C^α Compactness Theorem for Riemannian Manifolds with Bounds on Diameter, Injectivity Radius, and Some Integral Norms of Ricci Curvature

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0. Introduction. Let (M, g) be an n dimensional Riemannian manifold. In [AC], Anderson and Cheeger have introduced the notion of L_1^p harmonic radius.

Definition. Given $p \in (n, \infty)$ and $\eta > 0$, the L_1^p harmonic radius of (M, g) is the largest number $r_H = r_H(M)$ such that there is a harmonic coordinate $u = (u^1, \dots, u^n) : B_x(r_H) \rightarrow \mathbb{R}^n$ on any geodesic ball $B = B_x(r_H)$ of radius r_H , and the metric coefficients are L_1^p controlled in this coordinate; that is, let $g_{ij} = g(\partial/\partial u^i, \partial/\partial u^j)$ and g^{ij} be its inverse matrix, then we have

$$\begin{aligned} \Delta u^k &= 0 & (\text{or, equivalently, } g^{ij} \left(2 \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right) &= 0), \\ (0.1) \quad (1 + \eta)^{-1} \delta_{ij} &\leq g_{ij} \leq (1 + \eta) \delta_{ij} & (\text{as bilinear forms}), \\ r_H^{1-n/p} \left\| \frac{\partial g_{ij}}{\partial u^k} \right\|_{L_1^p(B)} &\leq \eta. \end{aligned}$$

They have proved that there exists a positive lower bound on L_1^p harmonic radius $r_H(M)$ in terms of the bounds $\text{Ric}_M \geq -\lambda_0$ for some $\lambda_0 \geq 0$ and $\text{inj}(M) \geq i_0$ for some $i_0 > 0$. They used this result to show a C^α compactness theorem for the class of compact Riemannian manifolds which have a lower bound on the Ricci curvature, a lower bound on the injectivity radius, and an upper bound on the diameter. The main purpose of this paper is to find a positive lower bound on L_1^p harmonic radius $r_H(M)$ under weaker conditions.

We have little hope to weaken the condition $\text{inj}(M) \geq i_0$ (see [AC]). However, there are some results based on integral norms of the Ricci curvature. For any $x \in M$, let $-\lambda(x)$ be the lowest nonpositive eigenvalue of the Ricci curvature at x (if $\text{Ric}_M > 0$ at x , we define $\lambda(x) = 0$). Our main result is the following theorem.

Theorem 0.1. *Given $p > n$, $\lambda_0 \geq 0$, and $i_0 > 0$. Let (M, g) be an n dimensional complete Riemannian manifold satisfying*

$$\int_M \lambda^{p/2} dv \leq \lambda_0, \quad \text{inj}(M) \geq i_0.$$

Then the L_1^p harmonic radius $r_H(M)$ is bounded from below by a positive number which depends only on η , n , p , λ_0 , and i_0 .

Consequently, we obtain the corresponding compactness theorem for the space of Riemannian manifolds.

Theorem 0.2. *Given $p > n$, $\lambda_0 \geq 0$, $i_0 > 0$, and $d_0 > 0$. The space of n dimensional compact Riemannian manifolds (M, g) satisfying*

$$(0.2) \quad \int_M \lambda^{p/2} dv \leq \lambda_0, \quad \text{inj}(M) \geq i_0, \quad \text{diam}(M) \leq d_0,$$

is precompact in the $C^{\alpha'}$ topology for an $\alpha' < \alpha = 1 - n/p$ (or, in the weak L_1^p topology).

We say a sequence of n dimensional compact Riemannian manifolds $\{(M_j, g_j)\}$ converges to (M, g) in the C^α (or strong L_1^p , or weak L_1^p) topology, if there exist diffeomorphisms $\varphi_j : M \rightarrow M_j$ such that the pullbacks $\varphi_j^* g_j$ converge to g in the C^α (or strong L_1^p , or weak L_1^p respectively) topology with respect to some coordinate atlas on M .

The passage from Theorem 0.1 to Theorem 0.2 is found in [A2] or [K]. We refer the readers to these papers for the details. The point is that we can embed the manifolds M into $\mathbb{R}^N$ by using harmonic coordinate atlas with the bound (0.1) and a standard packing argument based on the bound (0.2) (Proposition 1.5). It might be worth noticing that we can choose a harmonic coordinate atlas whose transition functions are controlled in the L_3^p topology. As a matter of fact, if $u : B_x(r) \rightarrow \mathbb{R}^n$ is a harmonic coordinate with the bounds (0.1), then the Laplace operator can be written in the following formula:

