Crossed Product Decompositions of a Purely Infinite Von Neumann Algebra with Faithful, Almost Periodic Weight

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ABSTRACT. For $\mathcal{M}$ a separable, purely infinite von Neumann algebra with almost periodic weight φ , Connes' decomposition of $\mathcal{M}$ as a crossed product of a semifinite von Neumann algebra by a trace-scaling action of a countable abelian group is described, and factoriality is related to ergodicity of a certain partial action. Then Takasaki's continuous decomposition of the same algebra is related to the above discrete decomposition via Takesaki's notion of induced action, but here one induces up from a dense subgroup. The above results are used to give a model for the one-parameter trace-scaling action of $\mathbf{R}_+^*$ on the injective II_∞ factor. Finally, another model of the same action, due to work of Aubert and explained by Jones, is described.

Introduction. A crucial part of the present-day understanding of type III factors is their decomposition as crossed products of type II_∞ von Neumann algebras by groups of trace-scaling (or trace-decreasing) automorphisms. This was accomplished by Connes in [4] and by Takesaki in [15]. Connes defined the classification of type III factors as type III_λ , $0 \leq \lambda \leq 1$ and showed that a type III_λ factor where $0 \leq \lambda < 1$ is isomorphic to the crossed product of a type II_∞ von Neumann algebra, $\mathcal{N}$, by a single automorphism (i.e., by the group $\mathbf{Z}$). When $\lambda > 0$, $\mathcal{N}$ can be chosen to be a factor and the automorphism trace-scaling, and in the case $\lambda = 0$ the automorphism is ergodic on the center of $\mathcal{N}$ and a trace can be chosen such that the automorphism is strictly decreasing of the trace.

Takesaki developed the theory of crossed products of a von Neumann algebra by actions of locally compact groups, including his duality theory and his theory of induced actions. He thereby proved the continuous decomposition for a factor $\mathcal{M}$: if $\mathcal{M}$ is type III_1 , then $\mathcal{M}$ is the crossed product of a type II_∞ factor $\mathcal{N}$ by a one-parameter group of trace-scaling automorphisms.

Almost periodic weights (the definition is reviewed in Section 1.3) were defined by Connes in [4] and can be used to elucidate the structure of certain type III_1 factors. Connes defined the invariant Sd for a full type III_1 factor in terms of its almost periodic weights in [6], where he also showed that there is a type III_1 factor having no almost periodic weights. However, many type III_1 factors of interest have them. For example, the injective type III_1 factor, which was shown to be unique by Haagerup [10], has many almost periodic weights, (cf Section 4). Also, the free products of certain finite dimensional algebras with respect to certain states that are not traces are known to be type III_1 factors (see [3], [7] and [8]) and the free product states on them are almost periodic.

In this paper, given a purely infinite, separable von Neumann algebra, $\mathcal{M}$, with almost periodic weight, φ , we recall how results of Connes imply that $\mathcal{M}$ can be written as a discrete crossed product of a type II_∞ von Neumann algebra $\mathcal{N}$ by the action of a countable abelian group.

The main contribution of this paper (Corollary 3.4) is to describe the continuous decomposition for $\mathcal{M}$ in terms of the above discrete decomposition via a straightforward generalization of Takesaki's definition of induced action (Definition 3.1). This generalization is to induce up from a possibly dense subgroup $H \subseteq K$ rather than just a closed subgroup. Of course, one must consider a topology on H , different from the restriction of the topology on K , with respect to which H is locally compact. One thereby obtains, when $\mathcal{M}$ is a type III_1 factor, a description of the continuous core type II_∞ factor and the one-parameter group action on it. This description will be interesting for the study free product factors of type III_1 , [8].

However, also in the injective case, a concrete description of the (unique by Haagerup [10]) one-parameter trace-scaling action on the (unique by Connes [5]) injective II_∞ factor is obtained. It was explained to the author by V.F.R. Jones that there is another description of this action, which will be described in the last section of this paper.

This paper has five sections. In Section 1, some preliminaries are briefly reviewed and our notation is explained, including Section 1.1, Takesaki's definition of crossed products; Section 1.2, the Arveson–Connes spectral analysis for a compact abelian group action; Section 1.3, Connes' compact group action on a von Neumann algebra with almost periodic weight. In Section 2, the discrete crossed product decomposition for a purely infinite von Neumann algebra with almost periodic weight is described, and factoriality is related to ergodicity of a certain partial action. In Section 3, Takesaki's continuous decomposition for a purely infinite von Neumann algebra with almost periodic weight is shown to arise via an induced action from the discrete action of Section 2. In Section 4, the previous results are used to give a concrete description of the one-parameter trace-scaling action on the injective II_∞ factor. In Section 5, the previously known description of this action is briefly described.

Acknowledgements. The work embodied in this paper was prompted by a desire to understand the one-parameter trace-scaling action on the injective II_∞ factor. This desire in turn stemmed from a discussion with M. Takesaki, for which I am grateful. I would also like to thank J. Feldman for helpful discussions and V.F.R. Jones for telling me about the model in Section 5.

1. Preliminaries and notation. Throughout this paper, all groups and Hilbert spaces will be separable and all von Neumann algebras will have separable preduals. Although these restrictions avoid measure theoretic difficulties, it is likely that similar results hold in more general circumstances.

1.1. Crossed products (Takesaki [15]). Let $\mathcal{M}$ be a von Neumann algebra and $\alpha : G \rightarrow \text{Aut}(\mathcal{M})$ a continuous action (Arveson [1]) of a locally compact group G on $\mathcal{M}$. (Henceforth, the word “action” will be used to mean a continuous action.) Suppose σ is a normal, faithful representation of $\mathcal{M}$ as operators on the Hilbert space $\mathcal{H}$. Consider the von Neumann algebra acting on $L^2(G) \otimes \mathcal{H}$ that is generated by

$$\{\lambda_g \otimes 1 \mid g \in G\} \cup \{\pi_{\alpha, \sigma}(a) \mid a \in \mathcal{M}\},$$

where $\lambda_g \xi(h) = \xi(g^{-1}h)$ for $\xi \in L^2(G)$ and $g, h \in G$, and where $\pi_{\alpha, \sigma}(a)\zeta(h) = \alpha_{h^{-1}}(a)\zeta(h)$ for $h \in G$ and $\zeta \in L^2(G, \mathcal{H})$, where $L^2(G, \mathcal{H})$ is the space of square-integrable, measurable functions from G into $\mathcal{H}$, and is identified with $L^2(G) \otimes \mathcal{H}$. The measure on G is left Haar measure. Takesaki proved that the choice of representation σ does not affect the isomorphism class, so that one may define the *crossed product* von Neumann algebra, which we will denote $\mathcal{M} \rtimes_\alpha G$, to be generated by elements denoted

$$\{\lambda_g \otimes 1 \mid g \in G\} \cup \{\pi_\alpha(a) \mid a \in \mathcal{M}\}$$

in such a way that if σ is a representation of $\mathcal{M}$ as above, then there is a faithful, normal representation of $\mathcal{M} \rtimes_\alpha G$ given by sending $\lambda_g \otimes 1$ to $\lambda_g \otimes 1$ and $\pi_\alpha(a)$ to $\pi_{\alpha, \sigma}(a)$.

For G abelian, Takesaki defined the *dual action* $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{M} \rtimes_\alpha G$ to be given by $\hat{\alpha}_\gamma(\lambda_g \otimes 1) = \overline{\langle g, \gamma \rangle} \lambda_g \otimes 1$ and $\hat{\alpha}_\gamma(\pi_\alpha(a)) = \pi_\alpha(a)$. *Takesaki duality* is the theorem that

$$(\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \simeq \mathcal{M} \otimes B(L^2(G)).$$

If $\mathcal{M}$ is a purely infinite von Neumann algebra with normal, faithful, semifinite weight φ , let σ^φ be the modular action of $\mathbf{R}$ on $\mathcal{M}$. Let $\mathbf{R}_+^*$ be the dual group of $\mathbf{R}$ with the pairing $\langle t, \mu \rangle = \mu^{it}$. Then by Takesaki duality, $\mathcal{M} \simeq \mathcal{M} \otimes B(L^2(\mathbf{R})) \simeq (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}) \rtimes_{\hat{\sigma}^\varphi} \mathbf{R}_+^*$. Takesaki showed that $\mathcal{M}_0 = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ is a type II_∞ von Neumann algebra that admits a n.f.s. trace τ satisfying

$\tau \circ \widehat{\sigma^\varphi}_\mu = \mu^{-1}\tau$, and that the W^* -dynamical system $(\mathcal{M}_0, \mathbf{R}_+^*, \widehat{\sigma^\varphi})$ does not depend on φ (up to isomorphism). We will refer to this dynamical system, giving rise to $\mathcal{M} \simeq \mathcal{M}_0 \rtimes_{\widehat{\sigma^\varphi}} \mathbf{R}_+^*$, as *Takesaki's continuous (crossed product) decomposition* of $\mathcal{M}$. If $\mathcal{M}$ is a type III₁ factor, then $\mathcal{M}_0$ is a factor and this is the crossed product decomposition of a type III₁ factor that was mentioned in the introduction.

