On Exhaustion of Domains

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ABSTRACT. We give examples of non-homogeneous bounded domains in $\mathbb{C}^n$ which can exhaust no other domains. And we prove that every bounded domain in $\mathbb{C}^n$ with C^2 boundary can exhaust the ball.

1. Introduction. Let D, G be bounded domains in $\mathbb{C}^n$, $n \geq 1$. We say that D can exhaust G if for any compact $K \Subset G$ there exists a biholomorphic imbedding $f : D \rightarrow G$ such that $f(D) \supset K$. This is equivalent to the following: There exists a sequence $f_k : D \rightarrow G$ of biholomorphic imbeddings such that the sets $f_k(D) \rightarrow G$ as $k \rightarrow \infty$, i.e., for any $K \Subset G$ there exists a number N that for $k \geq N$, $f_k(D) \supset K$. Below we call such a sequence $\{f_k\}$ an *exhausting sequence*. We use the notation $G \sim D$ to indicate that G is biholomorphically equivalent to D . Obviously if $G_1 \sim G_2$ and G_1 can be exhausted by D , then so can G_2 ; therefore for the question of exhaustion one can consider the whole class $\{G_1\}$ of domains biholomorphically equivalent to G_1 . We now introduce the set $E(D)$ of all classes $\{G\}$, where G is a bounded domain in $\mathbb{C}^n$ that can be exhausted by D . Obviously $E(D)$ is not empty: it always contains $\{D\}$. By $\#E(D)$ we denote the number of elements (different classes) in $E(D)$.

The problem of exhausting one domain by another has been considered in a number of papers ([2–7]). In this paper we are considering only bounded domains. One of the important problems is the following one: describe the set $E(D)$ for a specific D . It can be shown that if D is strongly pseudoconvex, then $\#E(D) \leq 2$: if $\{G\} \in E(D)$, then either $G \sim D$ or $G \sim B_n$ —the unit ball in $\mathbb{C}^n$ (see [5]). In [6] a universal domain is constructed: that is such a domain D that any G can be exhausted by it: the set $E(D)$ in this case contains all classes of bounded domains of $\mathbb{C}^n$ ($\#E(D) = \infty$).

If D is a homogeneous domain, then $\#E(D) = 1$: any G that can be exhausted by D is biholomorphically equivalent to D (see [5]). A natural question is: *if a bounded domain D can exhaust only the domains that are biholomorphically equivalent to D , is D necessarily homogeneous?*

We present two results in this direction. First we present two kinds of counterexamples. The first kind are domains (see Theorem 1.1) that have non-smooth

boundaries, and form an everywhere dense (in the Hausdorff metric) collection of domains. The second kind are domains (see Theorem 1.2) that are contractible, have piecewise smooth boundaries, but are not pseudoconvex. Later we give an affirmative answer to the question above for the case where the boundary ∂D is twice continuously differentiable.

Theorem 1.1. *Let M be a bounded domain in $\mathbb{C}^n$. Then for any neighborhood of the boundary $U \supset \partial M$, there exists a countable set $S \subset U \cap M$ of isolated points such that $D = M \setminus S$ has the property $\#E(D) = 1$.*

Theorem 1.2. *For $n \geq 2$ there exists a contractible domain Ω with piecewise smooth boundary such that $\#E(\Omega) = 1$ and Ω is not homogeneous.*

In [5] it is proved that if a complete hyperbolic manifold M can be exhausted by a bounded C^3 strictly pseudoconvex domain D in $\mathbb{C}^n$, then M is biholomorphic either to D or to B_n . In light of this result, it is natural to ask whether every bounded C^3 strongly pseudoconvex domain can exhaust the ball. In this note we answer the question above in the affirmative. In fact we will prove a stronger statement:

Theorem 1.3. *Let D be a domain in $\mathbb{C}^n$ contained in a ball U (of finite radius) such that ∂D touches ∂U at a point p and ∂D is C^2 in a neighborhood of p . Then D can exhaust the unit ball in $\mathbb{C}^n$.*

Remark. A seemingly possible approach to prove this theorem is to blow up D at p directly by Möbius transformations of U . This does not work for $n \geq 2$. (See the remark at the end of paper).

