Periodic Solutions for Degenerate Diffusion Equations

NORIKO MIZOGUCHI

ABSTRACT. In this paper, we are concerned with the existence of nonnegative periodic solutions of

$$u_t = \Delta u^m + h(t)f(u) \quad \text{in } \Omega \times \mathbb{R}$$

with the Dirichlet boundary condition, where $m > 1$, Ω is a smoothly bounded domain in $\mathbb{R}^N$ and h is a given positive periodic function on $\mathbb{R}$. The forcing term in this paper satisfies

$$\liminf_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} > \lambda_1 \quad \text{uniformly in } t \in \mathbb{R};$$

for example, $f(u) = u^p$ with $p \geq m$ and $h > \lambda_1$. We adapt the Leray-Schauder degree theory in the proof. To do that, we need an a priori estimate for solutions in L^∞ . Inequalities of Harnack type and the blow up (or scaling) argument play crucial parts.

1. Introduction. We are concerned with the existence of nonnegative solutions of the following problem

$$(1) \quad \begin{cases} u_t = \Delta u^m + h(t)f(u) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+T) = u(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

where $m > 1$, Ω is a smoothly bounded domain in $\mathbb{R}^N$, h is a positive T -periodic function on $\mathbb{R}$ and f is a nonnegative function on $\mathbb{R}_+$, which is superlinear or asymptotically linear; for example, $f(u) = u^p$ with $p \geq m$. Here $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The problem (1) without periodicity was investigated by many authors (see, e.g., [1], [5], [12], [14], [15]). On the other

hand, we know few articles on periodic problems. Nokao [13] obtained periodic solutions in a sublinear case. It seems that there are no results when

$$\liminf_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} > \lambda_1 \quad \text{uniformly in } t \in \mathbb{R},$$

where λ_1 is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition in Ω . In [3] and [4], Esteban showed the existence of positive solutions for a semilinear version of (1) with Δu^m replaced by Δu .

Putting $v = u^m$ for a nonnegative solution u of (1), v is nonnegative and solves the following quasilinear problem

$$(2) \quad \begin{cases} v_t = mv^\gamma \{\Delta v + h(t)\tilde{f}(v)\} & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

where $\gamma = 1 - 1/m$ and $\tilde{f}(\rho) = f(\rho^{1/m})$. It is clear that the converse implication also holds. Therefore, we shall show the existence of a nonnegative solution of (1) by treating (2) in the present paper. Our purpose is to prove the following two results.

Theorem 1. *Let Ω be convex and $m < p < \infty$ if $N = 1$ or Ω is an open ball in $\mathbb{R}^2$ and $m < p < (Nm + 2)/N$ if $N \geq 2$. Suppose that $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a nonnegative function with $f'(\rho)\rho^{1-m} \in C(\mathbb{R}_+)$ such that*

$$(3) \quad 0 < \lim_{\rho \rightarrow \infty} \frac{f(\rho)}{\rho^p} < \infty,$$

$$(4) \quad \limsup_{\rho \rightarrow \infty} \frac{f'(\rho)}{\rho^{p-1}} < \infty$$

and

$$(5) \quad \lim_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} < \lambda_1 \quad \text{uniformly in } t \in \mathbb{R}.$$

If $h : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic positive function of class C^1 , then there exists a nonnegative nontrivial weak solution of (1).

Here a weak solution of (1) means a function u such that $u^m \in L_{\text{loc}}^2(\mathbb{R}; H_0^1(\Omega)) \cap L^\infty(\Omega \times \mathbb{R})$, $(u^m)_t \in L_{\text{loc}}^2(\mathbb{R}; L^2(\Omega))$, $u^{2m-1} \in C(\mathbb{R}; L^p(\Omega))$ for all $1 \leq p < \infty$, $u(t+T) = u(t)$ a.e. in Ω for all $t \in \mathbb{R}$ and

$$\iint \{u\varphi_t - \nabla u^m \nabla \varphi + h(t)f(u)\varphi\} dx dt = 0$$

for each $\varphi \in C_0^\infty(\Omega \times (0, T))$.

Theorem 2. Assume that $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a nonnegative function with $f'(\rho)\rho^{1-m} \in C(\mathbb{R}_+)$ satisfying

$$(6) \quad \lambda_1 < \liminf_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} \leq \limsup_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} < \infty \quad \text{uniformly in } t \in \mathbb{R}$$

and the assumptions (4) with $p = m$ and (5) in Theorem 1. If $h : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic positive function of class C^1 , then the problem (1) possesses a nonnegative nontrivial weak solution.

We mention the sublinear case at the end of this paper.

We consider an approximate equation of (2) and seek for a nonnegative weak solution of (2) as a limit of the positive approximate solutions. The Leray-Schauder degree theory is adapted to solve the approximate equation. We need an upper bound for positive solutions of equations appearing in certain homotopy to calculate the value of degree in a large ball. An a priori upper estimate for solutions is crucial in this paper. Estimates in L^∞ were obtained for global solutions of initial value problems, for example in [5], [14] and [15], but they depend on initial values. We have to derive a priori estimates not depending on initial values. Our tool to get an upper bound is the blow-up (or scaling) argument which was used in [3], [4] and [7]. Our blow-up argument is more complicated since (2) is quasilinear and not semilinear. We make use of inequalities of Harnack type in time and in the space direction for the homotopic equations which are derived in Section 2. Section 3 and Section 4 are devoted to proofs of Theorem 1 and Theorem 2, respectively.

2. Harnack type inequalities. Let $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ be a solution of

$$(7) \quad \begin{cases} w_t = mw^\gamma \{\Delta w + g(w, x, t)\} & \text{in } \Omega \times \mathbb{R}, \\ w = \varepsilon & \text{on } \partial\Omega \times \mathbb{R}, \\ w(t+T) = w(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$. Similar inequalities of Harnack type were derived in [8] in the case of $\gamma \geq 1$, but they cannot apply to our situation because of $\gamma < 1$ in this paper. Put $z = w_t$. Differentiating (7) in t yields

$$z_t = \frac{\gamma}{w} z^2 + mw^\gamma \Delta z + mw^\gamma g_w z + mw^\gamma g_t,$$

where g_w and g_t denote the derivatives of g with respect to w and t , respectively. Letting $z(x_0, t_0) = \min_{\bar{\Omega} \times \mathbb{R}} z$, we have

$$\gamma z^2 + mw^{\gamma+1} g_w z + mw^{\gamma+1} g_t \leq 0 \quad \text{at } (x_0, t_0)$$

and hence

$$(8) \quad \min z \geq -\frac{m(g_w)_+}{\gamma} M^{\gamma+1} - \left(\frac{m}{\gamma} \|g_t\|_\infty \right)^{1/2} M^{(\gamma+1)/2},$$

where $M = \max_{\bar{\Omega} \times \mathbb{R}} w$, $f_+ = \max(f, 0)$ and $\|\cdot\|_\infty$ denotes the supremum norm.