1.2. The Arveson–Connes spectral analysis for a compact abelian group action (Arveson [1] and Connes [4]). Let G be a compact abelian group and let Γ denote its dual group, with pairing $\langle g, \gamma \rangle$, ($g \in G$, $\gamma \in \Gamma$). For $\gamma \in \Gamma$ the Arveson–Connes spectral subspace for an action, α , of G on $\mathcal{M}$ is denoted $\mathcal{M}^\alpha(\gamma)$. Hence $\mathcal{M}^\alpha(1_\Gamma)$ is the fixed-point subalgebra of $\mathcal{M}$ under α , also simply denoted $\mathcal{M}^\alpha$, and moreover

$$(1) \quad \mathcal{M}^\alpha(\gamma_1)\mathcal{M}^\alpha(\gamma_2) \subseteq \mathcal{M}^\alpha(\gamma_1\gamma_2)$$

and

$$(2) \quad \mathcal{M}^\alpha(\gamma)^* = \mathcal{M}^\alpha(\gamma^{-1}).$$

Also, for $a \in \mathcal{M}$ let

$$E_\gamma(a) = \int_G \langle g, \gamma \rangle \alpha_g(a) dg,$$

where dg is Haar measure.

Lemma 1.2.1. *For $a \in \mathcal{M}$ the following are equivalent:*

- (i) $a \in \mathcal{M}^\alpha(\gamma)$
- (ii) $\alpha_g(a) = \overline{\langle g, \gamma \rangle} a \ \forall g \in G$
- (iii) $E_\gamma(a) = a$.

Proof. The implication (i) $\Rightarrow$ (ii) is an easy case of Connes [4], Lemma 2.3.5 and (ii) $\Rightarrow$ (i) follows directly from the definition of $\mathcal{M}^\alpha(\gamma)$. The equivalence of (ii) and (iii) is easily verified. $\square$

Lemma 1.2.2. *E_γ is a weak*-continuous projection of norm 1 from $\mathcal{M}$ onto $\mathcal{M}^\alpha(\gamma)$. If $a \in \mathcal{M}$, $b \in \mathcal{M}^\alpha(\gamma_1)$ and $c \in \mathcal{M}^\alpha(\gamma_2)$ then*

$$(3) \quad E_\gamma(bac) = bE_{\gamma_1^{-1}\gamma\gamma_2^{-1}}(a)c.$$

Proof. In the notation of Connes [4] Section 2.1, $E_\gamma = \alpha(\langle \cdot, \gamma \rangle)$, hence is a weak* continuous linear mapping from $\mathcal{M}$ into $\mathcal{M}$, It is clearly of norm 1. That $E_\gamma \circ E_\gamma = E_\gamma$ is easily verified from the definition. That E_γ is onto $\mathcal{M}^\alpha(\gamma)$ follows from Lemma 1.2.1(iii). Equation (3) holds because $\alpha_g(bac) = \langle g, \gamma_1\gamma_2 \rangle b\alpha_g(a)c$. $\square$

It makes sense to think of $E_\gamma(a)$ as the γ th term in the Fourier series for a . To see that this generalizes the usual notion of Fourier series, let $\mathcal{M} = L^\infty(\mathbf{T})$, $G = \mathbf{T}$ and $\alpha_z = \text{rotation by } z$.

Lemma 1.2.3. *The linear span of $\bigcup_{\gamma \in \Gamma} \mathcal{M}^\alpha(\gamma)$ is a strongly dense $*$ -subalgebra of $\mathcal{M}$.*

Proof. From (1) and (2) it is clear that $\bigcup \mathcal{M}^\alpha(\gamma)$ is a $*$ -subalgebra of $\mathcal{M}$. Suppose $a \in \mathcal{M}$ is such that $\text{Sp}_\alpha(a)$ is a finite subset of Γ . Consider the function on G , $f(g) = \sum_{\gamma \in \text{Sp}_\alpha(a)} \langle g, \gamma \rangle$. Then $\alpha(f)a = \sum_{\gamma \in \text{Sp}_\alpha(a)} E_\gamma(a)$, and since the Fourier transform $\hat{f}$ is identically 1 on $\text{Sp}_\alpha(a)$ and zero elsewhere, it follows from Connes [4] Lemma 2.1.3(b) that $a = \sum E_\gamma(a) \in \text{span} \bigcup_{\gamma \in \Gamma} \mathcal{M}^\alpha(\gamma)$. But the collection of $a \in \mathcal{M}$ such that $\text{Sp}_\alpha(a)$ is finite is by Connes [4] Lemma 2.1.4 strongly dense in $\mathcal{M}$. $\square$

1.3. Connes' compact group action associated to an almost periodic weight (Connes [6]).

Definition 1.3.1. (Connes [4]). A normal, faithful, semifinite (n.f.s.) weight φ on a von Neumann algebra $\mathcal{M}$ is *almost periodic* if the modular operator Δ_φ on $L^2(\mathcal{M}, \varphi)$ is diagonalizable, i.e., the set of eigenvectors of Δ_φ has dense linear span in $L^2(\mathcal{M}, \varphi)$.

In this paper, when we say φ is almost periodic we will always exclude the trivial case when the point spectrum of Δ_φ is $\{1\}$, i.e., when φ is a trace.

Let φ be an almost periodic n.f.s. weight on a von Neumann algebra $\mathcal{M}$. Let $\Gamma \subseteq \mathbf{R}_+^*$ be the subgroup generated by the point spectrum of Δ_φ and endowed with the discrete topology. Let $\hat{\iota} : \Gamma \hookrightarrow \mathbf{R}_+^*$ denote the inclusion map. Let G be the compact abelian group whose dual is Γ , with pairing denoted as in Section 1.2. Also, consider $\mathbf{R}_+^*$ to be the dual of $\mathbf{R}$ under the pairing $\langle t, \mu \rangle = \mu^{it}$ for $t \in \mathbf{R}$, $\mu \in \mathbf{R}_+^*$. Then there is a group homomorphism $\iota : \mathbf{R} \rightarrow G$ determined by $\langle \iota(t), \gamma \rangle = \langle t, \hat{\iota}(\gamma) \rangle \forall \gamma \in G, t \in \mathbf{R}$. Note since $\hat{\iota}$ is injective that the image of ι is dense in G and also that ι is injective if and only if Γ is dense in $\mathbf{R}_+^*$.

Lemma 1.3.2. (Connes [6], Proposition 1.1). *There is an action α of G on $\mathcal{M}$ such that*

- (i) $\varphi \circ \alpha_g = \varphi \forall g \in G$
- (ii) $\alpha_{\iota(t)} = \sigma_t^\varphi \forall t \in \mathbf{R}$,

where σ^φ is the modular automorphism group associated to φ . The action above is thus given by

$$\alpha_g(a) = \langle g, \hat{\iota}(\gamma) \rangle a \quad \forall a \in \mathcal{M}_\varphi(\gamma) = \mathcal{M}^\alpha(\gamma).$$

The following lemma is standard and well known.

Lemma 1.3.3. *Let $a \in \mathcal{M}$ and $\gamma \in \Gamma$. Then a belongs to the spectral subspace $\mathcal{M}^\alpha(\gamma)$ if and only if*

$$\varphi(ba) = i(\gamma)\varphi(ab) \quad \forall b \in \mathcal{M}.$$

Proof. This can be proved using the KMS condition exactly like in Takesaki [14] Lemma 1.6, or see Connes [4] Lemma 3.7.5. $\square$

The discrete decomposition. In this section, we describe the discrete decomposition of a purely infinite von Neumann algebra with respect to an almost periodic weight, implicit in Connes [6], we develop a useful picture of the Π_∞ core of this decomposition, and we show that factoriality of our original algebra is equivalent to ergodicity of a certain partial action of the discrete group on the center of the core.

Let $\mathcal{M}$ be a purely infinite von Neumann algebra and φ be a n.f.s. almost periodic weight on $\mathcal{M}$. Let α be Connes' action of the compact group G on $\mathcal{M}$ and $\Gamma = \hat{G}$ as in Section 1.3.

