Periodic Automorphisms of $\mathbb{C}^n$

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ABSTRACT. The problem under consideration is whether every periodic holomorphic map F from $\mathbb{C}^n$ to $\mathbb{C}^n$ is conjugate to a linear one. It is proved that the answer is affirmative when $n = 2$ and F is a composition of finitely many so-called overshears.

Introduction. For each positive integer n we let $\text{Aut}(\mathbb{C}^n)$ be the group of all holomorphic one-to-one maps F from $\mathbb{C}^n$ onto $\mathbb{C}^n$, with composition as group operation. The iterates of F are defined by setting $F^1 = F$ and $F^k = F \circ F^{k-1}$ for $k \geq 2$. If $F^m = I$ (the identity map) for some $m \geq 1$ we say that F is *periodic*, with period m .

$\text{Aut}(\mathbb{C}^n)$ is a large and complicated group when $n \geq 2$. (See [10], for example.) One way to begin to understand its structure is to study its finite subgroups and, in particular, its periodic members.

It is of course easy to find all periodic *linear* transformations L on $\mathbb{C}^n$: The Jordan canonical form of such an L shows that L is diagonal with respect to some basis of $\mathbb{C}^n$ and that the diagonal entries are roots of unity. If one conjugates L with some $\Phi \in \text{Aut}(\mathbb{C}^n)$ the result, $\Phi^{-1}L\Phi$, is also periodic. The question arises whether all periodic automorphisms of $\mathbb{C}^n$ can be obtained in this way. One can turn this around and ask

Question 1. *If $F \in \text{Aut}(\mathbb{C}^n)$ is periodic, must there be a $\Phi \in \text{Aut}(\mathbb{C}^n)$ such that $\Phi^{-1}F\Phi$ is linear?*

In other words, can F be *globally linearized*? A very easy local result is proved in Part III.

A more ambitious question is:

Question 2. *Is every finite subgroup of $\text{Aut}(\mathbb{C}^n)$ conjugate (within $\text{Aut}(\mathbb{C}^n)$) to a group of linear transformations on $\mathbb{C}^n$?*

With “isomorphic” in place of “conjugate” we show in Part II that the answer to Question 2 is yes.

As regards Question 1, we have an affirmative answer in a special case, namely when $n = 2$ and F lies in the so-called “overshear subgroup” Ω of $\text{Aut}(\mathbb{C}^n)$.

To define this, we recall that a *shear* on $\mathbb{C}^n$ is a map that sends (x, y) to $(x + h(y), y)$ or to $(x, y + h(x))$, where $h : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic. We call F an *overshear* on $\mathbb{C}^2$ (this term was first used in [1]) if

$$(1) \quad F(x, y) = (g(y)x + h(y), y)$$

or

$$(2) \quad F(x, y) = (x, g(x)y + h(x))$$

where h is as above and $g : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ is holomorphic, and we let Ω be the group of all finite compositions of overshears. (In [1] it is proved, for all $n \geq 2$, that the overshear subgroup of $\text{Aut}(\mathbb{C}^n)$ is dense in $\text{Aut}(\mathbb{C}^n)$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}^n$. Thus Ω is a “large” group.)

Part I is devoted to the proof of our

Main result. *If $F \in \Omega$ and F is periodic, then $\Phi^{-1}F\Phi$ is linear for some $\Phi \in \Omega$.*

Let $\text{Aut}_P(\mathbb{C}^2)$ be the group of *polynomial* automorphisms of $\mathbb{C}^2$. In view of Jung’s theorem [7], [9] which states that every $F \in \text{Aut}_P(\mathbb{C}^2)$ is a finite composition of shears and linear maps, our main result (Theorem 1.5) has the following

Corollary. *Every periodic $F \in \text{Aut}_P(\mathbb{C}^2)$ has the form $\Phi L \Phi^{-1}$, for some linear L and some $\Phi \in \text{Aut}_P(\mathbb{C}^2)$.*

That Φ can be taken in $\text{Aut}_P(\mathbb{C}^2)$ and not merely in Ω can be seen by examining our proof of Theorem 1.5 under the assumption that F is polynomial.

Following Friedland and Milnor [5] (they did this in $\text{Aut}_P(\mathbb{C}^2)$) we make some further definitions.

- (i) The group of all *affine* automorphisms of $\mathbb{C}^2$ will be denoted by $\mathcal{A}$.
- (ii) The group consisting of all maps of the form

$$(3) \quad F(x, y) = (g(y)x + h(y), \beta y + \gamma)$$

where g and h are as in (1), and $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, will be denoted by $\mathcal{E}$ (for “elementary”).