Lemma 1. *If $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ is a solution of (7) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$, then*

$$w(x, t) \geq w(x, s) - C_M(t - s)$$

for all $s, t \in \mathbb{R}$ with $s \leq t$ and $x \in \bar{\Omega}$, where

$$C_M = \frac{m}{\gamma} G_1 M^{\gamma+1} + \left(\frac{m}{\gamma}\right)^{1/2} G_2 M^{(\gamma+1)/2}, \quad M = \max w;$$

$$G_1 = \max\{(g_w)_+ : 0 < w \leq M, (x, t) \in \bar{\Omega} \times \mathbb{R}\},$$

$$G_2 = \max\{\|g_t\|_\infty^{1/2} : 0 < w \leq M, (x, t) \in \bar{\Omega} \times \mathbb{R}\}.$$

Proof. Since $w_t \geq -C_M$ in $\bar{\Omega} \times \mathbb{R}$ from (8), it suffices to integrate this inequality over (s, t) . $\square$

Lemma 2. *Let $N = 1$. If $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ is a solution of (7) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$, then*

$$w^{\gamma+1}(x, t_0) \geq w^{\gamma+1}(x_0, t_0) - \frac{\gamma+1}{2} \tilde{C}_M (x - x_0)^2 \quad \text{for } x \in \bar{\Omega},$$

where

$$\tilde{C}_M = \frac{1}{m} C_M + M^\gamma G_3, \quad M = \max w = w(x_0, t_0),$$

$$G_3 = \max\{g : 0 < w \leq M, (x, t) \in \bar{\Omega} \times \mathbb{R}\}.$$

Proof. Fix $t \in \mathbb{R}$ arbitrarily. From (8), it follows that

$$w^\gamma w_{xx} = \frac{1}{m} w_t - w^\gamma g \geq -\tilde{C}_M \quad \text{in } \Omega,$$

and hence

$$(9) \quad (w^{\gamma+1})_{xx} = (\gamma+1)(\gamma w^{\gamma-1} w_x^2 + w^\gamma w_{xx}) \geq -(\gamma+1) \tilde{C}_M \quad \text{in } \Omega.$$

Integrating this inequality over (x_0, x) and then the obtained inequality over (x_0, x) , we have the desired inequality. $\square$

Lemma 3. *Let $N \geq 2$. If $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ is a solution of (7) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$, then*

$$w^{\gamma+1}(x_0, t) \leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} w^{\gamma+1}(x, t) dx + \frac{(\gamma+1) \tilde{C}_M}{2(N+2)} \rho^2$$

for each $x_0 \in \Omega$, $t \in \mathbb{R}$ and $0 < \rho < \text{dist}(x_0, \partial\Omega)$, where ω_N denotes the volume of unit balls in $\mathbb{R}^N$.

Proof. Fix $t \in \mathbb{R}$ arbitrarily. We get

$$(10) \quad \Delta w^{\gamma+1} \geq -(\gamma+1)\tilde{C}_M \quad \text{in } \Omega$$

in the same way as in (9). That is, it holds that

$$\Delta \left(w^{\gamma+1} + \frac{(\gamma+1)\tilde{C}_M}{2N} \|x - x_0\|^2 \right) \geq 0 \quad \text{in } \Omega$$

for every $x_0 \in \Omega$. According to the mean value theorem, it follows that for each $0 < \rho < \text{dist}(x_0, \partial\Omega)$,

$$\begin{aligned} \omega^{\gamma+1}(x_0, t) &\leq \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \left\{ \omega^{\gamma+1}(x, t) + \frac{(\gamma+1)\tilde{C}_M}{2N} \|x - x_0\|^2 \right\} dx \\ &= \frac{1}{\omega_N \rho^N} \int_{B_\rho(x_0)} \omega^{\gamma+1}(x, t) dx + \frac{(\gamma+1)\tilde{C}_M}{2(N+2)} \rho^2. \end{aligned} \quad \square$$

3. Proof of Theorem 1. To solve the problem (2), we consider the following approximate equation

$$(11) \quad \begin{cases} v_t = m(v + \varepsilon)^\gamma \{ \Delta v + h(t) \tilde{f}(u) \} & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\varepsilon > 0$. Fix a positive constant k . Set

$$\Phi(\rho, x, t) = k\rho_+ + h(t)\tilde{f}(\rho_+)$$

and

$$\tilde{\Phi}(\rho, x, t) = k\rho_+ + \lambda_1\rho_+ + h(t)\tilde{f}(\rho_+) + 1$$

for all $(\rho, x, t) \in \mathbb{R} \times \bar{\Omega} \times \mathbb{R}$. The Leray-Schauder degree theory is adapted to obtain approximate solutions. As seen later, to calculate the value of degree in a large ball, we need an a priori upper bound for positive solutions of

$$(12) \quad \begin{cases} v_t = m(v + \varepsilon)^\gamma \{ \Delta v - kv + \tau\Phi(v) + (1-\tau)\tilde{\Phi}(v) \} & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\tau \in [0, 1]$. By standard regularity theory and the maximum principle, every solution v is a classical positive solution and satisfies $\Delta v_t \in C(\bar{\Omega} \times \mathbb{R})$ under the

regularity assumptions of f and h . Putting $w = v + \varepsilon$ for each positive solution v of (12), w satisfies

$$(13) \quad \begin{cases} w_t = mw^\gamma \{ \Delta w + h(t)\tilde{f}(w - \varepsilon) + (1 - \tau)\lambda_1 w \\ \quad + (1 - \tau)(1 - \lambda_1 \varepsilon) \} & \text{in } \Omega \times \mathbb{R}, \\ w = \varepsilon & \text{on } \partial\Omega \times \mathbb{R}, \\ w(t + T) = w(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\tau \in [0, 1]$. From the assumption (3), $\tilde{f}$ is written as

$$\tilde{f}(\rho) = c\rho^q + \varphi(\rho) \quad \text{for } \rho \geq 0$$

with $c > 0$, $q = p/m$ and $\varphi \in o(\rho^q)$ as $\rho \rightarrow \infty$. By the hypotheses on h , there are $\alpha, \beta > 0$ such that

$$\alpha \leq h(t) \leq \beta \quad \text{for all } t \in \mathbb{R}.$$

Let $Q = \Omega \times (0, T)$.