$$\Delta = -g^{k\ell} \frac{\partial^2}{\partial u^k \partial u^\ell}$$

where $g^{k\ell}$ is the inverse matrix of $g_{k\ell} = g(\partial/\partial u^k, \partial/\partial u^\ell)$ and controlled in the L_1^p topology. Thus, if $f = (f^1, \dots, f^n)$ is a transition function from the coordinate u to another harmonic coordinate, f satisfies the following differential equation:

$$g^{k\ell} \frac{\partial^2}{\partial u^k \partial u^\ell} f = 0.$$

Therefore, by the standard elliptic regularity theorem, we obtain $\|f\|_{L^p_3(B')} \leq C\|f\|_{L^1(B)}$ for some constant C which depends only on n, p, η , and the domain $B' \Subset B$. Eventually, M is locally realized as a graph in $\mathbb{R}^N$ for some N depending only on n, p, η , and the bounds (0.2), and its graph functions are controlled in the L^p_3 topology. Now we may choose a sequence $\{(M_i, g_i)\}$ from the space of compact Riemannian manifolds satisfying the bounds (0.2) such that, as submanifolds, $M_i \subset \mathbb{R}^N$ converge to an L^p_3 submanifold M in the weak L^p_3 topology. Moreover, for sufficiently large i , we can choose L^p_3 diffeomorphisms $\varphi_i : M \rightarrow M_i$, such that, by taking appropriate harmonic coordinate atlases $\{u_{i,\mu}\}$ on M_i and $\{u_\mu\}$ on M , the maps $u_{i,\mu} \circ \varphi_i \circ u_\mu^{-1}$ converge to the identity map in the weak L^p_3 topology. This implies that the coefficients of the metrics $\phi_i^* g_i$ with respect to the atlas $\{u_\mu\}$ are bounded in the L^p_1 topology. Theorem 0.2 follows from the Sobolev embedding theorem which claims that the embedding $L^p_1 \subset C^{\alpha'}$ is compact for $\alpha' < \alpha = 1 - n/p$.

1. Estimate for the volume of geodesic balls. We begin with the estimate for the volume of geodesic balls in terms of the $L^{p/2}$ norm of the function λ . The results of this section are mainly due to Yang [Y] (originating from [G]). We will repeat his proof, because we will use the idea of his argument in the next section.

Let $S_x \subset T_x M$ denote the space of unit tangent vectors at x . The exponential map defines a normal coordinate map

$$\begin{aligned} \Psi : [0, \infty) \times S_x &\rightarrow M, \\ (t, \vartheta) &\mapsto \exp_x t\vartheta. \end{aligned}$$

Let $t_+(\vartheta)$ be the upper bound of the set T such that the geodesic $[0, T) \ni t \mapsto \Psi(t, \vartheta)$ is minimizing. We define $J(t, \vartheta)$ by pulling back the volume form dv of M to $U_x = \{(t, \vartheta) \mid 0 < t < t_+(\vartheta), \vartheta \in S_x\}$, i.e.,

$$dv = J(t, \vartheta) dt d\vartheta$$

where $d\vartheta$ is the standard volume form on $S_x = S^{n-1}$. Let ρ denote the distance function from x and let $(\cdot)'$ denote the differentiation with respect to the argument t . Then we have

$$J' = -\Delta \rho J.$$

On the other hand, Bochner-Weitzenböck's formula implies

$$|\nabla d\rho|^2 \leq \langle d\rho, d\Delta\rho \rangle + \lambda.$$

Notice that $(J'/J)' = -\langle d\rho, d\Delta\rho \rangle$. Since $|\Delta\rho|^2 \leq (n-1)|\nabla d\rho|^2$, we have

$$\frac{1}{n-1}(J'/J)^2 \leq -(J'/J)' + \lambda.$$

Let $a = J^{1/(n-1)}$. Then it comes to the well-known facts about the function a :

$$a'' \leq \frac{\lambda a}{n-1}, \quad a(0, \vartheta) = 0, \quad a'(0, \vartheta) = 1.$$

The following estimate is due to Yang.

Lemma 1.1. *Given $p > n$, $(t, \vartheta) \in U_x$,*

$$(a' - 1)a^{-\nu}(t, \vartheta) \leq \left(\gamma \int_0^t \lambda^{p/2} J(s, \vartheta) ds \right)^{1/(p-1)}$$

where

$$\nu = \frac{p-n}{p-1}, \quad \gamma = 2 \left(\frac{p-1}{p(n-1)} \right)^{p/2} \left(\frac{p-2}{p-n} \right)^{(p-2)/2}.$$

Proof. Assume $a'(t, \vartheta) > 1$; otherwise, the claim is clear. Let (t_0, t) be the open interval, on which $a'(s, \vartheta) > 1$ and $a'(t_0, \vartheta) = 1$. Then, on this interval, we have

$$((a' - 1)a^{-\nu})' + \nu a^{\nu-1} ((a' - 1)a^{-\nu})^2 \leq \frac{1}{n-1} \lambda a^{1-\nu}.$$