(iii) A *reduced product* is a map of the form

$$(4) \quad \Phi = \Phi_k \circ \Phi_{k-1} \circ \dots \circ \Phi_2 \circ \Phi_1$$

in which the Φ_i ’s alternate between members of $\mathcal{A}$ and $\mathcal{E}$, and no Φ_i lies in $\mathcal{A} \cap \mathcal{E}$.

It is clear that $\mathcal{A} \subset \Omega$ and $\mathcal{E} \subset \Omega$, and it is easy to see that every $F \in \Omega$ which is not in $\mathcal{A} \cap \mathcal{E}$ can be represented as a reduced product. What is not at all obvious, and what we shall need to know, is that the identity map is *not* a reduced product (Theorem 1.4). Friedland and Milnor [5] proved this, in the polynomial context, by using degrees of polynomials as a measure of growth. As was done in [1], we instead use some properties of the Nevanlinna characteristic of functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$. One of these, namely that the characteristic of the gradient of an entire function f is dominated by that of f , has previously been proved via the much more difficult so-called “lemma of the logarithmic derivative” in several variables [1], [11]. We give a very simple proof of the gradient inequality in an Appendix. It uses nothing more sophisticated than Green’s theorem on a ball.

In Part II we show that the groups $\text{Aut}(\mathbb{C}^n)$ and $\text{Aut}(\mathbb{C}^m)$ have different finite subgroups if $m \neq n$. They are therefore not isomorphic, which answers a question raised a few years ago by Lee Rubel.

Part III presents some examples of periodic automorphisms in $\mathbb{C}^3$.

Part I. As was mentioned above, the principal aim of this part is Theorem 1.5. We begin with the simplest case, which works just as well for all $n \geq 1$.

Theorem 1.1. *If F is affine and periodic on $\mathbb{C}^n$, then $T^{-1}FT$ is linear, for some translation T .*

Proof. There is a linear $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and a vector $c \in \mathbb{C}^n$ such that $F(z) = Lz + c$, and there is an $m \geq 1$ such that $F^m = I$. Iteration shows that

$$(1) \quad z = F^m(z) = L^m z + (L^{m-1} + \dots + L + I)c$$

for all $z \in \mathbb{C}^n$. Hence $L^m = I$ and

$$(2) \quad (I + L + \dots + L^{m-1})c = 0.$$

Since $1 - x$ and $1 + x + \dots + x^{m-1}$ are relatively prime in $\mathbb{C}[x]$, there are polynomials $P(x)$ and $Q(x)$ such that

$$(3) \quad (1 - x)P(x) + Q(x)(1 + x + \dots + x^{m-1}) = 1.$$

Replace x by L in (3) (and 1 by I), apply the resulting linear operator to c , put $z_0 = P(L)c$, and note, by (2), that then

$$(4) \quad (I - L)z_0 = (I - L)P(L)c = c.$$

Thus F has a fixed point:

$$(5) \quad F(z_0) = Lz_0 + c = z_0.$$

Define $Tz = z + z_0$. Then $T^{-1}FT = L$.

The next step is more substantial.

Theorem 1.2. *If $F \in \mathcal{E}$ and F is periodic, then $\Phi^{-1}F\Phi$ is linear, for some $\Phi \in \mathcal{E}$.*

Proof. Since $F \in \mathcal{E}$,

$$(1) \quad F(x, y) = (g(y)x + h(y), \beta y + \gamma)$$

where g and h are entire, g has no zero, β and γ are constants, $\beta \neq 0$. Choose $m \geq 1$ so that $F^m = I$.

As in Theorem 1.1, our first objective is to prove that F has a fixed point.

We claim that $\beta y_0 + \gamma = y_0$ for some $y_0 \in \mathbb{C}$. This is obvious when $\beta \neq 1$. But when $\beta = 1$, then $F^m = I$ forces $\gamma = 0$, and any y_0 will do.

If $g(y_0) \neq 1$, there is a unique x_0 such that $g(y_0)x_0 + h(y_0) = x_0$; hence $F(x_0, y_0) = (x_0, y_0)$.

If $g(y_0) = 1$, then iteration shows that

$$(2) \quad (x, y_0) = F^m(x, y_0) = (x + mh(y_0), y_0).$$

Hence $h(y_0) = 0$ and $F(x, y_0) = (x, y_0)$ for every $x \in \mathbb{C}$.

Thus F has a fixed point, and there is a translation T such that $T^{-1}FT$ fixes the origin. Note that $T \in \mathcal{E}$.