Lemma 4. *Under the assumptions of Theorem 1, there exist positive constants K_1, K_2, K_3 , such that*

$$\int \int_Q w(x, t)^q dx dt \leq K_1 \quad \text{and} \quad \int \int_Q \frac{w_t^2}{w^\gamma} dx dt \leq K_2 M + K_3$$

for each solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$, where $M = \max w$.

Proof. It suffices to prove the existence of $C_1, C_2, C_3 > 0$ such that

$$\int \int_Q v^q dx dt \leq C_1 \quad \text{and} \quad \int \int_Q \frac{v_t^2}{(v + \varepsilon)^\gamma} dx dt \leq C_2 M + C_3$$

for each positive solution v of (12) and $0 < \varepsilon \leq 1$. Let ϕ_1 be a positive eigenfunction corresponding to λ_1 . Multiplying $\phi_1/(v + \varepsilon)^\gamma$ with (12) and integrating over Q yields

$$\int \int_Q \{ -\lambda_1 v + h(t)\tilde{f}(v) + (1 - \tau)(\lambda_1 v + 1) \} \phi dx dt = 0.$$

Since $|\varphi(\rho)| \leq (\alpha c/2\beta)\rho^q + c_0$ for all $\rho \geq 0$ with some $c_0 > 0$ from $\varphi \in o(\rho^q)$ as $\rho \rightarrow \infty$, we get

$$\alpha c \int \int_Q v^q \phi_1 dx dt \leq \int \int_Q \left\{ \lambda_1 v + \beta \left(\frac{\alpha c}{2\beta} v^q + c_0 \right) \right\} \phi_1 dx dt,$$

so there is $c_1 > 0$ such that

$$(14) \quad \int \int_Q v^q \phi_1 dx dt \leq c_1.$$

By the same argument as Step 2 in the proof of Theorem 1.1 in [2], there are positive constants δ_0, c_2 such that

$$(15) \quad v \leq c_2 \quad \Omega_{\delta_0} \times \mathbb{R}$$

for each positive solution v of (12), where $\Omega_{\delta_0} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta_0\}$. Therefore, we obtain some positive constant C_1 such that

$$\int \int_Q v^q dx dt \leq C_1$$

for each positive solution v of (12). On the other hand, multiplying $v_t/(v + \varepsilon)^\gamma$ with (12) and integrating over Q yields

$$\int \int_Q \frac{v_t^2}{(v + \varepsilon)^\gamma} dx dt = -m \int \int h_t \tilde{F}(v) dx dt,$$

where

$$\tilde{F}(\rho) = \int_0^\rho \tilde{f}(r) dr \quad \text{for } \rho \geq 0.$$

Since the growth condition of $\tilde{f}$ implies the existence of some $c_3, c_4 > 0$ such that

$$\tilde{F}(\rho) \leq c_3 \rho^{q+1} + c_4 \quad \text{for all } \rho \geq 0,$$

we derive the second inequality. □

Lemma 5. *Let $N = 1$. Under the hypotheses of Theorem 1, there exist positive constants δ_1, ρ_1, C_1 such that if $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ is a solution of (13) with $w_{xxt} \in C(\bar{\Omega} \times \mathbb{R})$, $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $M = \max w = w(x_0, t_0) > C_1$, then*

$$w(x, t) \geq \frac{1}{4}M \quad \text{for } x \in \mathbb{R} \text{ with } |x - x_0| \leq \delta_1 M^{(1-q)/2} \text{ and}$$

$$t_0 \leq t \leq t_0 + \rho_1 M^{1-\gamma-q}.$$

Proof. Set

$$g(\rho, x, t) = h(t)\tilde{f}(\rho - \varepsilon) + (1 - \tau)\lambda_1\rho + (1 - \tau)(1 - \lambda_1\varepsilon).$$

By the assumptions (3), (4), we have positive constants c_0, c_1, c_2 such that

$$C_M \leq c_0 M^{\gamma+q} \quad \text{and} \quad \tilde{C}_M \leq c_1 M^{\gamma+q}$$

if $M \geq c_2$, where C_M and $\tilde{C}_M$ are as in Lemma 2. Set

$$\delta_1 = \left\{ \frac{2}{c_1(\gamma+1)} \left(1 - \left(\frac{1}{2} \right)^{\gamma+1} \right) \right\}^{1/2}.$$

Since x_0 is the middle point of Ω from [6], $\{x \in \mathbb{R} : |x - x_0| \leq \delta_1 M^{(1-q)/2}\} \subset \Omega$ if $M > (2\delta_1/|\Omega|)^{2/(q-1)}$, where $|\Omega|$ denotes the length of Ω . Put $C_1 = \max(c_2, (2\delta_1/|\Omega|)^{2/(q-1)})$. From Lemma 2, for each $x \in \mathbb{R}$ with $|x - x_0| \leq \delta_1 M^{(1-q)/2}$, we have

$$w^{\gamma+1}(x, t_0) \geq M^{\gamma+1} - \frac{\gamma+1}{2} c_1 M^{\gamma+q} (x - x_0)^2 \geq \left(\frac{M}{2} \right)^{\gamma+1},$$

that is,

$$w(x, t_0) \geq \frac{M}{2} \quad \text{for each } x \in \mathbb{R} \text{ with } |x - x_0| \leq \delta_1 M^{(1-q)/2}.$$

Putting $\rho_1 = 1/4c_0$, Lemma 1 implies

$$w(x, t) \geq w(x, t_0) - c_0 M^{\gamma+q} (t - t_0) \geq \frac{1}{4} M$$

if $M > C_1$, $|x - x_0| \leq \delta_1 M^{(1-q)/2}$ and $t_0 \leq t \leq t_0 + \rho_1 M^{1-\gamma-q}$. $\square$

Lemma 6. *Let $N \geq 2$. Under the hypotheses of Theorem 1, there exist positive constants δ_2, ρ_2, C_2 such that if $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ is a solution of (13) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$, $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $M = \max w = w(x_0, t_0) > C_2$,*

$$\int_{B_\delta(x_0)} w^{\gamma+1}(x, t) dx \geq \left(\frac{1}{2} \right)^{\gamma+2} \omega_N \delta^N M^{\gamma+1}$$

for $0 < \delta \leq \delta_2 M^{(1-q)/2}$ and $t_0 \leq t \leq t_0 + \rho_2 M^{1-\gamma-q}$.