The following statements are direct consequences of Theorem 1.3.

Corollary 1.4. *Every bounded domain in $\mathbb{C}^n$ with C^2 boundary can exhaust the unit ball in $\mathbb{C}^n$.*

Corollary 1.5. *If ∂D is C^2 and $\#E(D) = 1$, then $D \sim B_n$.*

Corollary 1.6. *Every bounded convex domain in $\mathbb{C}^n$ that has a C^2 strongly convex boundary point can exhaust the ball. So, if D is a bounded convex domain in $\mathbb{C}^n$ that has a C^2 strongly convex boundary point and if $\#E(D) = 1$, then $D \sim B_n$.*

Corollary 1.7. *Let D be a region in $\mathbb{C}$ such that there exists a disk V so that $V \cap \partial D$ is a Jordan arc. Then D can exhaust the unit disk.*

Remark. We are still unable to answer the following question for a *non-convex* strictly pseudoconvex domain. Let D be a such a domain and $p \in \partial D$.

Does there exist an exhausting sequence $\{f_j\}$ from D to the unit ball B_n such that $f_j^{-1}(0) \rightarrow p$?

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2. Proof of Theorems 1.1 and 1.2. In order to prove the theorems we will need several lemmas.

Lemma 2.1. *Let $f_k : D \rightarrow G$ be an exhausting sequence. Suppose that for a point $w \in G$, $\{f_k^{-1}(w)\} \nsubseteq D$ (that is, the closure of this sequence is compact in D). Then G is biholomorphically equivalent to D .*

Proof. This follows from a normal family argument. Also, see [5]. □

We will need the definition of the Carathéodory metric on a complex manifold (see [1]). Let Δ be the unit disc in $\mathbb{C}$, and $P(\cdot, \cdot)$ the Poincaré distance on Δ . Let $H(D, \Delta)$ denote the family of holomorphic maps from D to Δ . Then the Carathéodory distance $C_D(w, z)$ between w and z is defined by

$$C_D(w, z) = \sup_{f \in H(D, \Delta)} P(f(w), f(z)).$$

It follows from the definition that C_D is invariant under biholomorphic transformations, and that C_D can be extended to the envelope of holomorphy of D . By $B_D^C(z, r)$ we denote the Carathéodory balls. The following two statements are direct consequences of the definition.

Lemma 2.2. *Let d, G be domains in $\mathbb{C}^n$, and $r > 0$. Suppose that $z \in D \subset G$. Then $B_D^C(z, r) \subset B_G^C(z, r)$.*

Lemma 2.3. *Let $D \subset M$ be bounded domains in $\mathbb{C}^n$ such that M is contained in the envelope of holomorphy of D . Then $C_D(z, w) = C_M(z, w)$ for $z, w \in D$.*

Definition 2.1. *Let D be a bounded domain in $\mathbb{C}^n$. Let $\mu : D \rightarrow (0, \infty]$ be defined by*

$$\mu(z) = \mu_D(z) = \sup\{r : B_D^C(z, r) \nsubseteq D\}.$$

Then μ is invariant under biholomorphic mappings. If a boundary point p of D lies in the envelope of D , then $\lim_{z \rightarrow p} \mu(z) = 0$.

Lemma 2.4. *Let D be a bounded domain in $\mathbb{C}^n$ such that*

$$(2.1) \quad \lim_{z \rightarrow \partial D} \mu_D(z) = 0.$$

Then $\#E(D) = 1$.