We now need a one-variable

Lemma. *Let β and ω be m^{th} roots of unity. If $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and*

$$(3) \quad \sum_{k=0}^{m-1} \omega^{-k} f(\beta^k \lambda) = 0$$

for all $\lambda \in \mathbb{C}$, then there is an entire function φ such that

$$(4) \quad \varphi(\beta\lambda) - \omega\varphi(\lambda) = f(\lambda).$$

Proof. The result of inserting $f(\lambda) = \sum_0^\infty f_n \lambda^n$ into (3) is

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{m-1} (\omega^{-1} \beta^n)^k \right\} f_n \lambda^n = 0.$$

Hence $f_n = 0$ for those n for which $\beta^n = \omega$.

Define $\varphi(\lambda) = \sum_0^\infty \varphi_n \lambda^n$ by setting $\varphi_n = 0$ when $\beta^n = \omega$, $\varphi_n = f_n / (\beta^n - \omega)$ otherwise. In the second case, $|\beta^n - \omega| \geq |1 - e^{2\pi i/m}| = \delta$, say. Thus $|\varphi_n| \leq \delta^{-1} |f_n|$ for all n , which shows that φ is entire. It is clear that (4) holds, so that the lemma is proved.

We return to the proof of Theorem 1.2.

Replacing F by $T^{-1}FT$, our new F fixes the origin, so that (1) now holds with $\gamma = 0$ and $h(0) = 0$. Since $F^m = I$ we have

$$(5) \quad \beta^m = 1,$$

and one sees that

$$(6) \quad \prod_{i=0}^{m-1} g(\beta^i y) = 1$$

and

$$(7) \quad \sum_{k=0}^{m-1} h(\beta^k y) \prod_{i=k+1}^{m-1} g(\beta^i y) = 0$$

for all $y \in \mathbb{C}$. (When $k = m - 1$, the product in (7) is 1.)

Put $\omega = g(0)$. By (6), $\omega^m = 1$. Since g has no zeros in $\mathbb{C}$, there is an entire function ψ such that

$$(8) \quad g(y) = \omega e^{\psi(y)}, \quad \psi(0) = 0,$$

so that (6) implies

$$(9) \quad \sum_{i=0}^{m-1} \psi(\beta^i y) = 0.$$

We will show that F is conjugate to

$$(10) \quad L(x, y) = (\omega x, \beta y)$$

by finding $\Phi \in \mathcal{E}$, of the form

$$(11) \quad \Phi(x, y) = (xe^{a(y)} + b(y), y)$$

for which $F \circ \Phi = \Phi \circ L$, i.e., $\Phi^{-1}F\Phi = L$.

Both $F \circ \Phi$ and $\Phi \circ L$ have βy as their second component. Their first components are, respectively,

$$(12) \quad \omega x e^{\psi(y)+a(y)} + \omega e^{\psi(y)}b(y) + h(y)$$

and

$$(13) \quad \omega x e^{a(\beta y)} + b(\beta y).$$

Thus we need entire functions a and b such that

$$(14) \quad a(\beta y) - a(y) = \psi(y)$$

and

$$(15) \quad b(\beta y) = \omega e^{\psi(y)}b(y) + h(y).$$

Since (9) holds, the lemma furnishes a solution of (14). Once we have the function a , we replace b by Be^a . This converts (15) to

$$(16) \quad B(\beta y) - \omega B(y) = h(y)e^{-a(\beta y)}.$$

It remains to be shown that there is an entire function B that satisfies (16).

Replace y in (14) by $\beta^{k+1}y, \dots, \beta^{m-1}y$, and add the resulting equations. Since $\beta^m = 1$ we get

$$(17) \quad a(y) - a(\beta^{k+1}y) = \sum_{i=k+1}^{m-1} \psi(\beta^i y).$$

(When $k = m - 1$, both sides of (17) are 0.)

By (8) and (17), (7) now becomes

$$(18) \quad \sum_{k=0}^{m-1} \omega^{-k} h(\beta^k y) \exp[-a(\beta^{k+1} y)] = 0$$

and a second appeal to the lemma shows that (16) can be solved by an entire function B .

This completes the proof of Theorem 1.2. □

Definition. Let S be the unit sphere in $\mathbb{C}^n$, centered at the origin, and let σ be the positive rotation-invariant Borel measure on S , normalized so that $\sigma(S) = 1$. We define the *Nevanlinna characteristic* of any $f : \mathbb{C}^n \rightarrow \mathbb{C}$ (not necessarily holomorphic or meromorphic) to be

$$m(r, f) = \int_S \log^+ |f(r\zeta)| d\sigma(\zeta) \quad (0 < r < \infty)$$

where, as usual, $\log^+ x = \max(0, \log x)$ for $x > 0$.