Applying the inequality,

$$\frac{p^{p/2}}{2(p-2)^{(p-2)/2}} y \leq \sup(0, 1+y)^{p/2} \quad \text{for } y \in \mathbb{R},$$

we obtain

$$\frac{(p(n-1))^{p/2}}{2} \left(\frac{\nu}{p-2} \right)^{(p-2)/2} ((a' - 1)a^{-\nu})^{p-2} ((a' - 1)a^{-\nu})' \leq \lambda^{p/2} a^{(p-1)(1-\nu)}.$$

This implies that

$$\{((a' - 1)a^{-\nu})^{p-1}\}' \leq \gamma \lambda^{p/2} a^{n-1}.$$

Integrating both sides from t_0 to t , we obtain the result. $\square$

It follows from this lemma that

$$J'(t, \vartheta) \leq (n-1)J(t, \vartheta)^{(n-2)/(n-1)} + (n-1)J(t, \vartheta)^{(p-2)/(p-1)} \left(\gamma \int_0^t \lambda^{p/2} J(s, \vartheta) ds \right)^{1/(p-1)}.$$

Let

$$J_+(t, \vartheta) = \begin{cases} J(t, \vartheta) & 0 < t < t_+(\vartheta), \\ 0 & t \geq t_+(\vartheta), \end{cases}$$

and define

$$b(t) = \left(\int_{S_x} J_+(t, \vartheta) d\vartheta \right)^{1/(n-1)}.$$

For $0 < t < R$, we obtain

$$b'(t) \leq \omega_{n-1}^{1/(n-1)} + \mu^{1/(p-1)} b(t)^\nu$$

where ω_{n-1} is the standard volume of S^{n-1} and $\mu = \gamma \int_{B_x(R)} \lambda^{p/2} dv$.

Given $\tau > 0$, let $w(t) = (1 + \tau)\omega_{n-1}^{1/(n-1)}t$. Then $w(t)$ satisfies the differential inequality

$$w'(t) \geq \omega_{n-1}^{1/(n-1)} + \mu^{1/(p-1)} w(t)^\nu \quad \text{for } 0 \leq t \leq r_0 = \frac{\tau^{1/\nu}}{1 + \tau} \left(\frac{\omega_{n-1}}{\mu} \right)^{1/(p-n)},$$

from which we easily obtain $b(t) \leq w(t)$ for $0 \leq t \leq \min(r_0, R)$. Given $\varepsilon > 0$, if we put

$$\tau = \max \left\{ \frac{1}{\varepsilon}, \left(\frac{((1 + \varepsilon)R)^{p-n} \mu}{\omega_{n-1}} \right)^{1/(n-1)} \right\},$$

we have

$$r_0 \geq \frac{1}{1 + \varepsilon} \left(\frac{\tau^{n-1} \omega_{n-1}}{\mu} \right)^{1/(p-n)} \geq R.$$

Now we can prove:

Theorem 1.2. For given $p > n$, $R > 0$, and $\varepsilon > 0$, let

$$\kappa(x, R) = R^{p-n} \int_{B_x(R)} \lambda^{p/2} dv,$$

$$\gamma = 2 \left(\frac{p-1}{p(n-1)} \right)^{p/2} \left(\frac{p-2}{p-n} \right)^{(p-2)/2},$$

$$\beta = \max \left\{ \frac{\omega_{n-1}}{n} \left(\frac{1+\varepsilon}{\varepsilon} \right)^{n-1}, \frac{(1+\varepsilon)^{p-1} \gamma}{n} \kappa(x, R) \right\}.$$

Then we have

$$\text{Vol}(B_x(r)) \leq \beta r^n$$

for $0 \leq r \leq R$.

Proof. The theorem follows from the facts that $\text{Vol}(B_x(r)) = \int_0^r b(t)^{n-1} dt$ and $b(t) \leq w(t) \leq (n\beta)^{1/(n-1)} t$. $\square$

Corollary 1.3. *Suppose that M is a compact n dimensional Riemannian manifold. For given $p > n$, there exists a constant C which depends only on n and p such that*

$$\text{Vol}(M) \leq C(1 + D^{p-n} \Lambda) D^n$$

where $D = \text{diam}(M)$ and $\Lambda = \int_M \lambda^{p/2} dv$.

For the estimate of $\text{Vol}(B_x(r))$ from below, we quote Croke's result [C].

Theorem 1.4. *For $0 \leq r \leq \text{inj}(M)/2$, we have*

$$\text{Vol}(B_x(r)) \geq \frac{2^{n-1} \omega_{n-1}^n}{n^n \omega_n^{n-1}} r^n$$

where ω_{n-1} is the standard volume of S^{n-1} and ω_n is that of S^n .