Proof. Suppose that a bounded domain G can be exhausted by D and $f_k : D \rightarrow G$ is an exhausting sequence. Pick a point $z \in G$ and $r > 0$ such that $B_G^C(z, 2r) \mathbf{b} G$. Then for large enough k , $B_G^C(z, r) \mathbf{b} f_k(D)$, hence $B_{f_k(D)}^C(z, r) \subset B_G^C(z, r) \mathbf{b} f_k(D)$ by Lemma 2.2. Therefore $r \leq \mu_{f_k(D)}(z) = \mu_D(f_k^{-1}(z))$. This, together with (2.1), implies that $\{f_k^{-1}(z)\} \mathbf{b} D$. By Lemma 2.1, G is biholomorphically equivalent to D . $\square$

Proof of Theorem 1.1. Pick $S = \{z_k\}$ as a sequence of isolated points in $U \cap M$ such that: $\overline{S} \cap M = S$ and for any $r > 0$ there exists an open neighborhood $V_r \supset \partial M$ such that if $z \in V_r \cap M$, then $B_M^C(z, r) \cap S \neq \emptyset$. Clearly there are many ways to construct such a sequence. Let $D = M \setminus S$. It follows that (2.1) holds for D . By Lemma 2.4, $\#E(D) = 1$. $\square$

Now let Δ^n denote the polydisc in $\mathbb{C}^n$. For $z \in \mathbb{C}^n$ let

$$\begin{aligned} M(z) &= \max(|z_1|, \dots, |z_n|), \\ m(z) &= \min(|z_1|, \dots, |z_n|), \\ a(z) &= (1/n)(|z_1|^2 + \dots + |z_n|^2). \end{aligned}$$

Let

$$\begin{aligned} D_1 &= \{z \in \Delta^n : M(z) - m(z) < (1 - M(z))^2\}; \\ D_2 &= \{z \in \Delta^n : (|z_1|^2 - a(z))^2 + \dots + (|z_n|^2 - a(z))^2 < (1 - a(z))^4\}. \end{aligned}$$

Then both D_1 and D_2 are Reinhardt domains with piecewise smooth boundary, they both have Δ^n as envelope of holomorphy, and $T^n \equiv \{z \in \mathbb{C}^n : |z_1| = \dots = |z_n| = 1\}$ as distinguished boundary. The boundary of D_2 is “smoother” than that of D_1 .

Lemma 2.5. *The domains D_1 and D_2 satisfy (2.1).*

Proof. The proofs for D_1 and D_2 are similar, but the proof for D_2 needs more computations. So we will only prove (2.1) for $D \equiv D_1$. The envelope of holomorphy of D is Δ^n . If $p \in \partial D \setminus T^n$, then p lies in that envelope, hence $\lim_{z \rightarrow p} \mu(z) = 0$.

So we only need to consider the points near T^n . We now estimate $\mu_D(z)$ for a $z = (z_1, \dots, z_n) \in D$ near T^n . Without loss of generality we may assume $z_1 \geq z_2 \geq \dots \geq z_n > 1/2$. Then $z_1 = M(z)$ and $z_n = m(z)$. Let $u = z_1 - (1 - z_1)^2$, and $q = (z_1, \dots, z_{n-1}, u)$. Then $u < z_n \leq z_1$ and $q \in \partial D$. Thus

$$\begin{aligned} \mu_D(z) &\leq C_D(z, q) = P(z_n, u) \leq P(z_1, u) \\ &= 2 \int_u^{z_1} \frac{dt}{1-t^2} \leq 2 \frac{z_1 - u}{1 - z_1^2} \\ &= 2 \frac{(1 - z_1)^2}{1 - z_1^2} \leq 2(1 - z_1) = 2(1 - M(z)). \end{aligned}$$

Therefore $\lim_{z \in D, z \rightarrow T^n} \mu_D(z) = 0$. □

Proof of Theorem 1.2. It follows from the construction that the domains D_1 and D_2 are contractible with piecewise smooth boundaries. By Lemmas 2.4 and 2.5, they satisfy $\#E(D_1) = 1$ and $\#E(D_2) = 1$. Since they are not pseudoconvex, they are not homogeneous. □