Theorem 1.3. Suppose f, g, h are entire functions on $\mathbb{C}^n$, h is not constant, and φ is an entire function on $\mathbb{C}$. Then

- (i) $m(r, f + g) \leq m(r, f) + m(r, g) + \log 2$,
- (ii) $m(r, fg) \leq m(r, f) + m(r, g)$,
- (iii) $m(r, 1/f) \leq m(r, f) + c$ for $r > 1$, where $c = -\int_S \log |f| d\sigma$,
- (iv) $\lim_{r \rightarrow \infty} \frac{m(r, \varphi \circ h)}{m(r, h)} = \deg \varphi$ if φ is a polynomial,
- (v) $\lim_{r \rightarrow \infty} \frac{m(r, \varphi \circ h)}{m(r, h)} = \infty$ if φ is transcendental,
- (vi) $m(r, (\partial f / \partial z_j)) \leq 3m(r, f) + n \log r$ ($1 \leq j \leq n$) for all r outside a set of finite linear measure.

Proof. (i) and (ii) are trivial. (iii) follows from

$$(1) \quad \log^+ \left(\frac{1}{x} \right) = \log^+ x - \log x \quad (x > 0)$$

and the subharmonicity of $\log |f|$. When $n = 1$, (iv) and (v) are theorems of Clunie [4], [6]. If one follows the suggestion on p. 54 of [6], one sees that if φ has at least k zeros in $\mathbb{C}$, at $w_1, \dots, w_k$, then

$$(2) \quad \sum_{j=1}^k m(r, h - w_j) \leq c + m(r, \varphi \circ h)$$

where c depends only on φ and $h(0)$.

If now $h : \mathbb{C}^n \rightarrow \mathbb{C}$ and $h_\zeta(\lambda) = h(\lambda\zeta)$ for $\zeta \in S$, $\lambda \in \mathbb{C}$, one can apply (2) to the slice functions h_ζ with the same c for all ζ . Integration of (2), with h_ζ in place of h , over S , leads then to (iv) and (v). (If φ has no zeros, apply the preceding argument to $\varphi + 1$, which has infinitely many zeros.)

Finally, (vi) follows from the theorem proved in the Appendix.

Theorem 1.4. The identity map I on $\mathbb{C}^2$ is not a reduced product.

Proof. If there were such a representation of I , then there would be a shortest one, and this shortest one must have an even number of factors, otherwise conjugation by the inverse of either the first or the last factor would produce a shorter one. Thus we have to prove that if

$$(1) \quad \Phi = A_k F_k \cdots A_1 F_1$$

with all A_i in $\mathcal{A} \setminus \mathcal{E}$, all F_i in $\mathcal{E} \setminus \mathcal{A}$, then $\Phi \neq I$.

The following lemma will allow us to change (1) into a more convenient form; see (5) and (7).

Lemma. *If $A \in \mathcal{A} \setminus \mathcal{E}$ and $\tau(x, y) = (y, x)$, then there exist $B, C \in \mathcal{A} \cap \mathcal{E}$ such that $A = B\tau C$.*

Let

$$(2) \quad A(x, y) = (ax + by + \lambda_1, cx + dy + \lambda_2).$$

Note that $c \neq 0$ since $A \notin \mathcal{E}$, and look at the matrix equation

$$\begin{pmatrix} -c & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & \lambda_1 \\ c & d & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & \delta & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & \delta & * \end{pmatrix}$$

where $\delta = ad - bc \neq 0$. The point is that the first and the last of these has 0 in the lower left corner. If we multiply on the left by the inverse of the first matrix, we obtain $A = B\tau C$, where

$$(3) \quad B(x, y) = \left(\frac{ay - x}{c}, y \right)$$

$$(4) \quad C(x, y) = (cx + dy + \lambda_2, \delta y + a\lambda_2 - c\lambda_1).$$

Note that B and C are in $\mathcal{E}$ because x is absent from their second components.

Now apply the lemma to each $A_i = B_i \tau C_i$ in (1), put $\tilde{F}_1 = C_1 F_1$ and $\tilde{F}_i = C_i F_i B_{i-1}$ for $2 \leq i \leq k$. Each $\tilde{F}_i$ is then in $\mathcal{E} \setminus \mathcal{A}$, and (1) becomes

$$(5) \quad \Phi = B_k \tau \tilde{F}_k \tau \cdots \tau \tilde{F}_1.$$

Setting $G_i = \tau \tilde{F}_i$, the definition of $\mathcal{E}$ shows that each G_i has the form

$$(6) \quad G_i(x, y) = (\beta_i y + \gamma_i, g_i(y)x + h_i(y))$$

where $\beta_i \neq 0$, g_i and h_i are entire functions, g_i has no zero, and h_i is not affine if g_i is constant. Moreover, (5) now turns into

$$(7) \quad \Phi = B_k \circ G_k \circ \cdots \circ G_2 \circ G_1.$$

Define functions u_i, v_i , for $1 \leq i \leq k$, by

$$(8) \quad (G_i \circ \cdots \circ G_2 \circ G_1)(x, y) = (u_i(x, y), v_i(x, y)).$$

It is easy to see that

$$(9) \quad \limsup_{r \rightarrow \infty} \frac{m(r, v_1)}{m(r, u_1)} > 1.$$

Because of (7) and (9), the following lemma, whose proof closely follows that of Proposition 8.6 in [1], will show that $\Phi \neq I$.