Proof. Let c_0, c_1, c_2 be as in the proof of Lemma 5. Putting $\rho_2 = 1/2c_0$, Lemma 1 implies

$$w(x_0, t) \geq \frac{M}{2} \quad \text{for } t_0 \leq t \leq t_0 + \rho_2 M^{1-\gamma-q}.$$

Set $\delta_2 = \{((N+2)/(\gamma+1)c_1)(\frac{1}{2})^{\gamma+1}\}^{1/2}$. Since Ω is convex, there is $\delta_0 > 0$ such that $\text{dist}(x_0, \partial\Omega) \geq \delta_0$ by [6], so $B_\delta(x_0) \subset \Omega$ if $\delta \leq \delta_2 M^{(1-q)/2}$ and $M >$

$(\delta_2/\delta_0)^{2/(q-1)}$. Putting $C_2 = \max(c_2, (\delta_2/\delta_0)^{2/(q-1)})$, Lemma 3 yields the desired inequality if $M > C_2$, $0 < \delta \leq \delta_2 M^{(1-q)/2}$ and $t_0 \leq t \leq t_0 + \rho_2 M^{1-\gamma-q}$. $\square$

We now derive upper bounds for solutions of (13) by the blow-up argument. The properties of solutions of (13) obtained in [6] play an important part in Lemmas 7–9.

Lemma 7. *Let Ω be an open interval in $\mathbb{R}$. Under the assumptions of Theorem 1, there is $M_0 > 0$ such that $\max w \leq M_0$ for every solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $w_{xxt} \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$.*

Proof. Fix $0 < \varepsilon \leq 1$. Assuming that there are no upper bounds, we get a sequence $\{w_n\} \subset C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of solutions of (13) with $\Delta w_{nt} \in C(\bar{\Omega} \times \mathbb{R})$ and $w_n > \varepsilon$ in $\Omega \times \mathbb{R}$ such that $M_n = \max w_n = w_n(x_n, t_n) \rightarrow \infty$ as $n \rightarrow \infty$. By [6], x_n is the middle point of Ω and $w_n(\cdot, t)$ is symmetric with respect to $x = x_n$ and strictly decreasing in $x > x_n$ for each $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Therefore, we may assume that $\Omega = (-a, a)$ without loss of generality. Let $k = (q-1)/2$ and $\ell = q + \gamma - 1$. Put $\lambda_n = 1/M_n$, $B_n = \{y \in \mathbb{R} : |y| < a/\lambda_n^k\}$ and $Q_n = B_n \times \mathbb{R}$. Define $v_n : Q_n \rightarrow \mathbb{R}$ by

$$(16) \quad v_n(y, s) = \lambda_n w_n(\lambda_n^k y, t_n + \lambda_n^\ell s) \quad \text{for } (y, s) \in Q_n.$$

Then it is immediate that $0 < v_n \leq 1$ and $\max v_n = 1 = v_n(0, 0)$. By straightforward calculation, we derive

$$(17) \quad v_{ns} = m v_n^\gamma \{v_{nyy} + ch(t_n + \lambda_n^\ell s) v_n^q + \lambda_n^q \tilde{\varphi}(\lambda_n^{-1} v_n)\},$$

where $\tilde{\varphi} \in o(\rho^q)$ as $\rho \rightarrow \infty$, i.e., for any $\varepsilon > 0$, $|\tilde{\varphi}(\rho)| \leq \varepsilon \rho^q + C_\varepsilon$ for all $\rho \geq 0$ with some $C_\varepsilon > 0$. Therefore, it holds that

$$(18) \quad -\frac{1}{1-\gamma} \iint v_n^{1-\gamma} \psi_s dy ds \\ = m \iint \{v_n \psi_{yy} + ch(t_n + \lambda_n^\ell s) v_n^q \psi + \lambda_n^q \tilde{\varphi}(\lambda_n^{-1} v_n) \psi\} dy ds$$

for each $\psi \in C_0^\infty(\mathbb{R}^2)$. Fix $R > 0$ arbitrarily. By Lemma 4, we get

$$\begin{aligned} \int_{-R}^R \int_{-R}^R (v_n^{1+\gamma})_s^2 dy ds &\leq (1+\gamma)^2 \int_{-R}^R \int_{-R}^R v_{ns}^2 dy ds \\ &= (1+\gamma)^2 \lambda_n^{\ell+2-k} \int_{t_n-\lambda_n^\ell R}^{t_n+\lambda_n^\ell R} \int_{-\lambda_n^k R}^{\lambda_n^k R} w_{nt}^2 dx dt \\ &\leq 2K_2 \lambda_n^{\ell+1-k-\gamma} + 2K_3 \lambda_n^{\ell+2-k-\gamma}. \end{aligned}$$

From $\ell + 1 - k - \gamma > 0$, it follows that

$$\int_{-R}^R \int_{-R}^R (v_n^{1+\gamma})_s^2 dy ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, for $0 < x \leq \lambda_n^k R$, integrating (9) over $(0, x)$ yields

$$(w_n^{\gamma+1})_x(x, t) \geq -(\gamma + 1)c_1 M_n^{\gamma+q} x \geq -(\gamma + 1)c_1 \lambda_n^{k-\gamma-q} R,$$

since $\tilde{C}_{M_n} \leq c_1 M_n^{\gamma+q}$ for all n with some $c_1 > 0$. From $(w_n^{\gamma+1})_x(x, t) < 0$, for $x > 0$ and the symmetry of $w_n(\cdot, t)$ with respect to $x = 0$, we get

$$\int_{-\lambda_n^k R}^{\lambda_n^k R} \int_{t_n - \lambda_n^\ell R}^{t_n + \lambda_n^\ell R} (w_n^{\gamma+1})_x^2 dx dt \leq 4R^2(\gamma + 1)^2 c_1^2 \lambda_n^{3k-2\gamma-2q+\ell} \quad \text{for all } n.$$

Therefore, we have

$$\int_{-R}^R \int_{-R}^R (v_n^{\gamma+1})_y^2 dy ds \leq 4(\gamma + 1)^2 R^2 c_1^2.$$