3. Proof of Theorem 1.3. We now consider $n \geq 2$. For $n = 1$, see the remark after the proof. We first introduce some notation and terminology. We write the coordinates of $\mathbb{C}^n$ as $z = (z_1, z'), z' = (z_2, \dots, z_n)$. Let

$$\begin{aligned} q_k(z) &= 2 \operatorname{Re} z_1 + |z_1|^2 + k|z'|^2, \quad k \in \mathbb{N} \\ Q_k &= \{z \in \mathbb{C}^n : q_k(z) < 0\}, \quad k \in \mathbb{N}, \\ B &= Q_1, \\ \alpha B &= \{z : |z_1 + \alpha|^2 + |z'|^2 < \alpha^2\}, \\ B(r) &= \{z \in \mathbb{C}^n : |z_1 + 2|^2 + |z'|^2 < r^2\}, \\ S &= Q_3 \cup B(1/8), \\ T &= Q_2 \cup B(1/7). \end{aligned}$$

Clearly, $Q_{k+1} \subset Q_k$, and all Q_k 's are biholomorphic to the unit ball. $B \equiv Q_1$ is the unit ball centered at $(-1, 0')$.

Definition 3.1. Let $N, k \in \mathbb{Z}$, $N \geq 4$, $0 \leq k \leq 2(N - 4)$. A domain $D \subset \mathbb{C}^n$ is said to be a domain of type $C(N, k)$ if D satisfies the following conditions:

- (a) $D \subset B$, and $0 \in \partial D$;
- (b) ∂D is C^2 near 0;
- (c) There is a C^2 defining function of D at 0 of the form

$$(3.1) \quad \rho(z) = 2 \operatorname{Re} z_1 + \operatorname{Re} \sum_{i,j=2}^n a_{ij} z_i z_j + d|z_1|^2 + N|z'|^2 + o(|z|^2),$$

where the numbers N , d and a_{ij} ($i, j \geq 2$) satisfy

$$d > 1,$$

$$(3.2) \quad \operatorname{Re} \sum_{i,j=2}^n a_{ij} z_i z_j + (N - 4 - k/2)|z'|^2 \geq 0, \quad z' \in \mathbb{C}^{n-1}. \quad \square$$

Note that (3.2) implies that

$$(3.3) \quad \left| \sum_{i,j=2}^n a_{ij} z_i z_j \right| \leq (N - 4 - k/2)|z'|^2, \quad z' \in \mathbb{C}^{n-1}.$$

Consider the maps $f_\varepsilon : B \rightarrow B$ defined by

$$(3.4) \quad \begin{aligned} w_1 &= \frac{\varepsilon z_1}{2 - \varepsilon + (1 - \varepsilon)z_1} \\ w' &= \frac{\sqrt{\varepsilon(2 - \varepsilon)}}{2 - \varepsilon + (1 - \varepsilon)z_1} z'. \end{aligned}$$

The f_ε 's are Möbius transformations of B . One can check that $f_\varepsilon(Q_k) = Q_k$.

We will also consider the entire analytic map G_b defined by $G_b(z) = (z_1 + \sum_{j=1}^n b_j z_1 z_j, z')$, where the b_j 's are complex numbers.

Lemma 3.1. *There exists a $\lambda > 0$ such that if b_j are complex numbers that satisfy*

$$(3.5) \quad \sum_{j=1}^n |b_j| \leq \lambda,$$

then G_b is an injective holomorphic map in a neighborhood of $\overline{B}$, and $G_b(B) \subset 2B$.

Proof. By direct computations. □

Lemma 3.2. *Let D be a domain as in Theorem 1.3. Then there is an integer $N \geq 4$ such that D is biholomorphic to a domain of type $C(N, 0)$.*

Proof. Without loss of generality we may assume that $D \subset B$ and $0 \in \partial D$. After a unitary transformation in the z' directions, the defining function of D at 0 can be written as

$$(3.6) \quad \begin{aligned} \rho(z) = & 2 \operatorname{Re} z_1 + \operatorname{Re} \sum_{i,j=1}^n a_{ij} z_i z_j \\ & + \operatorname{Re} \sum_{j=1}^n c_j z_1 \bar{z}_j + \sum_{j=2}^n N_j |z_j|^2 + o(|z|^2). \end{aligned}$$

The fact that $D \subset B$ implies that

$$\operatorname{Re} \sum_{i,j=2}^n a_{ij} z_i z_j + \sum_{j=2}^n N_j |z_j|^2 \geq |z'|^2, \quad z' \in \mathbb{C}^{n-1}.$$

Thus each $N_j \geq 1$.