Lemma. Suppose G is as in (6), without the subscripts, $G(u, v) = (s, t)$, and

$$(10) \quad \limsup_{r \rightarrow \infty} \frac{m(r, v)}{m(r, u)} > 1.$$

Then (10) holds with (s, t) in place of (u, v) .

More explicitly, we are given that

$$(11) \quad s = \beta v + \gamma, \quad t = (g \circ v)u + h \circ v.$$

There are now two cases. First, assume g is constant, say $g = c$. Then, using Theorem 1.3,

$$(12) \quad m(r, t) = m(r, cu + h \circ v) \geq m(r, h \circ v) - m(r, u) - O(1).$$

Since $m(r, s) = m(r, v) + O(1)$, it follows that

$$(13) \quad \limsup_{r \rightarrow \infty} \frac{m(r, t)}{m(r, s)} \geq \limsup_{r \rightarrow \infty} \frac{m(r, h \circ v)}{m(r, v)} - \liminf_{r \rightarrow \infty} \frac{m(r, u)}{m(r, v)}.$$

The first term on the right of (13) is ≥ 2 because h is not affine, the last term is < 1 by (10), and the left side of (13) is therefore > 1 .

Second, if g is not constant then, having no zeros, g is transcendental. The chain rule, applied to $(s, t) = G(u, v)$, gives

$$(14) \quad -\beta g \circ v = \left[\frac{\partial s}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial s}{\partial y} \frac{\partial t}{\partial x} \right] \bigg/ \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right]$$

because (6) shows that the Jacobian of G is $-\beta g$.

We now use assertions of Theorem 1.3, especially parts (iii) and (vi), to deduce from (14) that

$$\begin{aligned} m(r, g \circ v) &\leq 6(m(r, s) + m(r, t) + m(r, u) + m(r, v)) + O(\log r) \\ &\leq 12(m(r, t) + m(r, u) + m(r, v)) + O(\log r) \end{aligned}$$

for all r outside some set E of finite measure. Thus

$$(15) \quad \frac{m(r, t)}{m(r, s)} \geq \frac{1}{12} \cdot \frac{m(r, g \circ v)}{m(r, v)} - \frac{m(r, u)}{m(r, v)} - O(1)$$

outside E . The right side of (15) tends to ∞ as $r \rightarrow \infty$ since g is transcendental.

This proves the lemma and hence the theorem. $\square$

We finally come to our main result.

Theorem 1.5. *To every periodic F in the overshear group Ω corresponds a $\Phi \in \Omega$ such that $\Phi^{-1}F\Phi$ is linear.*

Proof. Let

$$(1) \quad \Gamma(F) = \{\Phi^{-1}F\Phi : \Phi \in \Omega\}.$$

If $\Gamma(F)$ contains an element of $\mathcal{A} \cap \mathcal{E}$, we are done, by Theorem 1.1. If not, then every $G \in \Gamma(F)$ has a reduced product representation. Let

$$(2) \quad G_0 = H_k \circ H_{k-1} \circ \cdots \circ H_1$$

be one with a minimal number of factors.

If k is odd and $k > 1$, then H_1 and H_k both lie in $\mathcal{A}$ or both lie in $\mathcal{E}$. In either case, G_0 is conjugate to

$$(3) \quad H_k^{-1}G_0H_k = H_{k-1} \circ \cdots \circ H_2 \circ (H_1H_k),$$

and this contradicts the minimality of k .

If k is even and F has period m , so does G_0 , and since H_k and H_1 belong to different subgroups,

$$(4) \quad I = G_0^m = (H_k \cdots H_1)(H_k \cdots H_1) \cdots (H_k \cdots H_1)$$

is a reduced product, contradicting Theorem 1.4.

Thus $k = 1$, which means that $G_0 \in \mathcal{A}$ or $G_0 \in \mathcal{E}$. Theorems 1.1 and 1.2 now finish the proof of Theorem 1.5. $\square$

Part II. We shall now use the customary notation $\text{GL}(n, \mathbb{C})$ for the group of all invertible linear transformations on $\mathbb{C}^n$.

If $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic, the power series of its components can be grouped so as to give the *homogeneous expansion* of F , namely

$$F(z) = F(0) + P_1(z) + \cdots + P_s(z) + \cdots$$

in which P_s is a polynomial map of $\mathbb{C}^n$ to $\mathbb{C}^n$, homogeneous of degree s . In particular, P_1 is linear, and we define it to be the *derivative* of F at 0 : $P_1 = F'(0)$.