These results will be applied to the packing argument, which is essential to the compactness theorem.

Proposition 1.5. *Let $R > 0$ and $0 < r \leq \text{inj}(M)/2$. If N is a subset of $B_x(R-r)$ such that the geodesic balls $\{B_y(r) \mid y \in N\}$ are mutually disjoint. Then we have*

$$\#N \leq C \left(\frac{R}{r} \right)^n$$

where C is a constant which depends only on n , p and $\kappa(x, R)$.

2. L^p estimate for $\Delta\rho$. We define $\kappa(M)$ by

$$\kappa(M) = \text{inj}(M)^{p-n} \sup_{x \in M} \kappa(x, \text{inj}(M)).$$

Note that $\kappa(M)$ is invariant under the homothetic change of the metric.

Let ρ be the distance function from a fixed point $x \in M$. Via the exponential map $(t, \vartheta) \mapsto \exp_x t\vartheta$, we consider ρ , $\Delta\rho$ and λ as functions on $(0, \text{inj}(M)) \times S_x$.

Let us begin with the following lemma.

Lemma 2.1. For $0 < t < \text{inj}(M)/2$,

$$|\Delta\rho|^{p-1}J(t, \vartheta) \leq C_1 t^{1-p}J(t, \vartheta) + C_2 \int_0^{2t} \lambda^{p/2} J(s, \vartheta) ds,$$

where

$$C_1 = \frac{1}{2} \left(\frac{4(n-1)(p-1)}{2p-n-1} \right)^{p-1}$$

and

$$C_2 = 2^{p-1} \left(\frac{p-1}{p} \right)^{p/2} \left(\frac{(p-2)(n-1)}{p-n} \right)^{(p-2)/2}.$$

Proof. To prove this lemma, we will use the basic differential inequality, $(\Delta\rho)' \geq (1/(n-1))|\Delta\rho|^2 - \lambda$, again. Here we denote by $(\cdot)'$ the differentiation with respect to the argument t . Let ϕ be a function satisfying $\phi' = 1$, then

$$\left(\phi^2 \Delta\rho + \frac{2(n-1)(p-1)}{2p-n-1} \phi \right)' \geq \frac{1}{n-1} \phi^2 |\Delta\rho|^2 + 2\phi \Delta\rho - \phi^2 \lambda + \frac{2(n-1)(p-1)}{2p-n-1}.$$

Since $(J^{1/(p-1)})' = -(1/(p-1))\Delta\rho J^{1/(p-1)}$, we have

$$\begin{aligned} & \left(\left(\phi^2 \Delta\rho + \frac{2(n-1)(p-1)}{2p-n-1} \phi \right) J^{1/(p-1)} \right)' \\ & \geq \frac{p-n}{(n-1)(p-1)} \phi^2 |\Delta\rho|^2 J^{1/(p-1)} + \frac{4(p-n)}{2p-n-1} \phi \Delta\rho J^{1/(p-1)} \\ & \quad - \phi^2 \lambda J^{1/(p-1)} + \frac{2(n-1)(p-1)}{2p-n-1} J^{1/(p-1)}. \end{aligned}$$

From this inequality, it follows that

$$\begin{aligned} (2.1) \quad & \left(\left(\phi^2 \Delta\rho + \frac{2(n-1)(p-1)}{2p-n-1} \phi \right) J^{1/(p-1)} \right)' \\ & - \frac{p-n}{(n-1)(p-1)} \left(\phi \Delta\rho + \frac{2(n-1)(p-1)}{2p-n-1} \right)^2 J^{1/(p-1)} \\ & \geq -\phi^2 \lambda J^{1/(p-1)} + \frac{2(n-1)^2(p-1)}{(2p-n-1)^2} J^{1/(p-1)}. \end{aligned}$$

Now we will show that

$$(2.2) \quad -\Delta\rho J^{1/(p-1)}(t, \vartheta) \leq \frac{1}{2}t^{-1}(2C_1 J(t, \vartheta))^{1/(p-1)} \\ + \frac{1}{2} \left(2C_2 \int_0^{2t} \lambda^{p/2} J(s, \vartheta) ds \right)^{1/(p-1)}.$$

Let $\phi(s) = s$ and set

$$f = -\phi^2 \Delta\rho J^{1/(p-1)} - \frac{2(n-1)(p-1)}{2p-n-1} \phi J^{1/(p-1)}.$$

Then $f(0) = 0$ and we have, from (2.1),

$$f' + \frac{p-n}{(n-1)(p-1)} \phi^{-2} J^{-1/(p-1)} f^2 \leq \phi^2 \lambda J^{1/(p-1)}.$$