Consider a biholomorphic map $H : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by

$$H_1(z) = z_1, \quad H_j(z) = z_j + (c_j/(2N_j))z_1, \quad j = 2, \dots, n.$$

The defining function $\rho \circ H^{-1}$ of the domain $D' \equiv H(D)$ at 0 again has the form (3.6), but with $c_j = 0$ for $j = 2, \dots, n$. The domain D' satisfies $D' \subset \{z : \operatorname{Re} z_1 < 0\}$, $\partial D' \cap \{z : \operatorname{Re} z_1 = 0\} = \{0\}$, and $\partial D'$ is strongly convex near 0. So, D' is contained in a big ball αB . Shrinking D' in all directions and rescaling it in the z' directions, we obtain a domain $\tilde{D}$ that is biholomorphic to D , that is contained in B , and that has a defining function of the form

$$(3.7) \quad \begin{aligned} \tilde{\rho}(z) = & 2 \operatorname{Re} z_1 + 2 \operatorname{Re} \sum_{j=1}^n b_j z_1 z_j \\ & + \operatorname{Re} \sum_{i,j=2}^n a_{ij} z_i z_j + d|z_1|^2 + M|z'|^2 + o(|z|^2). \end{aligned}$$

Consider the domain $f_\varepsilon^{-1}(\tilde{D}) \subset B$, whose defining function can be chosen as

$$\rho_\varepsilon(z) = \frac{|2 - \varepsilon + (1 - \varepsilon)z_1|^2}{\varepsilon(2 - \varepsilon)} (\tilde{\rho} \circ f_\varepsilon(z)).$$

One can check that ρ_ε again has the form (3.7), but the coefficients are changed to

$$a'_{ij} = a_{ij}; \quad b'_j = \sqrt{\frac{\varepsilon}{2-\varepsilon}} b_j, \quad (j \geq 2); \quad b'_1 = \frac{\varepsilon}{2-\varepsilon} b_1; \quad d' = 1 + \frac{\varepsilon}{2-\varepsilon} (d-1).$$

Thus, considering $f_\varepsilon^{-1}(\tilde{D})$ for a sufficiently small $\varepsilon > 0$ in place of $\tilde{D}$, if necessary, we can assume that the numbers b_j in (3.7) satisfy (3.5). By Lemma 3.1, $\hat{D} \equiv G_b(\tilde{D}) \subset 2B$. One can check that the defining function $\hat{\rho} \equiv \tilde{\rho} \circ G_b^{-1}$ of $\hat{D}$ has the form

$$\hat{\rho}(z) = 2 \operatorname{Re} z_1 + \operatorname{Re} \sum_{i,j=2}^n a_{ij} z_i z_j + d|z_1|^2 + M|z'|^2 + o(|z'|^2).$$

Choose an integer $N > 8M$, and let $\beta = M/N$. Then $\beta < 1/8$. Let $D_1 = \beta \hat{D}$. Then $D_1 \subset (1/4)B$. The defining function $\beta \hat{\rho}(z/\beta)$ of D_1 has the form (3.1) (with the a_{ij} 's, d different from those in $\hat{\rho}(z)$). Now the defining function of $(1/4)B$ can be chosen as $h(z) = 2 \operatorname{Re} z_1 + 4|z|^2$. The fact that $D_1 \subset (1/4)B$ implies (3.2) with $k = 0$. Thus D_1 is a domain of type $C(N, 0)$. $\square$

For complex numbers b_{ij} , $2 \leq i, j \leq n$, we define a biholomorphic mapping $F_b : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$(3.8) \quad F_b(z_1, z') = \left(z_1 + \sum_{i,j=2}^n b_{ij} z_i z_j, z' \right).$$

Lemma 3.3. *Let b_{ij} , $2 \leq i, j \leq n$, be complex numbers satisfying*

$$(3.9) \quad \left| \sum_{i,j=2}^n b_{ij} z_i z_j \right| \leq \frac{1}{4} |z'|^2, \quad z' \in \mathbb{C}^{n-1}.$$

Then $F_b(S) \subset T$.