By this boundedness of $\{v_n^{\gamma+1}\}$ in $H_{\text{loc}}^1(\mathbb{R}^2)$, we may assume that $\{v_n^{\gamma+1}\}$ converges to some nonnegative function ξ weakly in $H_{\text{loc}}^1(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. Putting $v = \xi^{1/(\gamma+1)}$, $v_n \rightarrow v$ a.e. in $\mathbb{R}^2$. Letting $n \rightarrow \infty$ in (18) yields

$$-\frac{1}{1-\gamma} \iint v^{1-\gamma} \psi_s dy ds = m \iint \{v \psi_{yy} + ch(t_0) v^q \psi\} dy ds$$

for all $\psi \in C_0^\infty(\mathbb{R}^2)$, where $t_0 = \lim_{n \rightarrow \infty} t_n$. From $(v^{\gamma+1})_s = 0$ a.e. in $\mathbb{R}^2$, $v^{\gamma+1}$ is independent of s and hence so is v . Therefore, it holds that

$$\iint \{v \tilde{\psi}_{yy} + ch(t_0) v^q \tilde{\psi}\} dy ds = 0$$

for all $\tilde{\psi} \in C_0^\infty(\mathbb{R})$. According to the standard regularity theory, v satisfies

$$v_{yy} + ch(t_0) v^q = 0 \quad \text{in } \mathbb{R}$$

in the classical sense. By Liouville's theorem, this implies $v \equiv 0$. On the other hand, Lemma 5 yields

$$\text{for } y \text{ with } |y| \leq \delta_1 \text{ and } (0, s) \in \bigcup_{t \leq \frac{1}{4} \rho_1} \{t\} \times \mathbb{R}.$$

This is a contradiction. □

Lemma 8. *Let Ω be an open ball in $\mathbb{R}^2$. Under the assumptions of Theorem 1, there is $M_0 > 0$ such that $\max w \leq M_0$ for each solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$.*

Proof. Without loss of generality, we may assume that $\Omega = B_a(0)$. Assume that there are no upper bounds and let $\{w_n\}$ be as in the proof of Lemma 7. By [6], for $n \in \mathbb{N}$, and $t \in \mathbb{R}$, $w_n(\cdot, t)$ is radially symmetric and $w_n(\rho, t)$ is strictly decreasing in ρ , where $\rho = \|x\|$. From (10),

$$\Delta w_n^{\gamma+1} \geq -(\gamma+1)c_1 M_n^{\gamma+q} \quad \text{in } \Omega$$

with some $c_1 > 0$. Since

$$\Delta w_n^{\gamma+1} = (w_n^{\gamma+1})_{\rho\rho} + \frac{1}{\rho}(w_n^{\gamma+1})_{\rho},$$

we get

$$(\rho(w_n^{\gamma+1})_{\rho})_{\rho} \geq -(\gamma+1)c_1 M_n^{\gamma+q} \rho \quad \text{for } 0 < \rho < a$$

and hence,

$$(w_n^{\gamma+1})_{\rho}(\rho) \geq -\frac{(\gamma+1)c_1}{2} M_n^{\gamma+q} \rho \quad \text{for } 0 < \rho < a.$$

It follows that

$$|\nabla w_n^{\gamma+1}|^2 = (w_n^{\gamma+1})_{\rho}^2 \leq \frac{(\gamma+1)^2 c_1^2}{4} M_n^{2(\gamma+q)} \rho^2$$

for $0 < \rho < a$ since $w_{n\rho}(\rho, t) < 0$ for $0 < \rho < a$ and $t \in \mathbb{R}$. Put

$$\lambda_n = \frac{1}{M_n}, \quad B_n = \left\{ y \in \mathbb{R}^2 : \|y\| \leq \frac{a}{\lambda_n^k} \right\} \quad \text{and} \quad Q_n = B_n \times \mathbb{R}.$$

Defining $v_n : Q_n \rightarrow \mathbb{R}$ by (16), we can show this lemma in the same way as in the proof of Lemma 7. $\square$

Lemma 9. *Let $N \geq 2$ and $m < p < (Nm+2)/N$. Under the assumptions of Theorem 1, there is $M_0 > 0$ such that $\max w \leq M_0$ for every solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$.*

Proof. Assuming that there are no upper bounds, let $\{w_n\}$ and $\{(x_n, t_n)\}$ be as in the proof of Lemma 7. By the convexity of Ω , we obtain $\delta_0 > 0$ such that

$$\text{dist}(x_n, \partial\Omega) \geq \delta_0 \quad \text{for all } n$$

from [6]. Put $\lambda_n = 1/M_n$, $B_n = \{y \in \mathbb{R}^N : \|y - x_n\| \leq \delta_0/\lambda_n^k\}$ and $Q_n = B_n \times \mathbb{R}$. Defining $v_n : Q_n \rightarrow \mathbb{R}$ by

$$v_n(y, s) = \lambda_n w_n(x_n + \lambda_n^k y, t + \lambda_n^\ell s) \quad \text{for } (y, s) \in Q_n,$$

it holds that

$$\left(\frac{1}{1-\gamma} v_n^{1-\gamma} \right)_s \geq m \{ \Delta v_n + (ca - \varepsilon) v_n^q - C_\varepsilon \lambda_n^q \}$$

by the same argument as in the proof of Lemma 7. Taking $\varepsilon = c\alpha/2$, we get

$$(19) \quad \left(\frac{1}{1-\gamma} v_n^{1-\gamma} \right)_s \geq m \left\{ \Delta v_n + \frac{c\alpha}{2} v_n^q - C_\varepsilon \lambda_n^q \right\}.$$

For each $R > 0$, let ν_R and ϕ_R denote the first eigenvalue of $-\Delta$ in $B_R(0) \subset \mathbb{R}^N$ with the Dirichlet boundary condition and the eigenfunction corresponding to ν_R with $\int_{B_R(0)} \phi_R dy = 1$. Then it is easily seen that

$$\nu_R = \frac{\nu_1}{R^2} \quad \text{and} \quad \phi_R(y) = \frac{1}{R^N} \phi_1 \left(\frac{y}{R} \right).$$

Multiplying ϕ_R with (19) and integrating over $B_R(0)$ yields

$$\frac{d}{ds} \int_{B_R(0)} \frac{1}{1-\gamma} v_n^{1-\gamma} \phi_R dy \geq m \int_{B_R(0)} \left\{ -\nu_R v_n \phi_R + \frac{c\alpha}{2} v_n^q \phi_R - C_\varepsilon \lambda_n^q \phi_R \right\} dy.$$

By Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{1-\gamma} \frac{d}{ds} \int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \\ & \geq m \int_{B_R(0)} v_n \phi_R dy \left\{ -\nu_R + \frac{c\alpha}{2} \left(\int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \right)^{(q-1)/(1-\gamma)} \right. \\ & \quad \left. - C_\varepsilon \lambda_n^q \left(\int_{B_r(0)} v_n^{1-\gamma} \phi_R dy \right)^{-1/(1-\gamma)} \right\}. \end{aligned}$$