We assume that

$$f(t) = -t^2 \Delta\rho J^{1/(p-1)}(t, \vartheta) - \frac{2(n-1)(p-1)}{2p-n-1} t J^{1/(p-1)}(t, \vartheta) > 0,$$

otherwise the claim is obvious. Let (t_0, t) be the open interval on which $f(s) > 0$ and $f(t_0) = 0$. Then, applying the same argument as in the proof of Lemma 1.1, we have

$$\frac{1}{2} \left(\frac{p}{p-1} \right)^{p/2} \left(\frac{p-n}{(p-2)(n-1)} \right)^{(p-2)/2} (f^{p-1})' \leq \phi^{2(p-1)} \lambda^{p/2} J$$

on (t_0, t) . Integrating both sides from t_0 to t , we have

$$f(t)^{p-1} \leq 2 \left(\frac{p-1}{p} \right)^{p/2} \left(\frac{(p-2)(n-1)}{p-n} \right)^{(p-2)/2} t^{2(p-1)} \int_0^t \lambda^{p/2} J(s, \vartheta) ds$$

and (2.2) follows.

It remains to show that

$$(2.3) \quad \Delta\rho J^{1/(p-1)}(t, \vartheta) \leq \frac{1}{2}t^{-1}(2C_1 J(t, \vartheta))^{1/(p-1)} \\ + \frac{1}{2} \left(2C_2 \int_0^{2t} \lambda^{p/2} J(s, \vartheta) ds \right)^{1/(p-1)}.$$

Let $\phi(s) = s - 2t$ and set

$$f = \phi^2 \Delta \rho J^{1/(p-1)} + \frac{2(n-1)(p-1)}{2p-n-1} \phi J^{1/(p-1)}.$$

Then $f(2t) = 0$ and

$$-f' + \frac{p-n}{(n-1)(p-1)} \phi^{-2} J^{-1/(p-1)} f^2 \leq \phi^2 \lambda J^{1/(p-1)}.$$

We assume that

$$f(t) = t^2 \Delta \rho J^{1/(p-1)}(t, \vartheta) - \frac{2(n-1)(p-1)}{2p-n-1} t J^{1/(p-1)}(t, \vartheta) > 0$$

as above. Let (t, t_0) be the open interval on which $f(s) > 0$ and $f(t_0) = 0$. Notice that $t_0 \leq 2t$. Following the same argument, we have

$$\frac{1}{2} \left(\frac{p}{p-1} \right)^{p/2} \left(\frac{p-n}{(p-2)(n-1)} \right)^{(p-2)/2} (f^{p-1})' \leq \phi^{2(p-1)} \lambda^{p/2} J$$

on (t, t_0) . Integrating both sides from t to t_0 , we have

$$f(t)^{p-1} \leq 2 \left(\frac{p-1}{p} \right)^{p/2} \left(\frac{(p-2)(n-1)}{p-n} \right)^{(p-2)/2} t^{2(p-1)} \int_t^{2t} \lambda^{p/2} J(s, \vartheta) ds.$$

This shows (2.3).

Combining (2.2) and (2.3), we obtain

$$\begin{aligned} |\Delta \rho|^{p-1} J(t, \vartheta) &\leq \frac{1}{2^{p-1}} \left\{ (2C_1 t^{-1} J(t, \vartheta))^{1/(p-1)} \right. \\ &\quad \left. + \left(2C_2 \int_0^{2t} \lambda^{p/2} J(s, \vartheta) ds \right)^{1/(p-1)} \right\}^{p-1}. \end{aligned}$$

This proves the lemma. $\square$

If $0 < R \leq \text{inj}(M)/4$ and $0 < s < R$, then, by integrating the inequality of Lemma 2.1 over S_x , we obtain

$$\int_{S_x} |\Delta \rho|^{p-1} J(t, \vartheta) d\vartheta \leq C_1 t^{1-p} b(t)^{n-1} + C_2 \int_{B_x(4R)} \lambda^{p/2} dv$$

for $0 < t \leq 2R$. Here $b(t)$ is defined in Section 1. Since $b(t)^{n-1} \leq n\beta(M)t^{n-1}$ for $0 < t \leq \text{inj}(M)$ with

$$\beta(M) = \frac{2^{n-1}\omega_{n-1}}{n} + \frac{2^{p-1}}{n} \left(\frac{p-1}{p(n-1)} \right)^{p/2} \left(\frac{p-2}{p-n} \right)^{(p-2)/2} \kappa(M),$$

we obtain the estimate

$$(2.4) \quad \int_{B_x(R+s) \setminus B_x(R-s)} |\Delta \rho|^{p-1} dv \\ \leq C_1 n \beta(M) \int_{R-s}^{R+s} t^{n-p} dt + 2s C_2 \kappa(M) \cdot (4R)^{n-p}.$$