Definition 2.1. We have by Takesaki duality that $\mathcal{M} \simeq (\mathcal{M} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \Gamma$. The dynamical system $(\mathcal{M} \rtimes_\alpha G, \Gamma, \hat{\alpha})$ is the *discrete decomposition* associated to φ whose *discrete core* is $\mathcal{M} \rtimes_\alpha G$.

This section is devoted to elucidating the von Neumann algebra $\mathcal{M} \rtimes_\alpha G$ and the action $\hat{\alpha}$. We will see that the core of the discrete decomposition is semifinite and the action $\hat{\alpha}$ is trace-scaling (for the embedding of Γ in $\mathbf{R}_+^*$). The following proposition gives a useful picture of the discrete core.

Proposition 2.2. *Let α be an action of a compact group G on a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ and let Γ be the dual group of G . Then the Fourier-Plancherel transform provides an isomorphism from $\mathcal{M} \rtimes_\alpha G$ onto the von Neumann algebra, $\mathcal{Q}$, acting on $\ell^2(\Gamma) \otimes \mathcal{H}$ that is generated by*

$$\{M_f \otimes 1 \mid f \in \ell^\infty(\Gamma)\} \cup \{\lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\},$$

(where M_f is the multiplication operator $(M_f\xi)(\gamma) = f(\gamma)\xi(\gamma)$ and $(\lambda_\gamma\xi)(\gamma') = \xi(\gamma^{-1}\gamma')$ for $\xi \in \ell^2(\Gamma)$). Under this isomorphism the dual automorphism $\hat{\alpha}_\gamma$ on $\mathcal{M} \rtimes_\alpha G$ corresponds to the automorphism $\text{Ad}(\lambda_{\gamma^{-1}} \otimes 1)$ on $\mathcal{Q}$.

Proof. By the definition of crossed product (see Section 1.1) and by Lemma 1.2.3, $\mathcal{M} \rtimes_\alpha G$ is isomorphic to the von Neumann algebra on $L^2(G) \otimes B(\mathcal{H})$ that is generated by

$$(4) \quad \{\lambda_g \otimes 1 \mid g \in G\} \cup \{\pi_{\alpha, id}(a) \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\}.$$

But for $a \in \mathcal{M}^\alpha(\gamma)$, since then $\alpha_{h^{-1}}(a) = \langle h, \gamma \rangle a$, we have

$$\pi_{\alpha, id}(a) = M_{\langle \cdot, \gamma \rangle} \otimes a,$$

where $\langle \cdot, \gamma \rangle$ is the function $G \ni h \mapsto \langle h, \gamma \rangle$. Let $\mathcal{F} : L^2(G) \rightarrow \ell^2(\Gamma)$ be the Fourier–Plancherel transform given by $(\mathcal{F}\xi)(\gamma) = \int_G \overline{\langle g, \gamma \rangle} \xi(g) dg \forall \gamma \in \Gamma$. We will conjugate the operators in (4) by $\mathcal{F} \otimes 1$. Now $\mathcal{F} M_{\langle \cdot, \gamma \rangle} \mathcal{F}^{-1} = \lambda_\gamma$ and $\mathcal{F} \lambda_g \mathcal{F}^{-1} = M_{\overline{\langle g, \cdot \rangle}}$, so $(\mathcal{F} \otimes id)(\mathcal{M} \rtimes_\alpha G)(\mathcal{F} \otimes id)^{-1}$ is the von Neumann algebra on $\ell^2(\Gamma) \otimes \mathcal{H}$ generated by

$$\{M_{\overline{\langle g, \cdot \rangle}} \otimes 1 \mid g \in G\} \cup \{\lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\}.$$

But $\{\overline{\langle g, \cdot \rangle} \mid g \in G\}$ generates $l^\infty(\Gamma)$, so conjugation by $\mathcal{F} \otimes id$ takes $\mathcal{M} \rtimes_\alpha G$ onto $\mathcal{Q}$.

The dual automorphism $\hat{\alpha}_\gamma$ on the von Neumann algebra generated by the set (4) is $\hat{\alpha}_\gamma = \text{Ad}(M_{\overline{\langle \cdot, \gamma \rangle}} \otimes 1)$. But $\mathcal{F} M_{\overline{\langle \cdot, \gamma \rangle}} \mathcal{F}^{-1} = \lambda_{\gamma^{-1}}$, proving the last sentence of the proposition. $\square$

Remark 2.3. $\mathcal{Q} \subseteq B(\ell^2(\Gamma)) \otimes \mathcal{M}$, and every element of $x \in B(\ell^2(\Gamma)) \otimes \mathcal{M}$ can be viewed as a generalized matrix, indexed over Γ and having entries in $\mathcal{M}$, where the γ_1, γ_2 entry $[x]_{\gamma_1, \gamma_2}$ is given by

$$\langle [x]_{\gamma_1, \gamma_2} v, w \rangle_{\mathcal{H}} = \langle x(\chi_{\{\gamma_2\}} \otimes v), \chi_{\{\gamma_1\}} \otimes w \rangle_{\ell^2(\Gamma) \otimes \mathcal{H}}, \forall v, w \in \mathcal{H},$$

where $\chi_{\{\gamma_i\}} \in \ell^2(\Gamma)$ is the characteristic function of $\{\gamma_i\}$. One can easily prove that $\mathcal{Q}$ is the set of all elements $x \in B(\ell^2(\Gamma)) \otimes \mathcal{M}$ that, when viewed as generalized matrices, satisfy $[x]_{\gamma_1, \gamma_2} \in \mathcal{M}^\alpha(\gamma_2^{-1} \gamma_1) \forall \gamma_1, \gamma_2 \in \Gamma$. One also checks that

$$[\text{Ad}(\lambda_{\gamma^{-1}} \otimes 1)x]_{\gamma_1, \gamma_2} = [(\lambda_{\gamma^{-1}} \otimes 1)x(\lambda_\gamma \otimes 1)]_{\gamma_1, \gamma_2} = [x]_{\gamma\gamma_1, \gamma\gamma_2}.$$

Thus the discrete action $\hat{\alpha}$ on the discrete core $\mathcal{Q}$ is given by a permutation matrix.

Proposition 2.4. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and φ a n.f.s. almost periodic weight on $\mathcal{M}$. Let G , α and Γ be as in Section 1.3, and let $\mathcal{Q}$ be as in Proposition 2.2. Then there is a n.f.s. trace $\text{Tr}_{\mathcal{Q}}$ on $\mathcal{Q}$ such that*

$$\text{Tr}_{\mathcal{Q}} \circ \hat{\alpha}_\gamma = i(\gamma)^{-1} \text{Tr}_{\mathcal{Q}}.$$

Proof. Since $\mathcal{Q} \subseteq B(\ell^2(\Gamma)) \otimes \mathcal{M}$ let $\text{Tr}_{\mathcal{Q}}$ be $\text{Tr}(M_i \cdot) \otimes \varphi$ restricted to $\mathcal{Q}$, where Tr is the n.f.s. trace on $B(\ell^2(\Gamma))$ and the density matrix M_i is the multiplication operator associated to the unbounded positive function i on Γ . Then $\text{Tr}_{\mathcal{Q}}$ is normal and semifinite because Tr and φ are n.f.s. and because M_i is affiliated to $\mathcal{Q}$. To see that $\text{Tr}_{\mathcal{Q}}$ is a trace, let $\mathfrak{m} = \{a \in \mathcal{M} \mid \varphi(a^*a) < +\infty\}$. Then there is a net of projections $e_i \in \mathcal{M}^\alpha$, ($i \in I$), increasing to one, which together with Lemma 1.2.3 shows that $\text{span} \bigcup_{\gamma \in \Gamma} (\mathcal{M}^\alpha(\gamma) \cap \mathfrak{m})$ is a strongly dense $*$ -subalgebra of $\mathcal{M}$. Hence it will suffice to show that $\text{Tr}_{\mathcal{Q}}(xy) = \text{Tr}_{\mathcal{Q}}(yx)$ whenever x and y are finite products of elements in

$$\{M_f \mid f \in l^\infty(\Gamma) \text{ having finite support}\} \cup \{\lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma) \cap \mathfrak{m}\}.$$