(In fact, $F'(0)$ is just the Fréchet derivative of F at 0.)

Theorem 2.1. *If G is a not necessarily finite group of periodic automorphisms of $\mathbb{C}^n$, and $F(0) = 0$ for every $F \in G$, then*

$$(1) \quad F \rightarrow F'(0)$$

is an isomorphism of G onto a subgroup of $\text{GL}(n, \mathbb{C})$.

Proof. Since every $F \in G$ fixes the origin, the chain rule shows that (1) is a homomorphism. That $F'(0)$ is invertible is seen by applying the chain rule to $F^{-1}F = I$. We show that the kernel of this homomorphism consists of I alone.

If $F \in G$ and $F'(0) = I$, its homogeneous expansion has the form

$$(2) \quad F(z) = z + P_k(z) + P_{k+1}(z) + \cdots$$

where we assume that $k \geq 2$ and that $P_s(z) = 0$ for all s satisfying $2 \leq s < k$. If F has period m , then (2) leads to

$$z = F^m(z) = z + mP_k(z) + \cdots$$

which implies that $P_k = 0$. Thus $F(z) = z$, by induction.

Theorem 2.2. *If Γ is an abelian subgroup of $\text{GL}(n, \mathbb{C})$, k is a positive integer, and $F^k = I$ for every $F \in \Gamma$, then $|\Gamma| \leq k^n$.*

We use the notation $|\Gamma|$ to denote the cardinality of Γ . Note that $|\Gamma| < \infty$ is *not* part of the hypothesis.

Proof. If $n = 1$, every $F \in \Gamma$ is multiplication by a k^{th} root of unity. Thus $|\Gamma| \leq k$.

Now suppose $n > 1$. If $|\Gamma| > k$, then some $F_0 \in \Gamma$ is not a scalar multiple of I . Thus F_0 has at least two distinct eigenvalues $\lambda_1, \dots, \lambda_r$. The Jordan canonical form of F_0 is diagonal, since F_0 is periodic. This implies that $\mathbb{C}^n$ is the direct sum of subspaces $X_1, \dots, X_r$, where X_i is the eigenspace of F_0 corresponding to the eigenvalue λ_i . Let $d_i = \dim X_i$. Then $d_i > 0$, $\sum d_i = n$, hence $d_i < n$ because $r \geq 2$.

Since Γ is abelian, $F(X_i) \subset X_i$ for each i and each $F \in \Gamma$. Using $d_i < n$ it follows now, by induction on the dimension n , that

$$|\Gamma| \leq k^{d_1} k^{d_2} \cdots k^{d_r} = k^n. \quad \square$$

Theorem 2.3. *If G is a finite abelian subgroup of $\text{Aut}(\mathbb{C}^n)$, p is a prime, and $F^p = I$ for every $F \in G$, then $|G| \leq p^n$.*

Proof. Since G is a finite p -group of homeomorphisms of $\mathbb{C}^n$, the elements of G have a common fixed point [3; Thm. 7.11] which, by a suitable choice of coordinates, can be taken to be the origin of $\mathbb{C}^n$. Our assertion is therefore an immediate consequence of Theorems 2.1 and 2.2 $\square$

Theorem 2.4. *If $1 \leq m < n$, then the groups $\text{Aut}(\mathbb{C}^m)$ and $\text{Aut}(\mathbb{C}^n)$ are not isomorphic.*

Proof. Choose a prime p , put $\omega = \exp(2\pi i/p)$, and consider the maps

$$(z_1, \dots, z_n) \rightarrow (\omega^{a_1} z_1, \omega^{a_2} z_2, \dots, \omega^{a_n} z_n)$$

where $0 \leq a_i < p$ and each a_i is an integer. These maps form an abelian group $G \subset \text{Aut}(\mathbb{C}^n)$, with $|G| = p^n$.

By Theorem 2.3, G is not isomorphic to any subgroup of $\text{Aut}(\mathbb{C}^m)$. This proves our assertion. $\square$

Part III.

3.1. Here is an *attempt* to answer Question 1 affirmatively for at least those periodic $F \in \text{Aut}(\mathbb{C}^n)$ that fix some point of $\mathbb{C}^n$. (The fixed-point theorem quoted in the proof of Theorem 2.3 does not cover all periodic homeomorphisms of $\mathbb{R}^n$. In fact, Kister [8] has proved that there is a fixed-point-free differentiable action of the cyclic group of order 6 on $\mathbb{R}^8$. For a more striking result along these lines, see [3; p. 61]. It is not known whether every periodic $F \in \text{Aut}(\mathbb{C}^n)$ has a fixed point.)

