(j))/\alpha + \tilde{h}_\alpha$ on K^* , where the function $h_{\alpha,j}$ is equal to $\sup\{(\log|f|)/\alpha : f \in \mathcal{E}_\alpha(K_j^*)\}$. Since the sequence $(c_\alpha(j))_\alpha$ decreases, we get

$$(**) \quad h_{\alpha,j} \leq \frac{1}{\alpha} \log c_1(j) + \tilde{h}_\alpha, \quad \text{on } K^*.$$

Letting α go to infinity and according to Theorem 1, we obtain that $g_{K^*} \leq \liminf_\alpha \tilde{h}_\alpha$ on K^* . Next, letting j go to infinity and according to the property (1), we get that $g_{K^*} \leq \liminf_\alpha \tilde{h}_\alpha$ on K^* . Consequently, in view of (i) we

derive that the sequence $(\tilde{h}_\alpha)$ converges pointwise to g_{K^*} on K^* . Moreover, from inequality $(**)$ we obtain

$$\gamma_\alpha(K_j^*, \xi) \geq c_1(j)^{-1/\alpha} \tilde{\gamma}_\alpha(K^*, \xi).$$

Since a countable union of pluripolar sets is again pluripolar [1], by Theorem 1 and by letting α next j go to infinity, we get that $\lim_\alpha \tilde{\gamma}_\alpha(K^*, \xi) \leq C_{K^*}(\xi)$, quasi-everywhere in $S(O, 1)$. With property (i), we conclude. $\square$

Property 4.8. *If K is a closed polydisk in $\mathbf{C}^n$ with center at O , then there exists ξ in $S(O, 1)$ such that $C_{K^*}(\xi) < C_K(\bar{\xi})^{-1}$.*

Proof. Let $K = \bar{P}(O, R)$ in $\mathbf{C}^n$. Since K is convex, it is a Martineau compact. K is also L -regular; its generalized Green function g_K and its ξ -directional harmonic capacity $C_K(\xi)$ are defined by

$$\begin{aligned} g_K(z) &= \sup_{1 \leq k \leq n} \log^+(|z_k|/R_k) \quad \text{on } \mathbf{C}^n, \\ C_K(\xi) &= \left(\sup_{1 \leq k \leq n} (|\xi_k|/R_k) \right)^{-1} \quad \forall \xi \in S(O, 1) \end{aligned}$$

$\overset{\circ}{K} = \bigcup_{j \geq 1} \bar{P}(O, R^{(j)})$, where $R^{(j)} = (R_1 - 1/j, \dots, R_n - 1/j)$. For all j sufficiently large, $K_j = \bar{P}(O, R^{(j)})$ is a Martineau compact, since it is convex as K .

Let us denote by H and H_j the functions defined on $\mathbf{C}^{n*}$ by $H(\tau) = \sum_{k=1}^n R_k |\tau_k|$ and $H_j(\tau) = \sum_{k=1}^n (R_k - 1/j) |\tau_k|$. Then the dual open sets K^* and K_j^* are defined by $K^* = \{\tau \in \mathbf{C}^{n*} : H(\tau) < 1\}$ and $K_j^* = \{\tau \in \mathbf{C}^{n*} : H_j(\tau) < 1\}$. These bounded open sets are convex and thus hyperconvex. Their pluricomplex Green functions with logarithmic pole at O are respectively defined by $g_{K^*} = \log H$ on K^* and $g_{K_j^*} = \log H_j$ on K_j^* . The ξ -directional harmonic capacities $C_{K^*}(\xi)$ and $C_{K_j^*}(\xi)$ are respectively equal to $H(\xi)^{-1}$ and $H_j(\xi)^{-1} \forall \xi \in S(O, 1)$. Moreover, let us remark that properties (1) and (2) are satisfied by (K_j^*) and K^* . Consequently, K is a regular Martineau compact in $\mathbf{C}^n$.

Then we verify that for $\xi = (1/\sqrt{3}, \sqrt{2/3}, 0, \dots, 0)$ and R_1 and R_2 such that $R_2 > \sqrt{2}R_1$, $C_{K^*}(\xi) = \sqrt{3}/(R_1 + \sqrt{2}R_2) < C_K(\bar{\xi})^{-1} = 1/(\sqrt{3}R_1)$. $\square$

5. Polynomial approximations of the exponential function on a compact set in $\mathbf{C}^n$. First, let us estimate the Fourier-Laplace transform of an analytic functional.