But such a finite product is equal to $(M_f \otimes 1)(\lambda_\gamma \otimes a)$ for some $f \in l^\infty(\Gamma)$ having finite support, some $\gamma \in \Gamma$ and $a \in \mathcal{M}^\alpha(\gamma)$. This in turn is the sum of operators of the form $(\chi_{\gamma'} \otimes 1)(\lambda_\gamma \otimes a)$, which when viewed as a generalized matrix as in Remark 2.3 has all entries equal to zero except the $\gamma', \gamma'\gamma^{-1}$ entry. Hence it suffices to show that $\text{Tr}_{\mathcal{Q}}(xy) = \text{Tr}_{\mathcal{Q}}(yx)$ when $[x]_{\gamma_1, \gamma_2} = a$ and all other entries of x are zero and $[y]_{\gamma_3, \gamma_4} = b$ and all other entries of y are zero. Both $\text{Tr}_{\mathcal{Q}}(xy)$ and $\text{Tr}_{\mathcal{Q}}(yx)$ are zero unless $\gamma_3 = \gamma_2$ and $\gamma_4 = \gamma_1$, so assume this is the case. Then $a \in \mathcal{M}^\alpha(\gamma_2^{-1}\gamma_1)$ and $b \in \mathcal{M}^\alpha(\gamma_1^{-1}\gamma_2)$, so by Lemma 1.3.3,

$$\text{Tr}_{\mathcal{Q}}(yx) = i(\gamma_2)\varphi(ba) = i(\gamma_1)\varphi(ab) = \text{Tr}_{\mathcal{Q}}(xy).$$

□

We now exhibit a partial action, S_γ , of Γ on the center of the centralizer of $\mathcal{M}$, and show that factoriality of $\mathcal{M}$ is equivalent to ergodicity of this partial action. Let us first take a closer look at spectral subspaces $\mathcal{M}^\alpha(\gamma)$. Taking $\mathcal{M}$, φ , etc. as in the previous proposition, by Lemma 1.3.3 $\mathcal{M}^\alpha$ equals the centralizer, $\mathcal{M}_\varphi$, of φ . Thus for $\gamma \in \Gamma$, we will also denote the spectral subspace $\mathcal{M}^\alpha(\gamma)$ by $\mathcal{M}_\varphi(\gamma)$. Let $Z_\varphi = Z(\mathcal{M}_\varphi)$ denote the center of the centralizer of φ and PZ_φ the set of (self-adjoint) projections in Z_φ . In general, for a an element in a von Neumann algebra $\mathcal{N}$, $C_{\mathcal{N}}(a)$ will denote the central carrier of a in $\mathcal{N}$.

Definition 2.5. For $\gamma \in \Gamma$ and $p \in PZ_\varphi$ let

$$S_\gamma(p) = \bigvee \{C_{\mathcal{M}_\varphi}(apa^*) \mid a \in \mathcal{M}_\varphi(\gamma)\} \in PZ_\varphi.$$

Lemma 2.6.

(i) $\forall \gamma \in \Gamma$ there is a partial isometry $v_\gamma \in \mathcal{M}_\varphi(\gamma)$ such that

$$S_\gamma(p) = C_{\mathcal{M}_\varphi}(v_\gamma p v_\gamma^*) \quad \forall p \in PZ_\varphi;$$

- (ii) one may arrange that $v_{\gamma^{-1}} = v_{\gamma}^*$.
- (iii) Let $\gamma \in \Gamma$; then $p \leq p' \Rightarrow S_{\gamma}(p) \leq S_{\gamma}(p') \quad \forall p, p' \in PZ_{\varphi}$;
- (iv) setting $p_{\gamma} = S_{\gamma^{-1}}(1)$ we have $S_{\gamma}(p) = 0$ if $p \perp p_{\gamma}$ and, for any $p \in PZ_{\varphi}$, $S_{\gamma^{-1}}S_{\gamma}(p) = pp_{\gamma}$;
- (v) if $p_1, p_2 \in p_{\gamma}PZ_{\varphi}$ then $S_{\gamma}(p_1) = S_{\gamma}(p_2)$ implies $p_1 = p_2$;
- (vi) $S_{\gamma}(p_1p_2) = S_{\gamma}(p_1)S_{\gamma}(p_2) \quad \forall p_1, p_2 \in PZ_{\varphi}$;
- (vii) if $p_1, p_2 \in PZ_{\varphi}$ and $p_1 \perp p_2$ then $S_{\gamma}(p_1 + p_2) = S_{\gamma}(p_1) + S_{\gamma}(p_2)$.
- (viii) S_{γ} is normal in the sense that if $(p_n)_{n=1}^{\infty}$ is an increasing family of projections in Z_{φ} then

$$S_{\gamma}\left(\bigvee_{n=1}^{\infty} p_n\right) = \bigvee_{n=1}^{\infty} S_{\gamma}(p_n);$$

Proof. For (i), let $(v_i)_{i \in I}$ be a family of partial isometries in $\mathcal{M}_{\varphi}(\gamma)$ that is maximal with respect to the property that $C_{\mathcal{M}_{\varphi}}(v_i^*v_i) \perp C_{\mathcal{M}_{\varphi}}(v_{i'}^*v_{i'})$ and $C_{\mathcal{M}_{\varphi}}(v_i v_i^*) \perp C_{\mathcal{M}_{\varphi}}(v_{i'} v_{i'}^*)$ whenever $i, i' \in I$, $i \neq i'$. Let $v_{\gamma} = \sum_{i \in I} v_i$. For $p \in PZ_{\varphi}$, let $q = C_{\mathcal{M}_{\varphi}}(v_{\gamma} p v_{\gamma}^*)$. Clearly $q \leq S_{\gamma}(p)$. Suppose for contradiction that $q \neq S_{\gamma}(p)$. Then $\exists a \in \mathcal{M}_{\varphi}(\gamma)$ such that $(1-q)ap \neq 0$. Taking the polar decomposition, let w be the polar part of $(1-q)ap \neq 0$. Then $w \in \mathcal{M}_{\varphi}(\gamma)$ is a partial isometry, $w^*w \leq p$ and $ww^* \leq (1-q)$. We will show that $C_{\mathcal{M}_{\varphi}}(ww^*) \perp C_{\mathcal{M}_{\varphi}}(v_i v_i^*)$ and $C_{\mathcal{M}_{\varphi}}(w^*w) \perp C_{\mathcal{M}_{\varphi}}(v_i^*v_i) \quad \forall i \in I$, which will contradict the maximality of $(v_i)_{i \in I}$. If for some $i \in I$, $C_{\mathcal{M}_{\varphi}}(ww^*)C_{\mathcal{M}_{\varphi}}(v_i v_i^*) \neq 0$ then $\exists a \in \mathcal{M}_{\varphi}$ such that $w^*av_i \neq 0$. But $w^*av_i \in \mathcal{M}_{\varphi}$ so $0 \neq pw^*(1-q)av_i = w^*a(1-q)v_i p$ and $(1-q)v_i p \neq 0$, contradicting the choice of q . Similarly, if $C_{\mathcal{M}_{\varphi}}(w^*w)C_{\mathcal{M}_{\varphi}}(v_i^*v_i) \neq 0$ then $\exists a \in \mathcal{M}_{\varphi}$ such that $wav_i^* \neq 0$, so $0 \neq (1-q)wpav_i^* = wapv_i^*(1-q)$, a contradiction.

For (ii), just note from the proof of (i) that any maximal family $(v_i)_{i \in I}$ will do, and use that $\mathcal{M}_{\varphi}(\gamma^{-1}) = \mathcal{M}_{\varphi}(\gamma)^*$.

Part (iii) is clear from the definition of S_{γ} . For (iv), we have from (i) and (ii) that $p_{\gamma} = C_{\mathcal{M}_{\varphi}}(v_{\gamma}^*v_{\gamma})$. If $p \in PZ_{\varphi}$ and $p \perp p_{\gamma}$, then $p \perp v_{\gamma}^*v_{\gamma}$, so $S_{\gamma}(p) = C_{\mathcal{M}_{\varphi}}(v_{\gamma} p v_{\gamma}^*) = 0$. For any $p \in PZ_{\varphi}$, $S_{\gamma}(p) \geq v_{\gamma} p v_{\gamma}^*$ so $S_{\gamma^{-1}}S_{\gamma}(p) = C_{\mathcal{M}_{\varphi}}(v_{\gamma}^*S_{\gamma}(p)v_{\gamma}) \geq C_{\mathcal{M}_{\varphi}}(p v_{\gamma}^*v_{\gamma}) = pp_{\gamma}$. But also

$$\begin{aligned} S_{\gamma^{-1}}S_{\gamma}(p) &= C_{\mathcal{M}_{\varphi}}(v_{\gamma}^*S_{\gamma}(p)v_{\gamma}) \\ &= C_{\mathcal{M}_{\varphi}}(v_{\gamma}^*C_{\mathcal{M}_{\varphi}}(v_{\gamma} p v_{\gamma}^*)v_{\gamma}) \\ &\leq C_{\mathcal{M}_{\varphi}}(v_{\gamma}^*v_{\gamma} p v_{\gamma}^*v_{\gamma}) = pp_{\gamma}, \end{aligned}$$

proving (iv). Part (v) is clear from (iv).