Since $(|\Delta \rho|^{p-2} \Delta \rho)' = (p-1)|\Delta \rho|^{p-2}(\Delta \rho)'$, we have

$$(|\Delta \rho|^{p-2} \Delta \rho)' \geq \frac{p-1}{n-1} |\Delta \rho|^p - (p-1)\lambda |\Delta \rho|^{p-2}.$$

Using $J' = -\Delta \rho J$,

$$(|\Delta \rho|^{p-2} \Delta \rho J)' \geq \frac{p-n}{n-1} |\Delta \rho|^p J - (p-1)\lambda |\Delta \rho|^{p-2} J.$$

Define a function φ by

$$\varphi(t) = \begin{cases} \frac{2}{s}(t-R+s) & R-s < t < R-\frac{s}{2}, \\ 1 & R-\frac{s}{2} \leq t \leq R+\frac{s}{2}, \\ -\frac{2}{s}(t-R-s) & R+\frac{s}{2} < t < R+s, \\ 0 & \text{otherwise.} \end{cases}$$

Integrating by parts yields

$$\frac{p-n}{n-1} \int_{R-s}^{R+s} \varphi |\Delta \rho|^p J(t, \vartheta) dt \leq - \int_{R-s}^{R+s} \varphi' |\Delta \rho|^{p-2} \Delta \rho J(t, \vartheta) dt \\ + (p-1) \int_{R-s}^{R+s} \varphi \lambda |\Delta \rho|^{p-2} J(t, \vartheta) dt.$$

From Jensen's inequality, we have, for any $\varepsilon > 0$,

$$\lambda |\Delta \rho|^{p-2} \leq \frac{(p-2)\varepsilon}{p} |\Delta \rho|^p + \frac{2\varepsilon^{(2-p)/2}}{p} \lambda^{p/2}.$$

Therefore,

$$\begin{aligned} \frac{p-n}{n-1} \int_{R-s/2}^{R+s/2} |\Delta\rho|^p J(t, \vartheta) dt &\leq \frac{2}{s} \int_{R-s}^{R+s} |\Delta\rho|^{p-1} J(t, \vartheta) dt \\ &\quad + \frac{(p-1)(p-2)\varepsilon}{p} \int_{R-s}^{R+s} |\Delta\rho|^p J(t, \vartheta) dt \\ &\quad + \frac{2(p-1)\varepsilon^{(2-p)/2}}{p} \int_{R-s}^{R+s} \lambda^{p/2} J(t, \vartheta) dt. \end{aligned}$$

Hence, by setting $\varepsilon = p(p-n)/(2(p-1)(p-2)(n-1))$ and integrating over S_x , we obtain

$$\begin{aligned} \int_{B_x(R+s/2) \setminus B_x(R-s/2)} |\Delta\rho|^p dv &\leq \frac{4(n-1)}{(p-n)s} \int_{B_x(R+s) \setminus B_x(R-s)} |\Delta\rho|^{p-1} dv \\ &\quad + \frac{2\varepsilon^{-p/2}}{(p-2)} \int_{B_x(R+s) \setminus B_x(R-s)} \lambda^{p/2} dv. \end{aligned}$$

Putting (2.4) into this inequality, we obtain the following proposition.

Proposition 2.2. *Let $y \in M$ and $x \neq y$. Suppose that $R = \text{dist}(x, y) \leq \text{inj}(M)/4$ and $0 < s < R$. Then we have*

$$\int_{B_y(s/2)} |\Delta\rho|^p dv \leq \frac{C_3(1 + \kappa(M))}{s} \int_{R-s}^{R+s} t^{n-p} dt + C_4 \kappa(M) \cdot R^{n-p}$$

where the constants C_3 and C_4 depend only on n and p .

3. Proof of Theorem 0.1. As $\kappa(M) \leq \text{diam}(M)^{p-n} \int_M \lambda^{p/2} dv$, it is clear that the following theorem implies Theorem 0.1.

Theorem 3.1. *Let (M, g) be a complete Riemannian manifold with $\kappa(M) \leq K$ for some $K \geq 0$. Then there exists a positive constant C_H depending only on η , n , p , and K , such that*

$$r_H(M)/\text{inj}(M) \geq C_H.$$

Proof. If the theorem were false, there would exist a sequence of n dimensional complete Riemannian manifolds $\{(\hat{M}_i, \hat{g}_i)\}$ with $\kappa(\hat{M}_i) \leq K$ and

$$\lim_{i \rightarrow \infty} r_H(\hat{M}_i) = 0, \quad \lim_{i \rightarrow \infty} \frac{r_H(\hat{M}_i)}{\text{inj}(\hat{M}_i)} = 0.$$