Proof. It is clear that $F_b(B(1/8)) \subset B(1/7)$. So we only need to show that $F_b(Q_3) \subset Q_2$, which is equivalent to $q_2 \circ F_b(z) \leq 0$ when $z \in \partial Q_3$. For $z \in \partial Q_3$, we have $q_3(z) = 0$, $|z_1 + 1| \leq 1$, $|z'|^2 \leq 1/3$, hence

$$\begin{aligned} q_2 \circ F_b(z) &= q_2 \circ F_b(z) - q_3(z) \\ &= \left| \sum_{i,j=2}^n b_{ij} z_i z_j \right|^2 + 2 \operatorname{Re} \left[(\bar{z}_1 + 1) \sum_{i,j=2}^n b_{ij} z_i z_j \right] - |z'|^2 \\ &\leq (1/16)(1/3)|z'|^2 + (1/2)|z'|^2 - |z'|^2 \\ &\leq 0. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.4. *Suppose that $N \geq 5$, $0 \leq k \leq 2(N-4)-1$. Then every domain D of type $C(N, k)$ is biholomorphic to a domain of type $C(N, k+1)$.*

Proof. By definition, D has a defining function of the form (3.1), where the numbers a_{ij} ($i, j \geq 2$), d , and N satisfy (3.2). The inequality (3.2) implies that for z sufficiently close to 0 and $z \in \partial Q_3$, $\rho(z) \geq 0$. Thus there exists an $\eta > 0$ such that $D \cap \{z \in B : \operatorname{Re} z_1 \geq -\eta\} \subset Q_3$. Choose an $\varepsilon > 0$ so small that $f_\varepsilon^{-1}(\{z \in B : \operatorname{Re} z_1 \leq -\eta\}) \subset B(1/8)$. Since

$$f_\varepsilon^{-1}(D \cap \{z \in B : \operatorname{Re} z_1 \geq -\eta\}) \subset f_\varepsilon^{-1}(Q_3) = Q_3,$$

we have $f_\varepsilon^{-1}(D) \subset S$.

Let

$$(3.10) \quad b_{ij} = \frac{1}{4(N-4-k/2)} a_{ij}, \quad 2 \leq i, j \leq n.$$

Then the numbers b_{ij} satisfy (3.9). By Lemma 3.3, $F_b(S) \subset T$.

Let $h(z_1, z') = ((14/15)z_1, \sqrt{14/15}z')$. One can check that $h(T) \subset B$. Therefore $\hat{D} \equiv h \circ F_b \circ f_\varepsilon^{-1}(D) \subset B$. One can check that the defining function of $\hat{D}$ near the origin has the form

$$\begin{aligned} \hat{\rho}(z) &= \frac{14}{15} \frac{|2 - \varepsilon + 15(1 - \varepsilon)z_1/14|^2}{\varepsilon(2 - \varepsilon)} \rho \circ f_\varepsilon \circ F_b^{-1} \circ h^{-1}(z) \\ &= 2 \operatorname{Re} z_1 + \operatorname{Re} \sum_{i,j=2}^n (a_{ij} - 2b_{ij}) z_i z_j + d' |z_1|^2 + N |z'|^2 + o(|z|^2), \end{aligned}$$

where

$$d' = \frac{15}{14} \left[1 + \frac{\varepsilon}{2 - \varepsilon} (d - 1) \right] > 1.$$