If μ is an analytic functional on a compact set K in $\mathbf{C}^n$, then we denote by $c_\sigma(\mu)$ the complex numbers $\mu(z^\sigma)$ ($\sigma \in \mathbf{N}^n$), called momenta of μ . In addition, we use the same notations as in Section 4 (see the beginning of the proof of Proposition 4.7). The following proposition generalizes the theorem of G. Henkin and A. Shananin [5] and Proposition 1 of [13] in the case $n = 1$.

Proposition 5.1. *Let K be a compact set in $\mathbf{C}^n$.*

- (1) *Suppose that its dual open set K^* is a bounded hyperconvex domain in $\mathbf{C}^{n*}$. Then $\forall N \in \mathbf{N}^*$, $\forall \mu \in H'(K)$ / $c_\sigma(\mu) = 0$ if $|\sigma| \leq N$, $\forall \xi \in S(O, 1)$, $\forall \lambda \in \mathbf{C}$, $\forall (\tau, \delta) \in \mathbf{R}^2$ with $0 \leq \tau < \delta$ such that $\lambda \xi e^{i\theta} / (N + 1) \in K_\delta^*$ $\forall \theta \in \mathbf{R}$ and $|z \cdot \lambda \xi| < N + 1 \forall z \in K$, we have*

$$(10) \quad |\hat{\mu}(\lambda \xi)| \leq \frac{\|\mu\|_\tau}{\text{dist}(K_\tau, K_\delta^*)} \left\{ \frac{e^{\delta+1} |\lambda| (1 + \varphi'_\xi(\lambda / (N + 1)))}{(N + 1) C_{K^*}(\xi)} \right\}^{(N+1)}$$

The function φ'_ξ , which arises in the previous inequality, depends only on the compact set K and ξ .

- (2) *Suppose now that the compact set K is a Martineau compact. Then its dual open set K^* is connected and we have, $\forall \xi \in S(O, 1)$, $\forall \lambda \in \mathbf{C}$, $\forall N \in \mathbf{N}$, $\forall \alpha \in \mathbf{N}^*$ with $N = \alpha N' + i$, where $i \in \{0, \dots, \alpha - 1\}$ and $N' \in \mathbf{N}$ such that $\lambda \xi e^{i\theta} / (N + 1) \in K^* \forall \theta \in \mathbf{R}$, there exists an analytic functional $\mu \in H'(K)$ such that $c_\sigma(\mu) = 0$ for all multi-index $\sigma / |\sigma| \leq N$ and*

$$(11) \quad |\hat{\mu}(\lambda \xi)| \geq \frac{|\lambda|^{N+1} \cdot e^{-|d_\alpha \lambda|/\alpha}}{(N + 1)! \gamma_\alpha(K^*, \xi)^{\alpha N'} \cdot \gamma_{i+1}(K^*, \xi)^{i+1}} \left(1 - O\left(\frac{|\lambda|^2}{\sqrt{N + 1}}\right) \right).$$

The absolute value of $O(|\lambda|^2 / \sqrt{N + 1})$ in the previous inequality is bounded above by $C|\lambda|^2 / \sqrt{N + 1}$, where C is a constant depending only on the compact set K , the integers α , i , and ξ .

Proof. Part A of this proposition is similar to Part A of the theorem in [5]. The goal is to estimate $|\mathcal{F}\mu|$ and next to estimate $|\hat{\mu}|$, by using an integral formula for $\hat{\mu}$. Part B of this proposition is inspired by Part B of the theorem in [5], but instead of using the ξ -directional harmonic capacity of the dual open set K^* , we introduce the ξ -directional analytic capacities of the same open set.

There exists a function $F_\alpha \in \mathcal{E}_\alpha(K^*)$ and a function $F_{i+1} \in \mathcal{E}_{i+1}(K^*)$ such that $d_\xi^{(\alpha)} F_\alpha(O) / \alpha! = \gamma_\alpha(K^*, \xi)^{-\alpha}$ and $d_\xi^{(i+1)} F_{i+1}(O) / (i+1)! = \gamma_{i+1}(K^*, \xi)^{-(i+1)}$. Then if we denote by Φ the function $(F_\alpha)^{N'} F_{i+1}$, $\Phi \in \mathcal{E}_{N+1}(K^*)$ and

$$d_\xi^{(N+1)} \Phi(0) / (N + 1)! = \gamma_\alpha(K^*, \xi)^{-\alpha N'} \gamma_{i+1}(K^*, \xi)^{-(i+1)}.$$