For (vi), $S_\gamma(p_1p_2) \leq S_\gamma(p_1)S_\gamma(p_2)$ from (iii). Using (iv),

$$\begin{aligned} p_\gamma p_1 p_2 &= S_{\gamma^{-1}} S_\gamma(p_1 p_2) \leq S_{\gamma^{-1}}(S_\gamma(p_1) S_\gamma(p_2)) \\ &\leq (S_{\gamma^{-1}} S_\gamma(p_1))(S_{\gamma^{-1}} S_\gamma(p_2)) = p_\gamma p_1 p_2. \end{aligned}$$

Hence the inequalities in the previous sentence are equalities, so by (v), $S_\gamma(p_1p_2) = S_\gamma(p_1)S_\gamma(p_2)$.

For (vii), we have from (vi) and (i) that $C_{\mathcal{M}_\varphi}(v_\gamma p_1 v_\gamma^*) \perp C_{\mathcal{M}_\varphi}(v_\gamma p_2 v_\gamma^*)$, hence

$$\begin{aligned} S_\gamma(p_1 + p_2) &= C_{\mathcal{M}_\varphi}(v_\gamma p_1 v_\gamma^* + v_\gamma p_2 v_\gamma^*) \\ &= C_{\mathcal{M}_\varphi}(v_\gamma p_1 v_\gamma^*) + C_{\mathcal{M}_\varphi}(v_\gamma p_2 v_\gamma^*) S_\gamma(p_1) + S_\gamma(p_2). \end{aligned}$$

Part (viii) holds because taking central carriers respects $\bigvee$ (see Kadison and Ringrose [11], 5.5.3). $\square$

Lemma 2.7.

- (i) S_γ extends to a normal $*$ -homomorphism, also denoted S_γ , from Z_φ into Z_φ ; the kernel of S_γ is $(1 - p_\gamma)Z_\varphi$ and the restriction of S_γ is an isomorphism from $p_\gamma Z_\varphi$ onto $p_{\gamma^{-1}} Z_\varphi$.
- (ii) We have $p_{1_\Gamma} = 1$ and $S_{1_\Gamma} = id$.
- (iii) Let $\gamma_1, \gamma_2 \in \Gamma$ and $q = S_{\gamma_2^{-1}}(p_{\gamma_1})$. Then $q \leq p_{\gamma_1 \gamma_2}$ and

$$S_{\gamma_1} S_{\gamma_2}|_{qZ_\varphi} = S_{\gamma_1 \gamma_2}|_{qZ_\varphi}.$$

Proof. For (i), the properties proved in Lemma 2.6 show that S_γ extends to a homomorphism from $\text{span } PZ_\varphi$ (i.e., the L^∞ functions taking only finitely many values) to $\text{span } PZ_\varphi$, with kernel $\text{span}(1 - p_\gamma)PZ_\varphi$ and that is 1–1 on $\text{span } p_\gamma PZ_\varphi$. Extending to all of PZ_φ is now standard measure theory, making use of part (viii) of Lemma 2.6. Part (ii) is clear.

For (iii), let us show that

$$(5) \quad S_{\gamma_1} S_{\gamma_2}(p) = S_{\gamma_1 \gamma_2}(qp) \quad \forall p \in PZ_\varphi.$$

Denoting by $\text{supp}(b)$ the support in $\mathcal{M}_\varphi$ of a self-adjoint element $b \in \mathcal{M}_\varphi$, we have

$$\begin{aligned} S_{\gamma_1} S_{\gamma_2}(p) &= \bigvee_{a_1 \in \mathcal{M}_\varphi(\gamma_1)} \text{supp}\left(a_1 \left(\bigvee_{a_2 \in \mathcal{M}_\varphi(\gamma_2)} \text{supp}(a_2 p a_2^*) \right) a_1^*\right) \\ &\leq \bigvee_{\substack{a_1 \in \mathcal{M}_\varphi(\gamma_1) \\ a_2 \in \mathcal{M}_\varphi(\gamma_2)}} a_1 a_2 p a_2^* a_1^* \\ &\leq \bigvee_{a \in \mathcal{M}_\varphi(\gamma_1 \gamma_2)} a p a^* = S_{\gamma_1 \gamma_2}(p). \end{aligned}$$

Hence

$$S_{\gamma_1} S_{\gamma_2}(p) = S_{\gamma_1}(p_{\gamma_1} S_{\gamma_2}(p)) = S_{\gamma_1}(p_{\gamma_1} p_{\gamma_2^{-1}} S_{\gamma_2}(p)) = S_{\gamma_1} S_{\gamma_2}(qp) \leq S_{\gamma_1 \gamma_2}(qp).$$

But $qp = S_{\gamma_2^{-1}}(p_{\gamma_1} S_{\gamma_2}(p))$ so

$$S_{\gamma_1 \gamma_2}(qp) \leq S_{\gamma_1}(p_{\gamma_1} S_{\gamma_2}(p)) = S_{\gamma_1} S_{\gamma_2}(p),$$

which proves (5).

Now it only remains to show that $q \leq p_{\gamma_1 \gamma_2}$. For this it will suffice by part (i) to show that $p \in PZ_\varphi$, $p \leq q$ and $S_{\gamma_1 \gamma_2}(p) = 0$ implies $p = 0$. Then by (5), the assumptions imply $S_{\gamma_1} S_{\gamma_2}(p) = 0$, so by part (i) $S_{\gamma_2}(p) \perp p_{\gamma_1}$. Hence (because $p \leq q \leq p_{\gamma_2}$) by Lemma 2.6iv, $p = S_{\gamma_2^{-1}} S_{\gamma_2}(p) \perp S_{\gamma_2^{-1}}(p_{\gamma_1}) = q$, so $p = 0$. $\square$

Remark 2.8. From the lemmas above we see by standard arguments that $Z_\varphi = L^\infty(X, \mu)$ where X is a standard Borel space with measure μ and for each $\gamma \in \Gamma$, p_γ is the characteristic function of $A_\gamma \subseteq X$ and $S_\gamma(f) = f \circ T_{\gamma^{-1}}$ where $T_{\gamma^{-1}} : A_{\gamma^{-1}} \rightarrow A_\gamma$ is a bijective measurable mapping. Moreover, $T_{\gamma^{-1}} = (T_\gamma)^{-1}$ and $T_{\gamma_2} \circ T_{\gamma_1}|_B = T_{\gamma_2 \gamma_1}|_B$ where $B = T_{\gamma_1^{-1}}(A_{\gamma_1^{-1}} \cap A_{\gamma_2}) \subseteq A_{\gamma_1 \gamma_2}$.

Thus we can describe $\Gamma \ni \gamma \mapsto T_\gamma$ as a *partial action* of Γ on X . Let us define a measurable subset $F \subseteq X$ to be *invariant* under the partial action T if $T_\gamma(F \cap A_\gamma) = F \cap A_{\gamma^{-1}}$ $\forall \gamma \in \Gamma$, and the partial action T to be *ergodic* if every invariant subset is null or conull. An equivalent condition for the ergodicity of T is in terms of measurable equivalence relations of Feldman and Moore [9]: the union of the graphs of T_γ as γ ranges over all of Γ is a measurable equivalence relation. Then T is ergodic if and only if the equivalence relation is ergodic.

The following proposition says that T is ergodic if and only if $\mathcal{M}$ is a factor; but in order to avoid using the techniques of measure theory, the condition of ergodicity is expressed in terms of the maps S_γ .

Proposition 2.9. *Let $x \in \mathcal{M}$. Then x is in the center of $\mathcal{M}$ if and only if x is in the center of $\mathcal{M}_\varphi$ and*

$$(6) \quad S_\gamma(x) = xp_{\gamma^{-1}} \quad \forall \gamma \in \Gamma.$$

Proof. Since both $Z(\mathcal{M})$ and the set of $x \in \mathcal{M}$ satisfying (6) are von Neumann subalgebras of $\mathcal{M}$, we may assume x is a projection in $\mathcal{M}$. To show necessity, suppose $p \in PZ(\mathcal{M})$. Then $p \in PZ_\varphi$. For all $\gamma \in \Gamma$ we have

$$\begin{aligned} S_\gamma(p) &= \bigvee \{C_{\mathcal{M}_\varphi}(apa^*) \mid a \in \mathcal{M}_\varphi(\gamma)\} \\ &= \bigvee \{pC_{\mathcal{M}_\varphi}(aa^*) \mid a \in \mathcal{M}_\varphi(\gamma)\} = pp_{\gamma^{-1}}. \end{aligned}$$