Let $F \in \text{Aut}(\mathbb{C}^n)$ have $F(0) = 0$ and $F^m = I$, put $A = F'(0)$, and define

$$(1) \quad \Phi = \frac{1}{m} \sum_{k=0}^{m-1} A^{-k} F^k.$$

The linearity of A^{-1} shows that

$$(2) \quad A^{-1} \Phi F = \Phi,$$

and (1) implies also that $\Phi(0) = 0$, $\Phi'(0) = I$. Thus Φ is locally invertible: there exist neighborhoods U and V of 0 such that Φ is a biholomorphic map from U onto V . If $\Phi^{-1} : V \rightarrow U$ is defined by $\Phi^{-1}(\Phi(z)) = z$ for all $z \in U$, then it follows from (2) that

$$(3) \quad \Phi F \Phi^{-1}(x) = Ax \quad \text{for all } x \in V.$$

In other words, F is *locally linearized* by Φ .

Unfortunately, even though (2) holds in all of $\mathbb{C}^n$, there seems to be no reason why any such Φ should be globally one-to-one. In the following example, $n = 2$ and $m = 2$.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial, with $f(0) = 0$, $\deg f \geq 2$. Put

$$(4) \quad G(x, y) = (x + f(y), y), \quad \tau(x, y) = (y, x), \quad F = G^{-1}\tau G.$$

Then $F \in \text{Aut}(\mathbb{C}^2)$, $F^2 = I$, $A = F'(0) = \tau$, so that (1) becomes

$$(5) \quad \Phi = \frac{1}{2}(I + A^{-1}F) = \frac{1}{2}(I + \tau F).$$

A short calculation shows that the Jacobian of Φ is

$$(6) \quad (J\Phi)(x, y) = 1 - \frac{1}{4}f'(x + f(y))f'(y).$$

This is a nonconstant polynomial, hence has zeros in $\mathbb{C}^2$, and Φ is therefore not globally one-to-one.

One can try to rescue this attempt by replacing (1) by

$$(7) \quad T\Psi = \frac{1}{m} \sum_{k=0}^{m-1} A^{-k} \Psi F^k$$

where $\Psi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic, $\Psi(0) = 0$, $\Psi'(0) = I$. Then (2) still holds: $A^{-1}(T\Psi)F = T\Psi$.

It is easy to check that $T^2\Psi = T\Psi$. Thus T is a projection in the space of all Ψ 's. The question is whether the range of T contains an automorphism.

3.2. We now consider the map $\Phi : \mathbb{C}^3 \rightarrow \mathbb{C}^3$, $\Phi(x, y, z) = (u, v, w)$, given by

$$(1) \quad \begin{cases} u = x + 2y(y^2 - xz) + z(y^2 - xz)^2 \\ v = y + z(y^2 - xz) \\ w = z \end{cases}$$

Since $v^2 - uw = y^2 - xz$, it is not hard to see that $\Phi \in \text{Aut}(\mathbb{C}^3)$.

This Φ has long been a candidate for disproving the so-called *tame generator conjecture* [2; p. 299] which, if true, would extend Jung's theorem on polynomial automorphisms (stated in the Introduction) from $\mathbb{C}^2$ to all $\mathbb{C}^n$.

Φ is not periodic, but if v is replaced in (1) by $-v$ one obtains an automorphism of period 2. Rather than show this special case, we now look at a larger collection of such maps.

Let $m \geq 2$ be an integer and define $\Phi_\lambda(x, y, z) = (u, v, w)$, for $\lambda \in \mathbb{C}$, by

$$(2) \quad \begin{cases} u = x + \frac{[y + \lambda z(y^m - xz)]^m - y^m}{z}, \\ v = y + \lambda z(y^m - xz), \\ w = z. \end{cases}$$

This polynomial map reduces to (1) when $m = 2$, $\lambda = 1$. Setting

$$(3) \quad A(x, y, z) = (x, \omega y, z)$$

where $\omega = \exp(2\pi i/m)$, it is not hard to check that

$$(4) \quad A\Phi_\lambda = \Phi_{\omega\lambda}A \quad \text{and} \quad \Phi_\lambda\Phi_\mu = \Phi_{\lambda+\mu}.$$

Since $\Phi_0 = I$, it follows that each Φ_λ is invertible, with $\Phi_{-\lambda}$ as inverse. Thus $\{\Phi_\lambda : \lambda \in \mathbb{C}\}$ is a one-parameter subgroup of $\text{Aut}(\mathbb{C}^3)$.