Setting

$$\xi(s) = \int_{B_R(0)} v_n^{1-\gamma} \phi_R dy$$

for $s \in \mathbb{R}$, it holds that

$$\frac{1}{1-\gamma} \frac{d\xi}{ds} \geq m \int_{B_r(0)} v_n \phi_R dy \left(-\nu_R + \frac{c\alpha}{2} \xi^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \xi^{-1/(1-\gamma)} \right).$$

Suppose that

$$-\nu_R + \frac{c\alpha}{2} \xi^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \xi^{-1/(1-\gamma)} > 0$$

at some $s_0 \in \mathbb{R}$. Then we get

$$\frac{1}{1-\gamma} \frac{d\xi}{ds} \geq m \xi^{1/(1-\gamma)} \left(-\nu_R + \frac{c\alpha}{2} \xi^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \xi^{-1/(1-\gamma)} \right)$$

for $s \geq s_0$. This implies that ξ cannot be bounded on $\mathbb{R}$. Therefore, it follows from the boundedness of ξ on $\mathbb{R}$ that

$$-\nu_R + \frac{c\alpha}{2} \xi^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \xi^{-1/(1-\gamma)} \leq 0 \quad \text{for all } s \in \mathbb{R},$$

that is,

$$\begin{aligned} -\nu_R + \frac{c\alpha}{2} \left(\int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \right)^{(q-1)/(1-\gamma)} \\ - C_\varepsilon \lambda_n^q \left(\int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \right)^{-1/(1-\gamma)} \leq 0 \end{aligned}$$

for all $s \in \mathbb{R}$. Letting ρ_2, δ_2 be as in Lemma 6, we see

$$(20) \quad \int_{B_{\delta_2}(0)} v_n^{1-\gamma} dy \geq \left(\frac{1}{2} \right)^{\gamma+2} \omega_N \delta_2^N$$

for $0 < s < \rho_2$. We also get

$$(21) \quad \int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \geq c_1 R^{-N} \int_{B_{R/2}(0)} v_n^{1-\gamma} dy,$$

where c_1 is the value of ϕ_1 on $\partial B_{1/2}(0)$. It follows that

$$\int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \geq \left(\frac{1}{2} \right)^{\gamma+2} c_1 \omega_N \delta_2^N R^{-N}$$

for $R \geq 2\delta_2$. Therefore, we derive

$$-\nu_R + \frac{c\alpha}{2} \left(\int_{B_R(0)} v_n^{1-\gamma} \phi_R dy \right)^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \left(\frac{2^{\gamma+2} R^N}{c_1 \omega_N \delta_2^N} \right)^{1/(1-\gamma)} \leq 0.$$

Integrating this inequality over $(0, \rho_2)$ and using Hölder's inequality yields

$$(22) \quad -\nu_R + \frac{c\alpha}{2} \rho_2^{(2-q-\gamma)/(1-\gamma)} \left(\int_0^{\rho_2} \int_{B_R(0)} v_n^{1-\gamma} \phi_R dy ds \right)^{(q-1)/(1-\gamma)} - C_\varepsilon \lambda_n^q \left(\frac{2^{\gamma+2} R^N}{c_1 \omega_N \delta_2^N} \right)^{1/(1-\gamma)} \leq 0.$$

Since $\{v_n^{1-\gamma}\}$ converges to some w in L^∞ in the weak* topology, we get

$$\int_0^{\rho_2} \int_{B_R(0)} w \phi_R dy ds \leq c_2 R^{-2(1-\gamma)/(q-1)}$$

with some $c_2 > 0$, letting $n \rightarrow \infty$ in (22). In the same way as (21), it holds that

$$\int_0^{\rho_2} \int_{B_R(0)} w \phi_R dy ds \geq c_1 R^{-N} \int_0^{\rho_2} \int_{B_{R/2}(0)} w dy ds,$$

and hence

$$\int_0^{\rho_2} \int_{B_{R/2}(0)} w dy ds \leq \frac{c_2}{c_1} R^{N-2(1-\gamma)/(q-1)}.$$

From $p < (Nm + 2)/N$, it follows that

$$\int_0^{\rho_2} \int_{B_{R/2}(0)} w dy ds \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Integrating (20) over $(0, \rho_2)$ yields

$$\int_0^{\rho_2} \int_{B_{R/2}(0)} v_n^{1-\gamma} dy ds \geq \left(\frac{1}{2} \right)^{\gamma+2} \omega_N \rho_2 \delta_2^N$$

for $R \geq 2\delta_2$, which is a contradiction. $\square$

Let X be the space of functions in $C(\bar{\Omega} \times \mathbb{R})$ which are T -periodic in time. The following result was given in Lemma 5 in [9].

Lemma 10. *Assume that k is a positive constant and that a continuous function $a : \mathbb{R} \rightarrow \mathbb{R}$ satisfies*

$$\lambda \leq a(r) \leq \Lambda \quad \text{for } r \in \mathbb{R}$$

with some positive constants λ, Λ . Then for each $u \in X$, there exists a unique solution $v \in \bigcap_{p>1} W_p^{2,1}(Q)$ of

$$\begin{cases} v_t = a(v)(\Delta v - kv + u) & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}. \end{cases}$$

Moreover, for any $p > 1$, there is $C_p > 0$ depending only on λ, Λ, p such that

$$\|v\|_{W_p^{2,1}(Q)} \leq C_p \|u\|_\infty.$$

For $u \in X$, define $S(u)$ by the unique solution of

$$\begin{cases} v_t = ma_\varepsilon(v)(\Delta v - kv + u) & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

where $a_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function such that

$$a_\varepsilon(r) \geq \lambda_\varepsilon \quad \text{and} \quad 0 \leq \frac{da_\varepsilon}{dr} \leq \gamma \varepsilon^{\gamma-1} \quad \text{for } r \in \mathbb{R},$$

with some $\lambda_\varepsilon > 0$, and

$$a_\varepsilon(r) = (r + \varepsilon)^\gamma \quad \text{for } r \geq 0.$$

Since $\max S(u) \leq \|u\|_\infty$ by the maximum principle, S is well-defined and a compact operator from X into itself from Lemma 10.

Lemma 11. *Under the assumptions of Theorem 1, there is $r > 0$ such that*

$$\deg(I - S \circ \Phi, B_r(0), 0) = 1 \quad \text{for all } \varepsilon > 0,$$

where $\deg$ denotes the Leray-Schauder degree and $B_r(0)$ is the closed ball with radius r centered at 0 in X .