We define the new metric g_i on $M_i = \hat{M}_i$ by rescaling $g_i = r_H(\hat{M}_i)^{-2} \cdot \hat{g}_i$. Under this rescaling, we have $\kappa(M_i) = \kappa(\hat{M}_i)$, $\text{inj}(M_i) = \text{inj}(\hat{M}_i)/r_H(\hat{M}_i)$ and $r_H(M_i) = r_H(\hat{M}_i)/r_H(\hat{M}_i) = 1$. Thus we obtain a sequence of n dimensional Riemannian manifolds $\{(M_i, g_i)\}$ satisfying:

$$\begin{aligned} \kappa(M_i) &\leq K, \\ (3.1) \quad \lim_{i \rightarrow \infty} \text{inj}(M_i) &= \infty, \\ r_H(M_i) &= 1. \end{aligned}$$

We will show:

Proposition 3.2. *The sequence $\{(M_i, g_i)\}$ satisfying the bounds (3.1) has a subsequence which converges to the canonical flat Euclidean space in the strong L_1^p topology. More precisely, there exists a set of diffeomorphisms $\varphi_i : \mathbb{R}^n \supset B_0(R_i) \rightarrow \varphi_i(B_0(R_i)) \subset M_i$ such that $\lim_{i \rightarrow \infty} R_i = \infty$ and the pullbacks $\varphi_i^* g_i$ converge to the canonical flat metric in the strong L_1^p topology on any compact subset of $\mathbb{R}^n$.*

To see how this proposition leads to contradiction, note that the limit manifold (M, g) of the sequence $\{(M_i, g_i)\}$ must have harmonic radius 1, because the L_1^p harmonic radius r_H is continuous in the strong L_1^p topology (see [AC]). But, the canonical flat Euclidean space clearly has harmonic radius ∞ .

Proof of Proposition 3.2. Since $r_H(M_i) = 1$, using a packing argument based on Proposition 1.5, we can show that the sequence $\{(M_i, g_i)\}$ satisfying (3.1) has a subsequence which converges to a complete L_1^p Riemannian manifolds (M, g) in the weak L_1^p topology (see [AC], [A1], [A2]). This means that there exists an increasing sequence of compact subdomains $\{D_i\}$ of M with $\bigcup D_i = M$ and diffeomorphisms $\varphi_i : D_i \rightarrow \varphi_i(D_i) \subset M_i$ such that $\varphi_i^* g_i$ converge to g in the weak L_1^p topology. More precisely, for any point $y \in M$ and any points $y_i \in M_i$ with $\lim_{i \rightarrow \infty} \varphi_i^{-1}(y_i) = y$, we can choose a harmonic coordinate u_i around each y_i and a harmonic coordinate u around y such that the maps $u_i \circ \varphi_i \circ u^{-1}$ converge to the identity map in the weak L_3^p topology. Therefore, in order to show the claim, we will show that the metric coefficients of g_i in the coordinate u_i subconverge in the strong L_1^p topology to the metric coefficients of g in the coordinate u .

Via u_i and u , we view the metrics g_i and g as the metrics on a ball $B \subset \mathbb{R}^n$ of sufficiently small but fixed radius, and identify y_i and y with the center of B . We may assume that g_i converge to g weakly in $L_1^p(B)$. Let $v_i \in T_{y_i} M_i$ be the unit tangent vectors with respect to the metrics g_i , which point in the

same direction $v \in T_y B$ under the identification. Set $x_i = \exp_{y_i}(R_i \cdot v_i)$ with $R_i = \text{inj}(M_i)/4$. For sufficiently large i , the functions $\rho_i = \text{dist}_{M_i}(x_i, \cdot) - R_i$ (considered as functions on B) are smooth on B and bounded in the C^1 topology. We may assume that the sequence $\{\rho_i\}$ converges to a function ρ in the C^0 topology. Let Δ_i be the Laplace operator with respect to the metric g_i . Then Proposition 2.2 implies

$$(3.2) \quad \|\Delta_i \rho_i\|_{L^p(B)} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

The Sobolev embedding theorem and (0.1) imply that the coefficients of Δ_i converge to that of the Laplace operator Δ with respect to the metric g in the $C^{\alpha'}$ topology for $0 < \alpha' < 1 - n/p$. By the L^p estimate for the elliptic operator Δ_i , we have

$$\|\rho_i\|_{L_2^p(B')} \leq C(\|\Delta_i \rho_i\|_{L^p(B)} + \|\rho_i\|_{L^1(B')})$$

where C is a constant depending only on η , n , p , and domains $B' \Subset B$. Thus the L_2^p norms of the functions ρ_i are bounded. Hence we may assume that the convergence $\rho_i \rightarrow \rho$ is in the weak L_2^p topology. This means that $\Delta_i \rho_i$ converge to $\Delta \rho$ in the weak L^p topology. Combining this with (3.2), we have $\Delta \rho = 0$ and the convergence $\Delta_i \rho_i \rightarrow 0$ is in the strong L^p topology. Applying the L^p estimate:

$$\begin{aligned} \|\rho_i - \rho\|_{L_2^p(B'')} &\leq C(\|\Delta_i(\rho_i - \rho)\|_{L^p(B')} + \|\rho_i - \rho\|_{L^1(B')}) \\ &\leq C(\|\Delta_i \rho_i\|_{L^p(B')} + \|(\Delta_i - \Delta)\rho\|_{L^p(B')} + \|\rho_i - \rho\|_{L^1(B')}), \end{aligned}$$

we conclude that ρ_i converges to ρ in the strong L_2^p topology on $B'' \Subset B'$.

Now, as in [AC], choosing appropriate $(n+1)/2$ directions $v^m \in T_y M$ ($m = 1, 2, \dots, (n+1)/2$), we define sequences of functions $\{p_i^m\}$ as above, which have limit functions ρ^m in the strong L_2^p topology. Using $\{\rho_i^m\}$, we can write the metric coefficients of g_i in the coordinate u_i as rational expressions of $\{\partial \rho_i^m / \partial u_i^k\}$, the first derivatives of the corresponding functions ρ_i^m in the coordinate $u_i = (u_i^1, \dots, u_i^n)$. Thus we can show that the convergence $g_i \rightarrow g$ is actually strong in the L_1^p topology. Since functions ρ^m satisfy $\Delta \rho^m = 0$, i.e.,

$$g^{jk} \frac{\partial^2 \rho^m}{\partial u^j \partial u^k} = 0$$

and $g^{jk} \in L_1^p \subset C^{\alpha'}$ for $0 < \alpha' < 1 - n/p$, we have $\rho^m \in C^{2, \alpha'}$ by the Schauder estimate. As we can also write the metric coefficients of g as rational expressions of $\{\partial \rho^m / \partial u^k\}$, we have $g \in C^{1, \alpha'}$ and the standard bootstrap argument shows $g \in C^\infty$ and $\rho \in C^\infty$.

It remains to show that the limit Riemannian manifold is isometric to the canonical flat Euclidean space. Let ρ_i be the functions defined as above, and let ρ be their limit. If ∇_i and ∇ are the Levi-Civita connections with respect to g_i and g respectively, then $\nabla_i d\rho_i \rightarrow \nabla d\rho$ in the strong L^p topology, because we have shown $g_i \rightarrow g$ in the strong L_1^p topology. Fix $s > 0$ and let ϕ be a smooth function satisfying $0 \leq \phi \leq 1$, $\phi \equiv 0$ outside $B_{y_i}(s)$, $\phi \equiv 1$ on $B_{y_i}(s/2)$, and $|d\phi| \leq 4/s$. Bochner-Weitzenböck's formula implies that

$$\phi |\nabla_i d\rho_i|^2 \leq \phi |\Delta_i \rho_i|^2 - \Delta_i \rho_i \langle d\rho_i, d\phi \rangle + \phi \lambda_i.$$

Hence we have

$$\begin{aligned} \int_{B_{y_i}(s)} \phi |\nabla_i d\rho_i|^2 dv_i &\leq \int_{B_{y_i}(s)} \phi |\Delta_i \rho_i|^2 dv_i \\ &\quad + \int_{B_{y_i}(s)} |d\phi| |\Delta_i \rho_i| dv_i + \int_{B_{y_i}(s)} \phi \lambda_i dv_i. \end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned} \int_{B_{y_i}(s/2)} |\nabla_i d\rho_i|^2 dv_i &\leq \text{Vol}(B_{y_i}(s))^{(p-2)/p} \left(\int_{B_{y_i}(s)} |\Delta_i \rho_i|^p dv_i \right)^{2/p} \\ &\quad + \frac{4\text{Vol}(B_{y_i}(s))^{(p-1)/p}}{s} \left(\int_{B_{y_i}(s)} |\Delta_i \rho_i|^p dv_i \right)^{1/p} \\ &\quad + \text{Vol}(B_{y_i}(s))^{(p-2)/p} \left(\int_{B_{y_i}(s)} \lambda_i^{p/2} dv_i \right)^{2/p}. \end{aligned}$$

From Theorem 1.2, Proposition 2.2, and the inequality

$$\int_{B_{y_i}(s)} \lambda_i^p dv_i \leq \text{inj}(M_i)^{n-p} \kappa(M_i), \quad \text{for large } i,$$

we conclude that

$$\int_{B_{y_i}(s/2)} |\nabla_i d\rho_i|^2 dv_i \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

This implies that $\nabla d\rho = 0$ and that the limit metric g splits in every direction. Thus g is the canonical flat metric on $\mathbb{R}^n$.

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