Now fix λ , put $F_\lambda = A\Phi_\lambda$, and choose α so that $\alpha + \omega(\lambda - \alpha) = 0$. Then (4) shows that

$$\Phi_\alpha F_\lambda \Phi_{-\alpha} = \Phi_\alpha A \Phi_{\lambda-\alpha} = \Phi_{\alpha+\omega(1-\alpha)} A = A.$$

Since $A^m = I$, by (3), it follows that $F_\lambda^m = I$.

Each F_λ is thus an automorphism of $\mathbb{C}^3$, of period m , and is conjugate to a linear one.

Appendix. If $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic, the *gradient* of f at z is the vector

$$(\nabla f)(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right).$$

Its length is

$$|(\nabla f)(z)| = \left\{ \sum_{i=1}^n |\partial f / \partial z_i|^2 \right\}^{1/2}.$$

Theorem. *To every entire $f : \mathbb{C}^n \rightarrow \mathbb{C}$ corresponds a set $E \subset [0, \infty)$, of finite linear measure, such that*

$$(1) \quad m(r, |\nabla f|) \leq 2m(r, f) + 2\log m(r, f) + n \log r$$

for all r outside E .

Note that this gives Theorem 1.3(vi).

We precede the proof with two observations.

- (*) If $\varphi : [0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and increasing, and $E = \{x : \varphi'(x) > \varphi^2(x)\}$, then $m(E) < \infty$, because

$$m(E) \leq 1 + \int_1^\infty (\varphi' / \varphi^2) dx \leq 1 + 1/\varphi(1).$$

- (**) If $u > 0$ and $\int$ denotes integration with respect to some probability measure, then Jensen's inequality shows that

$$\begin{aligned} \int \log^+ u &\leq \int \log(1 + u) \leq \log \int (1 + u) \\ &= \log(1 + \int u) \leq \log 2 + \log^+ \int u. \end{aligned}$$

To prove the theorem we introduce the function

$$(2) \quad g = \log(1 + f\bar{f})$$

whose Laplacian is

$$(3) \quad \Delta g = 4 \sum_{i=1}^n \partial^2 g / \partial z_i \partial \bar{z}_i = \frac{4|\nabla f|^2}{(1 + f\bar{f})^2}$$

so that

$$(4) \quad |\nabla f| = \frac{1}{2}(1 + f\bar{f})\sqrt{\Delta g}.$$

Since $\log(1 + x^2) \leq \log 2 + 2\log^+ x$, it follows that

$$(5) \quad \log^+ |\nabla f| \leq \log 2 + 2\log^+ |f| + \frac{1}{2} \log^+ (\Delta g)$$

and now (**) leads to

$$(6) \quad m(r, |\nabla f|) \leq 2m(r, f) + \log 4 + \frac{1}{2} \log^+ \int_S (\Delta g)(r\zeta) d\sigma(\zeta).$$

All we need therefore to get (1) is a good upper bound for the integral

$$(7) \quad P(r) = \int_S (\Delta g)(r\zeta) d\sigma(\zeta).$$

Define

$$(8) \quad Q(r) = \int_{rB} (\Delta g) d\nu, \quad G(r) = \int_S g(r\zeta) d\sigma(\zeta).$$

Here B is the unit ball in $\mathbb{C}^n$ and ν is Lebesgue measure on $\mathbb{C}^n$, so normalized that $\nu(B) = 1$.

Green's formula

$$(9) \quad \int_{rB} (\Delta g) d\nu = 2nr^{2n-1} \int_S \frac{\partial g}{\partial r}(r\zeta) d\sigma(\zeta)$$

becomes now

$$(10) \quad Q(r) = 2nr^{2n-1}G'(r).$$

(To check that the multiplicative constant is correct in (9), replace g by $|z|^2$.) Differentiation of (8) (write the integral over rB in polar coordinates) gives

$$(11) \quad Q'(r) = 2nr^{2n-1}P(r).$$

Now let $E_1 = \{r : G'(r) > G^2(r)\}$, $E_2 = \{r : Q'(r) > Q^2(r)\}$, and $E_0 = E_1 \cup E_2$. By (*), E_0 has finite linear measure.

If $r \notin E_0$, (10) and (11) show that

$$(12) \quad \begin{aligned} P(r) &= (2n)^{-1}r^{1-2n}Q'(r) \leq (2n)^{-1}r^{1-2n}Q^2(r) \\ &= 2nr^{2n-1}G'(r)^2 \leq 2nr^{2n-1}G^4(r). \end{aligned}$$

For $r \notin E_0$, the last term in (6) is thus

$$(13) \quad \frac{1}{2}\log^+ P(r) \leq \frac{1}{2}\log(2n) + \left(n - \frac{1}{2}\right)\log^+ r + 2\log^+ G(r),$$

and since $G(r) \leq \log 2 + 2m(r, f)$, we obtain (1) from (6) and (13) for all sufficiently large r outside E_0 .

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