To show sufficiency, let $x = p \in PZ_\varphi$ such that (6) holds. To show $p \in Z(\mathcal{M})$ it will by Lemma 1.2.3 suffice to show that p commutes with a for every $\gamma \in \Gamma$ and $a \in \mathcal{M}_\varphi(\gamma)$. But if not then $(1-p)ap \neq 0$, so $(1-p)C_{\mathcal{M}_\varphi}(apa^*) \neq 0$, so $(1-p)S_\gamma(p) \neq 0$, contradicting $S_\gamma(p) = pp_{\gamma^{-1}}$. $\square$

Corollary 2.10. *$\mathcal{M}$ is a factor if and only if the partial action T is ergodic.*

Proposition 2.11. *Let $\mathcal{M}$, φ , G , α , Γ and $\mathcal{Q} \simeq \mathcal{M} \rtimes_{\alpha} G$ be as in Proposition 2.4, and regard elements of $\mathcal{Q}$ as generalized matrices as described in Remark 2.3. Let $x \in \mathcal{Q}$. Then x is in the center of $\mathcal{Q}$ if and only if*

- (1) $[x]_{\gamma_1, \gamma_2} = 0$ whenever $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$;
- (2) $[x]_{\gamma, \gamma} \in Z_{\varphi} \forall \gamma \in \Gamma$; and
- (3) $S_{\gamma_1}([x]_{\gamma, \gamma} p_{\gamma_1}) = [x]_{\gamma \gamma_1, \gamma \gamma_1} p_{\gamma_1}^{-1} \forall \gamma, \gamma_1 \in \Gamma$.

Proof. The set of $x \in \mathcal{Q}$ satisfying (1)–(3) is a von Neumann subalgebra of $\mathcal{Q}$, so we may assume $x = q$ is a projection in $\mathcal{Q}$. To prove necessity, suppose $q \in Z(\mathcal{Q})$. Then since every bounded, diagonal operator having entries in $\mathcal{M}_{\varphi}$ is in $\mathcal{Q}$, clearly (1) and (2) hold. Suppose for contradiction that (3) fails for some $\gamma_1, \gamma_2 \in \Gamma$. We may assume that $(1 - [q]_{\gamma \gamma_1, \gamma \gamma_1}) S_{\gamma_1}([q]_{\gamma, \gamma} p_{\gamma_1}) \neq 0$, (otherwise take $S_{\gamma_1}^{-1}$ of both sides). Hence $\exists a \in \mathcal{M}_{\varphi}(\gamma_1)$ such that $a[q]_{\gamma, \gamma} = a \neq 0$ and $aa^* = a[q]_{\gamma, \gamma} a^* \perp [q]_{\gamma \gamma_1, \gamma \gamma_1}$. Let $y \in \mathcal{Q}$ have entry $[y]_{\gamma \gamma_1, \gamma} = a$ and all other entries equal to 0. It follows that $[yq]_{\gamma \gamma_1, \gamma} = a[q]_{\gamma, \gamma} \neq 0$ and $[qy]_{\gamma \gamma_1, \gamma} = [q]_{\gamma \gamma_1, \gamma \gamma_1} a = 0$, contradicting that $q \in Z(\mathcal{Q})$.

For sufficiency, suppose $q \in P\mathcal{Q}$ satisfies (1)–(3). To show $q \in Z(\mathcal{Q})$ it will suffice to show that $yq = qy$ whenever $y \in \mathcal{Q}$ has only one nonzero entry, i.e., to show

$$(7) \quad a[q]_{\gamma, \gamma} = [q]_{\gamma \gamma_1, \gamma \gamma_1} a \quad \forall \gamma, \gamma_1 \in \Gamma \quad \forall a \in \mathcal{M}_{\varphi}(\gamma_1).$$

Since $a^*a \in \mathcal{M}_{\varphi}$, taking the polar decomposition of a , one sees that it suffices to show (7) for $a = v \in \mathcal{M}_{\varphi}(\gamma_1)$ a partial isometry. But $vv^* \leq p_{\gamma_1}^{-1}$, so using (3) it suffices to show $v[q]_{\gamma, \gamma} v^* = S_{\gamma_1}([q]_{\gamma, \gamma} p_{\gamma}) vv^*$, hence, since $v^*v \leq p_{\gamma}$, to show

$$(8) \quad vrv^* = S_{\gamma_1}(r)vv^* \quad \forall r \in p_{\gamma_1} PZ_{\varphi} \quad \forall \gamma_1 \in \Gamma$$

$\forall \text{partial isometries } v \in \mathcal{M}_{\varphi}(\gamma_1).$

But $vrv^* \leq S_{\gamma_1}(r)$ by definition, so $\leq$ holds in (8). Since $r = S_{\gamma_1}^{-1} S_{\gamma_1}(r)$ we may apply $\leq$ of (8) to get $vrv^* = v(v^* v S_{\gamma_1}^{-1}(S_{\gamma_1}(r)))v^* \geq v(v^* S_{\gamma_1}(r)v)v^* = S_{\gamma_1}(r)vv^*$. $\square$

Proposition 2.12. *The discrete core $\mathcal{Q}$ is a factor if and only if $\mathcal{M}_{\varphi}$ is a factor, and in that case Γ equals the point spectrum of Δ_{φ} and $\mathcal{Q} \simeq \mathcal{M}_{\varphi} \otimes B(\mathcal{H})$, where $\mathcal{H}$ is separable, infinite dimensional Hilbert space.*

Proof. Suppose that $\mathcal{M}_\varphi$ is not a factor, let p be a nontrivial projection in the center of $\mathcal{M}_\varphi$ and let $q \in \mathcal{Q}$, satisfying (1)–(3) of the proposition, be such that $[q]_{\gamma,\gamma} = S_\gamma(p)$. Then by Proposition 2.11, q is in the center of $\mathcal{Q}$ and $\mathcal{Q}$ is not a factor.

Suppose that $\mathcal{M}_\varphi$ is a factor. Then for each $\gamma \in \Gamma$, $p_\gamma = 0$ or 1 , and $p_\gamma = 0$ if and only if $\mathcal{M}_\varphi(\gamma) = \{0\}$ if and only if γ is not in the point spectrum of Δ_φ . By Lemma 2.7iii, $\Gamma' = \{\gamma \in \Gamma \mid p_\gamma = 1\}$ is a subgroup of Γ . But Γ' equals the point spectrum of Δ_φ and Γ was taken to be the group generated by Γ' , so $\Gamma' = \Gamma$. Choose any $\gamma_0 \in \Gamma$ and let $q \in \mathcal{Q}$ be such that $[q]_{\gamma_0,\gamma_0} = 1$ and all other entries of q are zero. Then since $p_\gamma = 1 \ \forall \gamma \in \Gamma$, it follows that the central carrier of q in $\mathcal{Q}$ equals the identity of $\mathcal{Q}$. But $q\mathcal{Q}q \simeq \mathcal{M}_\varphi$, so $\mathcal{Q}$ is a factor. Looking to Proposition 2.4 and noting that Γ is an infinite set, we see that $\mathcal{Q}$ is a type II_∞ factor and $\mathcal{Q} \simeq q\mathcal{Q}q \otimes B(\mathcal{H})$. $\square$

3. The continuous decomposition. In this section, for $\mathcal{M}$ a purely infinite von Neumann algebra with n.f.s. almost periodic weight φ , Takesaki's continuous decomposition for $\mathcal{M}$ (see Section 1.1) is related to the discrete decomposition obtained in Section 2. We shall see that Takesaki's continuous decomposition can be viewed as the induced representation of the discrete decomposition, if one broadens Takesaki's definition of induced representation to include actions induced up from dense subgroups. For this notion of induced action, one must consider a locally compact topology on the subgroup, which may be different from the restriction topology, an example of which is $\mathbf{Q} \subseteq \mathbf{R}$, with the usual separable topology on $\mathbf{R}$ and the discrete topology on $\mathbf{Q}$. What follows is Takesaki's definition [15] of induced action, except that he required H to be a closed subgroup with the restriction topology.

Definition 3.1. Let K be a locally compact abelian group, $H \subseteq K$ a subgroup. Suppose α is an action of H on a von Neumann algebra $\mathcal{N}$, continuous with respect to some locally compact topology on H that is finer than the restriction of the topology on K . Consider the von Neumann algebra $L^\infty(K) \otimes \mathcal{N} \simeq L^\infty(K, \mathcal{N})$, equal to the set of bounded measurable functions from K into $\mathcal{N}$, and the action β of H on $L^\infty(K, \mathcal{N})$ given by $(\beta_h f)(k) = \alpha_h(f(kh))$ for $h \in H$, $f \in L^\infty(K, \mathcal{N})$ and $k \in K$. Let $\mathcal{M}$ be the fixed-point subalgebra of $L^\infty(K, \mathcal{N})$ under the action β of H . The action, τ , of K on $L^\infty(K, \mathcal{N})$ given by $(\tau_k f)(k') = f(k^{-1}k')$ leaves $\mathcal{M}$ globally invariant. The action *induced up* to K from the action α of H on $\mathcal{N}$ is the action τ of K restricted to $\mathcal{M}$.