Proof. We consider

$$(23) \quad \begin{cases} v_t = ma_\varepsilon(v)(\Delta v - kv + \tau\Phi(v)) & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t+T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\tau \in [0, 1]$. By standard regularity theory and the maximum principle, each nontrivial solution v of (23) is a positive classical solution, so v satisfies

$$(24) \quad v_t = m(v + \varepsilon)^\gamma \{ \Delta v - (1 - \tau)kv + \tau h(t)\tilde{f}(v) \} \quad \text{in } \Omega \times \mathbb{R}.$$

From the assumption (5), there are positive constants c_1, ρ_1 with $C_1 < \lambda_1$ such that

$$h(t)\tilde{f}(\rho) \leq c_1\rho \quad \text{for all } 0 < \rho \leq \rho_1 \text{ and } t \in \mathbb{R}.$$

Then there is $\delta > 0$ such that $\max v \geq \delta$ for each nontrivial solution v of (23). In fact, on the contrary, assume that $\max v_n$ converges to 0 as $n \rightarrow \infty$ for some sequence $\{v_n\}$ of nontrivial solutions of (23). Multiplying $\phi_1/m(v_n + \varepsilon)^\gamma$ with (24) with $v = v_n$ and integrating over Q yields

$$0 \leq \int \int_Q \{ \tau c_1 - \lambda_1 - (1 - \tau)c \} v_n \phi_1 \, dx \, dt < 0$$

for sufficiently large n . This is a contradiction. Taking $r < \delta$, $S \circ (\tau\Phi)$ has no fixed points on $\partial B_r(0)$ for all $\tau \in [0, 1]$. According to the homotopy invariance of degree, we obtain

$$\deg(I - S \circ \Phi, B_r(0), 0) = \deg(I, B_r(0), 0) = 1. \quad \square$$

Lemma 12. *Under the hypothesis of Theorem 1, there is $R > r$ such that*

$$\deg(I - S \circ \Phi, B_R(0), 0) = 0 \quad \text{for all } 0 < \varepsilon \leq 1.$$

Proof. By Lemma 7–9, there is $M_0 > 0$ such that $\max v \leq M_0$ for each positive solution v of (12) and $0 < \varepsilon \leq 1$. Taking $R > \max(M_0, r)$, $S \circ (\tau\Phi + (1 - \tau)\tilde{\Phi})$ does not possess fixed points on $\partial B_R(0)$ for $\tau \in [0, 1]$. From the homotopy invariance of degree, it follows that

$$\deg(I - S \circ \Phi, B_R(0), 0) = \deg(I - S \circ \tilde{\Phi}, B_R(0), 0).$$

Since $S \circ \tilde{\Phi}$ has no fixed points in X , we see

$$\deg(I - S \circ \tilde{\Phi}, B_R(0), 0) = 0.$$

This completes the proof. □

Proof of Theorem 1. By Lemmas 11 and 12, we see

$$\deg(I - S \circ \Phi, B_R(0) \setminus B_r(0), 0) = -1,$$

which yields a positive solution v_ε of (11) such that

$$r < \max v_\varepsilon \leq R \quad \text{for each } 0 < \varepsilon \leq 1.$$

Multiplying $v_\varepsilon/m(v_\varepsilon + \varepsilon)^\gamma$ and $v_{\varepsilon t}/(v_\varepsilon + \varepsilon)^\gamma$ with (11) with $v = v_\varepsilon$ and integrating over Q , respectively, we see that $\{v_\varepsilon\}$ is bounded in $H^1(Q)$. Therefore, we may assume that $\{v_\varepsilon\}$ converges to some v weakly in $H^1(Q)$ and a.e. in Q . It is clear that $\max v \geq r$ and $v \geq 0$ a.e. in Q . For every $\varphi \in C_0^\infty(Q)$, it holds that

$$\int \int_Q \left\{ \frac{1}{m(1-\gamma)} (v_\varepsilon + \varepsilon)^{1-\gamma} \varphi_t - \nabla v_\varepsilon \nabla \varphi + h(t) \tilde{f}(v_\varepsilon) \varphi \right\} dx dt = 0.$$

Letting $\varepsilon \rightarrow 0$, we get

$$(25) \quad \int \int_Q \left\{ \frac{1}{m(1-\gamma)} v^{1-\gamma} \varphi_t - \nabla v \nabla \varphi + h(t) \tilde{f}(v) \varphi \right\} dx dt = 0$$

for every $\varphi \in C_0^\infty(Q)$. Putting $w_\varepsilon = v_\varepsilon + \varepsilon$, there is $C_1 > 0$ such that

$$-\Delta w_\varepsilon^{\gamma+1} \leq C_1 \quad \text{in } \Omega$$

for all $t \in \mathbb{R}$ and $0 < \varepsilon \leq 1$, by (10). Therefore, we have

$$\int_\Omega |\nabla w_\varepsilon^{\gamma+1}|^2 dx \leq C_1 |\Omega| R^{\gamma+1} \quad \text{for all } t \in \mathbb{R} \text{ and } 0 < \varepsilon \leq 1.$$

It is immediate that

$$\int \int_Q |(w_\varepsilon^{\gamma+1})_t|^2 dx dt \leq C_2 \quad \text{for all } 0 < \varepsilon \leq 1$$

with some $C_2 > 0$. Then it follows that

$$\begin{aligned} \int_\Omega |w_\varepsilon^{\gamma+1}(t) - w_\varepsilon^{\gamma+1}(s)|^2 dx &\leq p \left| \int_\Omega \int_s^t |w_\varepsilon^{\gamma+1}(\tau) - w_\varepsilon^{\gamma+1}(s)|^{p-1} |(w_\varepsilon^{\gamma+1})_t| dx d\tau \right| \\ &\leq p(2R)^{p-1} (2C_2)^{1/2} |\Omega|^{1/2} |t - s|^{1/2} \end{aligned}$$

for $s, t \in \mathbb{R}$ with $|s - t| \leq T$ and $1 \leq p < +\infty$. This implies the equicontinuity of $\{w_\varepsilon^{\gamma+1}\}_{0 < \varepsilon \leq 1}$ from $\mathbb{R}$ into $L^p(\Omega)$. According to Ascoli's theorem, $\{w_\varepsilon^{\gamma+1}\}$ converges to $v^{\gamma+1}$ in $L^p(\Omega)$ uniformly on every bounded interval of $\mathbb{R}$. Therefore, we obtain $v^{\gamma+1} \in C(\mathbb{R}; L^p(\Omega))$ and $v^{\gamma+1}(t + T) = v^{\gamma+1}(t)$ in $L^p(\Omega)$ for all $t \in \mathbb{R}$. Setting $u = v^{1/m}$, u is our desired weak solution of (1). $\square$

4. Proof of Theorem 2. We can prove Theorem 2 by the same method as in Section 3, except for a priori upper estimates for solutions of (13). Therefore, it suffices to show the following result.