Also, for any λ and ξ as in the statements and for $t \in \mathbf{C}$ such that $|t| = N + 1$, we have

$$\begin{aligned} \Phi(\lambda\xi/t) &= \gamma_\alpha(K^*, \xi)^{-\alpha N'} \gamma_{i+1}(K^*, \xi)^{-(i+1)} \left(\frac{\lambda}{t}\right)^{N+1} \\ &\quad \cdot \left[\left(1 + \frac{d_\alpha \lambda}{t}\right)^{N'} \left(1 + \frac{d_{i+1} \lambda}{t}\right) + O\left(\frac{|\lambda|^2}{N+1}\right) \right], \end{aligned}$$

where $O(|t|)$ with $t \in \mathbf{C}$, depends on α and i but doesn't depend neither on λ , nor on N sufficiently large. Since K is a Martineau compact, there exists an unique analytic functional μ on K such that its Fantappi transform $\mathcal{F}\mu$ is Φ . Since $\Phi \in \mathcal{E}_{N+1}(K^*)$, $c_\sigma(\mu) = 0 \ \forall \sigma \in \mathbf{N}^n / |\sigma| \leq N$.

We are going to use the following formula, which expresses $\hat{\mu}$ by means of $\mathcal{F}\mu$ for any analytic functional μ on K . Let $p \in \mathbf{C}^n$ and R_0 be a positive real number such that $p/t \in K^* \ \forall t \in S(R_0) = \{t \in \mathbf{C} : |t| = R_0\}$ and $|z \cdot p| < R_0 \ \forall z \in K$. We have:

$$\hat{\mu}(p) = \frac{1}{2i\pi} \int_{S(R_0)} \frac{e^{-it}}{t} \cdot \mathcal{F}\mu(p/t) dt.$$

We prove this formula with the help of Cauchy formula in one variable [5]. Then, applying it to $p = \lambda\xi$ and $R_0 = N + 1$, we obtain $\hat{\mu}(\lambda\xi) = J_1 + J_2$, where

$$\begin{aligned} J_1 &= \frac{1}{2i\pi} \int_{S(N+1)} \frac{e^{-it}}{t} \gamma_\alpha(K^*, \xi)^{-\alpha N'} \gamma_{i+1}(K^*, \xi)^{-i-1} \cdot \left(\frac{\lambda}{t}\right)^{N+1} \\ &\quad \cdot \left(1 + d_\alpha \frac{\lambda}{t}\right)^{N'} \left(1 + d_{i+1} \frac{\lambda}{t}\right) dt, \\ J_2 &= \frac{1}{2i\pi} \int_{S(N+1)} \frac{e^{-it}}{t} \gamma_\alpha(K^*, \xi)^{-\alpha N'} \\ &\quad \cdot \gamma_{i+1}(K^*, \xi)^{-i-1} \left(\frac{\lambda}{t}\right)^{N+1} O\left(\frac{|\lambda|^2}{N+1}\right) dt. \end{aligned}$$

If we compute J_1 by the residus formula and note $P(\ell, N) = N'(N'-1) \dots (N'-\ell+1)/(N+\ell+1)(N+\ell) \dots (N+2) \ \forall \ell \in \{1, \dots, N'\}$ and $P(0, N) = 1$, we obtain

$$\begin{aligned} |J_1| &= \gamma_\alpha(K^*, \xi)^{-\alpha N'} \cdot \gamma_{i+1}(K^*, \xi)^{-i-1} \frac{|\lambda|^{N+1}}{(N+1)!} \\ &\quad \left| \sum_{\ell=0}^{N'} \frac{(-id_\alpha \lambda)^\ell}{\ell!} P(\ell, N) \left(1 - \frac{id_{i+1} \lambda}{N+\ell+2}\right) \right|. \end{aligned}$$