Let G be a compact abelian group with dual group Γ , α an action of G on a von Neumann algebra $\mathcal{M}$, E a locally compact abelian group, $\iota : E \rightarrow G$ a continuous homomorphism whose image is dense in G . Then $\alpha \circ \iota$ is an action

of E on $\mathcal{M}$. Let $\hat{\iota} : \Gamma \rightarrow \hat{E}$ be the injective homomorphism given by $\langle t, \hat{\iota}(\gamma) \rangle = \langle \iota(t), \gamma \rangle \forall t \in E, \gamma \in \Gamma$. Let

$$\begin{aligned}\mathcal{P} &= \mathcal{M} \rtimes_{\alpha} G \\ \tilde{\mathcal{P}} &= \mathcal{M} \rtimes_{\alpha \circ \iota} E.\end{aligned}$$

Proposition 3.2. *Let $\tilde{\pi}$ be a normal, faithful representation of $\mathcal{M}$ on the Hilbert space $\mathcal{H}$. Then the Fourier–Plancherel transform provides an isomorphism from $\tilde{\mathcal{P}}$ onto the von Neumann algebra, $\tilde{\mathcal{Q}}$, acting on $L^2(\hat{E}) \otimes \mathcal{H}$, that is generated by*

$$\{M_F \otimes 1 \mid F \in L^\infty(\hat{E})\} \cup \{\lambda_{\hat{\iota}(\gamma)} \otimes \tilde{\pi}(a) \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\},$$

(where M_F is the multiplication operator on $L^2(\hat{E})$). Under this isomorphism, the dual automorphism $(\alpha \circ \iota)_\mu^*$ on $\tilde{\mathcal{P}}$ for $\mu \in \hat{E}$ corresponds to $\text{Ad}(\lambda_{\mu^{-1}} \otimes 1)$ on $\tilde{\mathcal{Q}}$.

Proof. This proposition is proved just like Proposition 2.2, once one notes that if $a \in \mathcal{M}^\alpha(\gamma)$ and $t \in E$ then $(\alpha \circ \iota)_{t^{-1}}(a) = \langle t, \hat{\iota}(\gamma) \rangle a$, so that $\pi_{\alpha \circ \iota, \tilde{\pi}}(a) = M_{\langle \cdot, \hat{\iota}(\gamma) \rangle} \otimes \tilde{\pi}(a)$. $\square$

Proposition 3.3. *In the situation of Proposition 3.2, let the action of $\hat{E}$ that is induced up from the action $\hat{\alpha}$ of Γ on $\mathcal{P}$ be denoted the action τ of $\hat{E}$ on $\mathcal{R}$. Then there is an isomorphism from $\tilde{\mathcal{P}}$ onto $\mathcal{R}$ that intertwines the dual action $(\alpha \circ \iota)^*$ of $\hat{E}$ on $\tilde{\mathcal{P}}$ with the action τ .*

Proof. Let $\pi_{\alpha, id}$ be the representation of $\mathcal{M}$ on $L^2(G) \otimes \mathcal{H}$ used in the proof of Proposition 2.2 and let $\mathcal{F} : L^2(G) \rightarrow \ell^2(\Gamma)$ be the Fourier–Plancherel transform. Then $\tilde{\pi} = (\mathcal{F} \otimes 1)\pi_{\alpha, id}(\mathcal{F} \otimes 1)^{-1}$ is a faithful, normal representation of $\mathcal{M}$ on $\ell^2(\Gamma) \otimes \mathcal{H}$ and $\tilde{\pi}(a) = \lambda_\gamma \otimes a$ if $\gamma \in \Gamma$ and $a \in \mathcal{M}^\alpha(\gamma)$, so by Proposition 3.2, $\tilde{\mathcal{P}}$ is isomorphic to the von Neumann algebra acting on $L^2(\hat{E}) \otimes \ell^2(\Gamma) \otimes \mathcal{H}$ that is generated by

$$\{M_F \otimes 1 \otimes 1 \mid F \in L^\infty(\hat{E})\} \cup \{\lambda_{\hat{\iota}(\gamma)} \otimes \lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\}$$

and the dual automorphism $(\alpha \circ \iota)_\mu^*$ for $\mu \in \hat{E}$ is given by $\text{Ad}(\lambda_{\mu^{-1}} \otimes 1 \otimes 1)$.

By Proposition 2.2, $L^\infty(\hat{E}, \mathcal{P})$ is isomorphic to the von Neumann algebra acting on $L^2(\hat{E}) \otimes \ell^2(\Gamma) \otimes \mathcal{H}$ that is generated by

$$\begin{aligned}\{M_{\langle t, \cdot \rangle} \otimes 1 \otimes 1 \mid t \in E\} \cup \{1 \otimes M_{\langle g, \cdot \rangle} \otimes 1 \mid g \in G\} \\ \cup \{1 \otimes \lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\},\end{aligned}$$

the automorphism β_γ for $\gamma \in \Gamma$ is given by $\text{Ad}(\lambda_{i(\gamma)^{-1}} \otimes \lambda_{\gamma^{-1}} \otimes 1)$ and the automorphism τ_μ for $\mu \in \hat{E}$ is given by $\text{Ad}(\lambda_\mu \otimes 1 \otimes 1)$. Hence

$$\begin{aligned} \mathcal{R} = & \left(\{M_{\langle t, \cdot \rangle} \otimes M_{\langle g, \cdot \rangle} \otimes 1 \mid t \in E, g \in G\} \cup \{1 \otimes \lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\} \right)'' \\ & \cap \{\lambda_{i(\gamma)^{-1}} \otimes \lambda_{\gamma^{-1}} \otimes 1 \mid \gamma \in \Gamma\}'. \end{aligned}$$

Let U be the unitary on $L^2(\hat{E} \times \Gamma)$ given by $(U\xi)(\mu, \gamma) = \xi(i(\gamma)\mu^{-1}, \gamma)$ for $\mu \in \hat{E}$, $\gamma \in \Gamma$, so $U^* = U$. Then

$$\begin{aligned} U(M_{\langle t, \cdot \rangle} \otimes M_{\langle g, \cdot \rangle})U &= M_{\langle t, \cdot \rangle} \otimes M_{\langle i(t)g, \cdot \rangle}, \\ U(1 \otimes \lambda_\gamma)U &= \lambda_{i(\gamma)} \otimes \lambda_\gamma, \\ U(\lambda_{i(\gamma)^{-1}} \otimes \lambda_{\gamma^{-1}})U &= 1 \otimes \lambda_{\gamma^{-1}}, \\ U(\lambda_\mu \otimes 1)U &= \lambda_{\mu^{-1}} \otimes 1 \end{aligned}$$

and

$$\begin{aligned} URU &= (\{M_F \otimes M_f \otimes 1 \mid F \in L^\infty(\hat{E}), f \in \ell^\infty(\Gamma)\} \\ &\quad \cup \{\lambda_{i(\gamma)} \otimes \lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\})'' \cap \{1 \otimes \lambda_{\gamma^{-1}} \otimes 1 \mid \gamma \in \Gamma\}' \\ &= (\{M_F \otimes 1 \otimes 1 \mid F \in L^\infty(\hat{E})\} \cup \{\lambda_{i(\gamma)} \otimes \lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}^\alpha(\gamma)\})''. \end{aligned}$$

Moreover, the automorphism τ_μ of $\mathcal{R}$ corresponds to the automorphism $\text{Ad}(\lambda_{\mu^{-1}} \otimes 1 \otimes 1)$ of URU . This is precisely the picture for $\tilde{\mathcal{P}}$ and $(\alpha \circ \iota)_\mu$ obtained above. $\square$

Corollary 3.4. *Let $\mathcal{M}$ be a purely infinite von Neumann algebra having separable predual and φ a n.f.s. almost periodic weight on $\mathcal{M}$. Let Γ be the subgroup of $\mathbf{R}_+^*$ generated by the point spectrum of Δ_φ and let $(\mathcal{N}, \Gamma, \beta)$ giving rise to $\mathcal{M} \simeq \mathcal{N} \rtimes_\beta \Gamma$ be the discrete decomposition of $\mathcal{M}$ associated to φ . Let $(\tilde{\mathcal{N}}, \mathbf{R}_+^*, \tilde{\beta})$ giving rise to $\mathcal{M} \simeq \tilde{\mathcal{N}} \rtimes_{\tilde{\beta}} \mathbf{R}_+^*$ be Takesaki's continuous decomposition. Then the action $\tilde{\beta}$ of $\mathbf{R}_+^*$ on $\tilde{\mathcal{N}}$ is the action induced up to $\mathbf{R}_+^*$ from the action β of Γ on $\mathcal{N}$.*