Lemma 13. *Under the hypotheses of Theorem 2, there is $M_0 > 0$ such that $\max w \leq M_0$ for each solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$.*

Proof. We first get $C_0 > 0$ such that

$$(26) \quad \int_0^T w(x, t) dt \leq C_0 \quad \text{in } \Omega$$

for each solution $w \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of (13) with $\Delta w_t \in C(\bar{\Omega} \times \mathbb{R})$ and $w > \varepsilon$ in $\Omega \times \mathbb{R}$ and $0 < \varepsilon \leq 1$ in the same way as in the proof of Lemma 3 in [9]. Assuming that the class of such solutions of (13) has no upper bounds, there is a sequence $\{w_n\}$ in $C^{2,1}(\bar{\Omega} \times \mathbb{R})$ of solutions of (13) with $\varepsilon = \varepsilon_n$ ($0 < \varepsilon_n \leq 1$) such that $\Delta w_{nt} \in C(\bar{\Omega} \times \mathbb{R})$, $w_n > \varepsilon_n$ in $\Omega \times \mathbb{R}$ and

$$M_n = \max w_n = w_n(x_n, t_n) \rightarrow \Omega \quad \text{as } n \rightarrow \infty.$$

Since

$$C_{M_n} \leq c_1 M_n^{\gamma+1} \quad \text{for all } n$$

with some $c_1 > 0$, where C_{M_n} is as in Lemma 1, it follows that

$$\int_{t_n}^{t_n+T} w_n(x_n, t) dt \geq \int_{t_n}^{t_n+1/c_1 M_n^\gamma} \{M_n - c_1 M_n^{\gamma+1}(t - t_n)\} dt = \frac{1}{2c_1} M_n^{1-\gamma} \rightarrow \infty$$

as $n \rightarrow \infty$ from Lemma 1. This contradicts (26). $\square$

Remark 1.

- (i) By the same argument as in Remark 1 in [9], we see that there are no positive solutions of (1) (or (2)) if

$$\sup_{\rho \neq 0, t \in \mathbb{R}} h(t) \frac{f(\rho)}{\rho^m} < \lambda_1.$$

- (ii) If

$$\limsup_{\rho \rightarrow \infty} h(t) \frac{f(\rho)}{\rho^m} < \lambda_1$$

uniformly in $t \in \mathbb{R}$ and $f(0) \neq 0$, then there is a nonnegative weak solution of (1).

Indeed, we consider

$$(27) \quad \begin{cases} v_t = m(v + \varepsilon)^\gamma (\Delta v + \tau h(t) \tilde{f}(v)) & \text{in } \Omega \times \mathbb{R}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v(t + T) = v(t) & \text{in } \Omega \times \mathbb{R}, \end{cases}$$

with $\tau \in [0, 1]$. By the assumption, there are positive constants $\alpha < \lambda_1$, C_0 such that

$$h(t) \tilde{f}(\rho) \leq \alpha \rho + C_0 \quad \text{for all } \rho \geq 0 \text{ and } t \in \mathbb{R}.$$

Multiplying $1/m(v + \varepsilon)^\gamma$ with (27) and integrating over $(0, T)$ yields

$$\int_0^T \{\Delta v + \alpha v + C_0\} dt \geq 0.$$

Putting $V(x) = \int_0^T v(x, t) dt$, it follows that

$$(28) \quad \begin{cases} \Delta V + \alpha V + C_0 T \geq 0 & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

Then V is a subsolution of

$$(29) \quad \begin{cases} \Delta W + \alpha W + C_0 T = 0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that $\max W \leq C_1$ for every positive solution W of (29) with some $C_1 > 0$ (see [2]), so $\max V \leq C_1$. Therefore, we can obtain a nonnegative weak solution of (27) in the same way as in the proof of Theorem 2.

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REFERENCES

- [1] D. ARONSON, M. G. CRANDALL & L. A. PELETIER, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Anal. **6** (1982), 1001–1022.
- [2] D. G. DE FIGUEIREDO, P. L. LIONS & R. D. NUSSBAUM, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. **61** (1982), 41–63.
- [3] M. ESTEBAN, *On periodic solutions of periodic parabolic problems*, Trans. Amer. Math. Soc. **93** (1986), 171–189.

- [4] ———, *A remark of the existence of positive periodic solutions of superlinear parabolic problems*, Proc. Amer. Math. Soc. **102** (1988), 131–136.
- [5] M. FILA, *Boundedness of global solutions of nonlinear diffusion equations*, J. Differential Equations **98** (1992), 226–240.
- [6] B. GIDAS, W. M. NI & L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
- [7] Y. GIGA, *A bound for global solutions of semilinear heat equations*, Comm. Math. Phys. **103** (1986), 415–421.
- [8] Y. GIGA & N. MIZOGUCHI, *Existence of periodic solutions for equations of evolving curves*, (Preprint).
- [9] ———, *On the time periodic solutions of the Dirichlet problem for degenerate parabolic equations of non-divergence type*, (Preprint).
- [10] N. HIRANO & N. MIZOGUCHI, *Positive unstable periodic solutions for superlinear parabolic equations*, Proc. Amer. Math. Soc. **123** (1995), 1487–1495.
- [11] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV & N. N. URAL'ZEVA, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monograph, no. 23, Amer. Math. Soc., Providence, RI, 1968.
- [12] H. A. LEVINE & P. E. SACKS, *Some existence and nonexistence theorem for solutions of degenerate parabolic equations*, J. Differential Equations **52** (1984), 135–161.
- [13] M. NAKAO, *Periodic solutions of some nonlinear degenerate parabolic equations*, J. Math. Anal. Appl. **104** (1984), 554–467.
- [14] ———, *Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations*, Nonlinear Anal. **10** (1986), 299–314.
- [15] W. M. NI, P. E. SACKS & J. TAVANTZIS, *On the asymptotic behavior of solutions of certain quasilinear equations of parabolic type*, J. Differential Equations **54** (1984), 97–120.

Department of Mathematics
Tokyo Gakuhei University
Koganei
Tokyo, 184 Japan

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