The expression

$$\Sigma = \left| \sum_{\ell=0}^{N'} \frac{(-id_{\alpha}\lambda)^{\ell}}{\ell!} P(\ell, N) \left(1 - \frac{id_{i+1}\lambda}{N+l+2} \right) \right|$$

has a lower bound $[1 - O(|\lambda|/(N+1))]e^{-|d_{\alpha}\lambda|/\alpha}$, where $|O(|t|)| \leq D|t|$ in a neighbourhood of 0. The constant D depends on α and i , but doesn't depend on N sufficiently large. In addition $|J_2|$ satisfies the following inequality:

$$|J_2| \leq \gamma_{\alpha}(K^*, \xi)^{-\alpha N'} \gamma_{i+1}(K^*, \xi)^{-i-1} \left(\frac{e|\lambda|}{N+1} \right)^{N+1} O \left(\frac{|\lambda|^2}{N+1} \right).$$

Consequently, we get

$$\begin{aligned} |\hat{\mu}(\lambda\xi)| &\geq |J_1| - |J_2| \\ &\geq \gamma_{\alpha}(K^*, \xi)^{-\alpha N'} \gamma_{i+1}(K^*, \xi)^{-i-1} \frac{|\lambda|^{N+1}}{(N+1)!} \\ &\quad \left[\left(1 - O \left(\frac{|\lambda|}{N+1} \right) \right) e^{-|d_{\alpha}\lambda|/\alpha} - \frac{(N+1)!}{(N+1)^{N+1}} e^{N+1} O \left(\frac{|\lambda|^2}{N+1} \right) \right]. \end{aligned}$$

And according to Stirling's formula, we finally obtain inequality (11). □

The following proposition, which is a consequence of Proposition 5.1, and Theorem 1 gives us Theorem 3.

Proposition 5.2. *Let K be a compact set in $\mathbf{C}^n$.*

(A) *If K^* is a bounded hyperconvex domain then we have for every $\xi \in S(O, 1)$, $\lambda \in \mathbf{C}$ and $\delta \geq 0$:*

$$(12) \quad \limsup_{N \rightarrow +\infty} N \cdot E_N^{1/N}(K_{\delta}, e^{\lambda\xi \cdot z}) \leq e^{\delta+1} |\lambda| C_{K^*}(\xi)^{-1}.$$

(B) *If K is a Martineau compact in $\mathbf{C}^n$, then we have for all compact set K_0 in $\mathbf{C}^n$ such that $K \subset K_0$, for every $p \in \mathbf{N}^*$, $\xi \in S(O, 1)$ and $\lambda \in \mathbf{C}$:*

$$(13) \quad \liminf_{N \rightarrow +\infty} N \cdot E_N^{1/N}(K_0, e^{\lambda\xi \cdot z}) \geq e|\lambda| \gamma_p(K^*, \xi)^{-1}.$$

Let us remark that for $n = 1$, the first inequality results from [20] and the second one from [13].

Proof. First, let us prove Part A. Let $\delta \geq 0$ be fixed and $H_u(K_\delta)$ be the complex vector space of holomorphic functions on a neighbourhood of K_δ , provided with the topology of uniform convergence on K_δ . Let f be the entire function defined by $f(z) = e^{z \cdot \lambda \xi}$. $f \notin \mathcal{P}_N$ and $E_N(K_\delta, f) > 0$, $\forall N \geq 0$. Let Φ_1 be the continuous linear functional defined on the vector space $V(f, \mathcal{P}_N)$ (the subspace of $H_u(K_\delta)$ generated by the function f and the subspace $\mathcal{P}_N$) by $\Phi_1(tf + p) = t$, $\forall p \in \mathcal{P}_N$ and $\forall t \in \mathbf{C}$. Then $\|\Phi_1\|_{V'(f, \mathcal{P}_N)} = \sup\{|\Phi_1(g)| : g \in V(f, \mathcal{P}_N), \|g\|_{K_\delta} \leq 1\} = \sup\{(\|p/t + f\|_{K_\delta})^{-1} : p \in \mathcal{P}_N, p \neq 0, t \in \mathbf{C} \setminus \{0\}\} = E_N(K_\delta, f)^{-1}$. By Hahn-Banach's theorem, this linear functional Φ_1 can be extended as a continuous linear functional Φ on $H_u(K_\delta)$, such that $\|\Phi\|_{H'_u(K_\delta)} = \|\Phi_1\|_{V'(f, \mathcal{P}_N)}$. Let μ be equal to $E_N(K_\delta, f)\Phi$. μ satisfies $\mu(f) = E_N(K_\delta, f)$, $\mu(p) = 0 \forall p \in \mathcal{P}_N$ and $\|\mu\|_{H'_u(K_\delta)} = \|\mu\|_\delta = 1$. Since the topology of the uniform convergence on K_δ is coarser than the inductive topology on $H(K_\delta)$, μ is an analytic functional on K_δ . Let us remark that $\mathcal{F}\mu \in H(K_\delta^*)$.