Proof. Apply Proposition 3.3 with α and ι as in Section 1.3. $\square$

4. A model for the one-parameter trace-scaling action on the injective II_∞ factor.

Lemma 4.1. *Let $\mathcal{N}_i$ be a von Neumann algebra with almost periodic n.f.s. weight ψ_i , ($i = 1, 2$). Let $\mathcal{M} = \mathcal{N}_1 \otimes \mathcal{N}_2$ be the tensor product von Neumann algebra and $\varphi = \psi_1 \otimes \psi_2$ the tensor product weight. Then φ is an almost periodic n.f.s. weight on $\mathcal{M}$ and the point spectrum of Δ_φ is the product of the point spectra of Δ_{ψ_1} and Δ_{ψ_2} . Let Γ_i (respectively Γ), taken with discrete topology, be the subgroup of $\mathbf{R}_+^*$ generated by the point spectrum of Δ_{ψ_i} (respectively Δ_φ), so $\Gamma = \Gamma_1 \Gamma_2$, and let $G_i = \widehat{\Gamma}_i$ (respectively $G = \widehat{\Gamma}$). Let α_i (respectively α) be the action of G_i on $\mathcal{N}_i$ (respectively G on $\mathcal{M}$) as in Section 1.3. Then for $\gamma \in \Gamma$, the spectral subspace is*

$$(9) \quad \mathcal{M}^\alpha(\gamma) = \overline{\bigoplus_{\{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \mid \gamma_1 \gamma_2 = \gamma\}} \mathcal{N}_1^{\alpha_1}(\gamma_1) \otimes \mathcal{N}_2^{\alpha_2}(\gamma_2)}^{\text{w}^*}.$$

Proof. From $L^2(\mathcal{M}, \varphi) = L^2(\mathcal{N}_1, \psi_1) \otimes L^2(\mathcal{N}_2, \psi_2)$ and $\Delta_\varphi = \Delta_{\psi_1} \otimes \Delta_{\psi_2}$ one sees that φ is almost periodic and has the desired point spectrum. Moreover, from the characterization in Lemma 1.3.3, one sees that $\supseteq$ holds in (9). But by Lemma 1.2.3, the direct sum of $\mathcal{N}_1^{\alpha_1}(\gamma_1) \otimes \mathcal{N}_2^{\alpha_2}(\gamma_2)$ over all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ is dense in $\mathcal{M}$, so equality must hold in (9). $\square$

Theorem 4.2. *Let $0 < \mu_1, \mu_2 < 1$ be such that $\log(\mu_1)/\log(\mu_2)$ is irrational. Let Γ be the multiplicative subgroup of $\mathbf{R}_+^*$ generated by μ_1 and μ_2 , let $\mathcal{N}$ be the injective II_∞ factor and let β be an action of Γ on $\mathcal{N}$ such that $\text{Tr}_{\mathcal{N}} \circ \beta_\gamma = \gamma \text{Tr}_{\mathcal{N}} \forall \gamma \in \Gamma$, where $\text{Tr}_{\mathcal{N}}$ is the n.f.s. trace on $\mathcal{N}$. Then the action of $\mathbf{R}_+^*$ induced up from the action β of Γ on $\mathcal{N}$ is the one-parameter trace-scaling action of $\mathbf{R}_+^*$ on the injective II_∞ factor. (See Definition 3.1.)*

Remark 4.3. Since a trace-scaling action β of Γ on $\mathcal{N}$ is relatively easy to construct, Theorem 4.2 gives an accessible picture of the one-parameter trace-scaling action on the injective II_∞ factor, heuristically as translation on the fixed point subalgebra of $L^\infty(\mathbf{R}_+^*, \mathcal{N})$ under a pair of commuting automorphisms (see Definition 3.1)

Proof of Theorem 4.2. By [12], all trace-scaling actions β of Γ on $\mathcal{N}$ are outer conjugate, so it suffices to show the theorem for a particular one. Let $\lambda_i = (1 + \mu_i)^{-1}$ and let $\mathcal{M}^{\lambda_i}$ be Powers' factor with Powers' state $\omega_i = \omega_{\lambda_i}$ [13]. It is well known (cf. [14], Section 5) that the centralizer $\mathcal{M}_{\omega_i}^{\lambda_i}$ is the hyperfinite II_1 -factor, R , and the point spectrum of Δ_{ω_i} equals $\{\mu_i^n \mid n \in \mathbf{Z}\}$. Consider the AFD (approximately finite dimensional), purely infinite factor $\mathcal{M} = \mathcal{M}^{\lambda_1} \otimes \mathcal{M}^{\lambda_2}$ and let $\varphi = \omega_1 \otimes \omega_2$ be the tensor product state. Then by Lemma 4.1 φ is

almost periodic, the point spectrum of Δ_φ is $\Gamma = \{\mu_1^{n_1} \mu_2^{n_2} \mid n_1, n_2 \in \mathbf{Z}\}$ and the centralizer $\mathcal{M}_\varphi = R \otimes R \simeq R$ is a factor. Now by Proposition 2.12, the discrete decomposition of $\mathcal{M}$ associated to φ is a trace-scaling action of Γ on the injective II_∞ factor. By the factoriality of the centralizer and Connes [4], 3.2.7., it follows that $\mathcal{M}$ is a type III_1 factor, so Takesaki's continuous decomposition is the one-parameter trace-scaling action of $\mathbf{R}_+^*$ on the injective type II_∞ factor. Now Corollary 3.4 finishes the proof. $\square$

Remark 4.4. A more concrete picture of the one-parameter trace-scaling action in terms of operators on Hilbert space can be had from the proof of Theorem 4.2. In [14], Section 5, Takesaki showed that ω_i is a homogeneous, periodic state. Thus, by general results of the same paper, there is an isometry, v_i , in the spectral subspace $\mathcal{M}_{\omega_i}^{\lambda_i}(\mu_i^{-1})$ such that

$$(10) \quad \mathcal{M}_{\omega_i}^{\lambda_i}(\mu_i^n) = \begin{cases} M_{\omega_i}^{\lambda_i} v_i^{-n} & \text{if } n \leq 0 \\ (v_i^*)^n M_{\omega_i}^{\lambda_i} & \text{if } n \geq 0. \end{cases}$$

Keeping in mind that the centralizers $\mathcal{M}_{\omega_i}^{\lambda_i}$ are copies of the hyperfinite II_1 factor, R , and noting that the spectral subspaces of the tensor product are

$$\mathcal{M}_\varphi(\mu_1^{n_1} \mu_2^{n_2}) = \mathcal{M}_{\omega_1}^{\lambda_1}(\mu_1^{n_1}) \otimes \mathcal{M}_{\omega_2}^{\lambda_2}(\mu_2^{n_2}),$$

one can use (10) to describe the spectral subspaces of $\mathcal{M}$ under φ . Let $\mathcal{H}$ be the Hilbert space on which $\mathcal{M}$ acts. Then by Proposition 3.2, the injective II_∞ factor is the von Neumann algebra on $L^2(\mathbf{R}_+^*) \otimes \mathcal{H}$ generated by

$$(L^\infty(\mathbf{R}_+^*) \otimes 1) \cup \{\lambda_\gamma \otimes a \mid \gamma \in \Gamma, a \in \mathcal{M}_\varphi(\gamma)\}$$

and the one-parameter trace-scaling action is given by $\text{Ad}(\lambda_t \otimes 1)$ for $t \in \mathbf{R}_+^*$, i.e., translation by t on the first component.

5. Another model for the one-parameter trace-scaling action on the injective II_∞ factor. We now explain a model which was described to the author by V. F. R. Jones, and depends on work of P.-L. Aubert. The usual (linear) action of $SL(2, \mathbf{Z})$ on $\mathbf{R}^2$ preserves Lebesgue measure. Let $\mathcal{N}$ be the crossed product von Neumann algebra of $L^\infty(\mathbf{R}^2)$ by the action of $SL(2, \mathbf{Z})$. From Lebesgue measure on $\mathbf{R}^2$, one gets a n.f.s. trace on $\mathcal{N}$. In [2], Aubert showed the somewhat surprising fact that $\mathcal{N}$ is the injective II_∞ factor. Then for $t \in \mathbf{R}_+^*$, the dilation of $\mathbf{R}^2$ by t gives an automorphism, α_t , of $\mathcal{N}$ that scales the trace by t . So $t \mapsto \alpha_t$ is the one-parameter trace-scaling action of $\mathbf{R}_+^*$ on the injective II_∞ factor.

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