Since

$$a_{ij} - 2b_{ij} = \frac{N - 4 - (k+1)/2}{N - 4 - k/2} a_{ij},$$

we have

$$(3.11) \quad \left| \sum_{i,j=2}^n (a_{ij} - 2b_{ij}) z_i z_j \right| \leq (N - 4 - (k+1)/2) |z'|^2, \quad z' \in \mathbb{C}^{n-1}.$$

Therefore $\hat{D}$ is a domain of type $C(N, k+1)$. □

Lemma 3.5. *Let D and G be bounded domains in $\mathbb{C}^n$. Suppose that G is starlike with respect to $p \in G$. If there is a sequence $\{h_k\}$ of injective holomorphic maps from D to $\mathbb{C}^n$ such that $h_k(D) \rightarrow G$ in the Hausdorff metric, then D can exhaust G .*

Proof. We may assume that $p = 0$. Let $K \Subset G$. Choose $\varepsilon > 0$ such that $[(1+\varepsilon)/(1-\varepsilon)]K \subset G$. Then choose k such that $(1-\varepsilon)G \subset h_k(D) \subset (1+\varepsilon)G$. Let $g = (1+\varepsilon)^{-1}h_k$. Then $K \subset g(D) \subset G$. $\square$

Proof of Theorem 1.3. By Lemma 3.2, we can assume that the domain D is of type $C(N, 0)$. Recall that $N \geq 4$. If $N \geq 5$, then by Lemma 3.4 and induction, D is biholomorphic to a domain of type $C(N, 2(N-4))$. (If $N = 4$, then $C(N, 0)$ is the same as $C(N, 2(N-4))$). The inequality (3.3) with $k = 2(N-4)$ forces $\sum a_{ij}z_i z_j = 0$. So, there exists a biholomorphic image $D_1 \subset B$ of D that has a defining function of the form

$$\rho_1(z) = 2 \operatorname{Re} z_1 + d|z_1|^2 + N|z'|^2 + o(|z|^2).$$

Direct computations give that

$$(3.12) \quad \lim_{\varepsilon \rightarrow 0} \left[\frac{|2 - \varepsilon + (1 - \varepsilon)z_1|^2}{\varepsilon(2 - \varepsilon)} \rho_1 \circ f_\varepsilon(z) \right] = q_N(z).$$

Thus $f_\varepsilon^{-1}(D_1) \rightarrow Q_N$ in the Hausdorff metric. Since Q_N is convex and therefore starlike, by Lemma 3.5, D_1 can exhaust Q_N . Hence D can exhaust the ball. This completes the proof. $\square$

Remark. A direct application of Möbius transformations (3.4) does not work for $n \geq 2$, the reason being that second order terms of the form $\sum a_{ij}z_i z_j$ would not be eliminated this way. However in case $n = 1$ (since those terms are not present) it produces the desired result.

REFERENCES

- [1] C. CARATHÉODORY, *Über die abbildungen, die durch systeme von analytischen funktionen von mehreren veränderlichen erzeugt werden*. Math. Z. **34** (1932), 758–792.
- [2] C. K. CHEUNG, *On some properties of exhaustion maps between bounded domains*. Pacific J. Math. **156** (1992), 107–117.
- [3] J. E. FORNÆSS AND E. L. STOUT, *Polydiscs in complex manifolds*. Math. Ann. **227** (1977), 145–153.

- [4] J. E. FORNÆSS AND N. SIBONY, *Increasing sequences of complex manifolds*. Math. Ann. **255** (1981), 351–360.
- [5] B. L. FRIDMAN, *Biholomorphic invariants of a hyperbolic manifold and some applications*. Trans. Amer. Math. Soc. **276** (1983), 685–698.
- [6] B. L. FRIDMAN, *A universal exhausting domain*. Proc. Amer. Math. Soc. **98** (1986), 267–270.
- [7] A. KODAMA, S. G. KRANTZ AND D. MA, *A characterization of generalized complex ellipsoids in $\mathbb{C}^n$ and related problems*. Indiana Univ. Math. J. **41** (1992), 173–195.

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