By the same way as for the part A of Proposition 5.1, we prove that $\forall N \in \mathbf{N}^*$, $\forall \xi \in S(O, 1)$, $\forall \lambda \in \mathbf{C}$, $\forall \varepsilon > 0$ such that $\lambda \xi e^{i\theta}/(N+1) \in K_{\delta+\varepsilon}^* \forall \theta \in \mathbf{R}$ and $|z \cdot \lambda \xi| < N+1 \forall z \in K$, $\hat{\mu}$ satisfies

$$|\hat{\mu}(-i\lambda\xi)| \leq \frac{\|\mu\|_\delta}{\text{dist}(K_\delta, K_{\delta+\varepsilon}^*)} \left[\frac{e^{\delta+\varepsilon+1} |\lambda| (1 + \varphi'_\xi(\lambda/(N+1)))}{(N+1)C_{K^*}(\xi)} \right]^{N+1}.$$

Since $\|\mu\|_\delta = 1$ and $\hat{\mu}(-i\lambda\xi) = \mu_z(e^{z \cdot \lambda \xi}) = E_N(K_\delta, f)$, we deduce that

$$\begin{aligned} N \cdot E_N(K_\delta, f)^{1/N} &\leq \\ \frac{N(N+1)^{-(N+1)/N}}{\text{dist}(K_\delta, K_{\delta+\varepsilon}^*)^{1/N}} &[e^{1+\delta+\varepsilon} |\lambda| (1 + \varphi'_\xi(\lambda/(N+1))) C_{K^*}(\xi)^{-1}]^{(N+1)/N}. \end{aligned}$$

Finally, letting N go to infinity and ε go to 0 in the previous inequality, we obtain (12).

Now let us prove Part B of this proposition. According to Proposition 5.1 B, we have : $\forall \xi \in S(O, 1)$, $\forall \lambda \in \mathbf{C}$, $\forall N \in \mathbf{N}$, $\forall \alpha \in \mathbf{N}^*$ with $N = \alpha N' + i$ where $i \in \{0, \dots, \alpha - 1\}$ and $N' \in \mathbf{N}$ such that $\lambda \xi e^{i\theta}/(N+1) \in K^* \forall \theta \in \mathbf{R}$, there exists an analytic functional μ on K such that $c_\sigma(\mu) = 0$ for all $\sigma \in \mathbf{N}^n / |\sigma| \leq N$ and satisfying inequality (11). Let us note $\|\mu\|_{K_0}^* = \sup\{|\mu(h)| : h \in H(K_0), \|h\|_{K_0} \leq 1\}$. Since $\hat{\mu}(-i\lambda\xi) = \mu_z(e^{z \cdot \lambda \xi}) = \mu_z(e^{z \cdot \lambda \xi} - p(z)) \forall p \in \mathcal{P}_N$, we deduce that $|\hat{\mu}(-i\lambda\xi)| \leq \|\mu\|_{K_0}^* \sup\{|e^{z \cdot \lambda \xi} - p(z)| : z \in K_0\}$, $\forall p \in \mathcal{P}_N$. Consequently,

$$E_N(K_0, f) \geq |\hat{\mu}(-i\lambda\xi)|/\|\mu\|_{K_0}^* \text{ and}$$

$$N \cdot E_N(K_0, f)^{1/N} \geq \frac{N|\lambda|^{(N+1)/N}}{((N+1)!)^{1/N}} \cdot \left[\frac{e^{-|d_\alpha \lambda|/\alpha}(1 - O(|\lambda|^2/\sqrt{N+1}))}{\|\mu\|_{K_0}^* \gamma_\alpha(K^*, \xi)^{\alpha N'} \gamma_{i+1}(K^*, \xi)^{i+1}} \right]^{1/N}.$$

Letting N go to infinity, we finally obtain the inequality (13). $\square$

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Institut de Mathématiques (UMR 9994 du C.N.R.S.)
Equipe d'Analyse Complexe
Université Pierre et Marie Curie
Tour 46-00—5^{ème} ét.—Boite 172
4, Place Jussieu—F-75252 Paris Cedex 05, France

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