Superoptimal Singular Values and Indices of Infinite Matrix Functions

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ABSTRACT. We study the problem of finding a best approximation of a given bounded operator function Φ on the unit circle $\mathbb{T}$ by bounded operator functions Q analytic in the unit disk. We minimize not only $\operatorname{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - Q(\zeta)\|$ but also the L^∞ norms of further singular values of $\Phi(\zeta) - Q(\zeta)$. We show that if the Hankel operator H_Φ is compact, then such an approximation (which is called *superoptimal*) exists and unique. We study related factorizations (*thematic factorizations*), establish invariance of their indices and prove an inequality between the singular values of the Hankel operator H_Φ and the superoptimal singular values $\operatorname{ess\,sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$.

0. Introduction. The classical Nehari problem is to find, for a given bounded scalar function φ on the unit circle $\mathbb{T}$, a function f in H^∞ such that $\|\varphi - f\|_{L^\infty(\mathbb{T})} = \operatorname{dist}_{L^\infty(\mathbb{T})}\{\varphi, H^\infty\}$. A trivial compactness argument proves that such a function f always exists. However, such a best approximation is not necessarily unique (see [AAK]). It is also well known ([Kh]) that if $\varphi \in H^\infty + C$, then φ has a unique best approximation by functions in H^∞ . The Nehari problem turned out to be very important in applications in control theory (see [Fr]). However in control theory it is essential to consider not only scalar functions but also matrix-valued functions and even operator-valued functions (i.e. functions which take values in the space $\mathcal{B}$ of bounded linear operators on Hilbert space or, more generally, the space of bounded linear operators from one Hilbert space to another one). A simple example $\Phi = \begin{pmatrix} 2\bar{z} & \mathbb{O} \\ 0 & \bar{z} \end{pmatrix}$ shows that in the matrix case we almost never have uniqueness. It was shown in [PY1] that for a matrix function $\Phi \in H^\infty + C$ if we minimize not only the L^∞ -norm of $\|\Phi(\zeta) - Q(\zeta)\|$, $\zeta \in \mathbb{T}$, but also the L^∞ -norm of all further singular values, we get uniqueness.

To be more precise, given a matrix function Φ , we put

$$\Omega_0 = \{Q \in H^\infty : Q \text{ minimizes } t_0 = \sup_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - Q(\zeta)\|\};$$

$$\Omega_j = \{Q \in \Omega_{j-1} : Q \text{ minimizes } t_j = \sup_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))\}.$$

Here s_j is the j th singular value, which is the distance to the set of operators (matrices) of rank at most j . Put

$$t_j \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta)), \quad Q \in \Omega_j.$$

It was proved in [PY1] that if Φ is an $n \times m$ matrix function in $H^\infty + C$, then the set $\Omega_{\min\{n,m\}-1}$ consists of one function which is called the *superoptimal approximation* of Φ by bounded analytic functions. It was also proved in [PY1] that for the superoptimal approximation Q all singular values $s_j(\Phi(\zeta) - Q(\zeta))$ are constant almost everywhere on $\mathbb{T}$ and coincide with the above numbers t_j which are called the *superoptimal singular values*.

The proof of the uniqueness given in [PY1] is based on certain special factorizations (thematic factorizations) and it reduces the problem of finding a superoptimal approximation to the same problem for a matrix function in $H^\infty + C$ of lower size. Later in [T2] another approach to this problem was found which also works for operator-valued functions in $H^\infty(\mathcal{B}) + C(\mathcal{K})$, ($\mathcal{K}$ is the class of compact operators on Hilbert space). If Φ is a weakly measurable bounded operator-valued function, a function Q in $H^\infty(\mathcal{B})$ is called a *superoptimal approximation* of Φ by bounded analytic operator functions if $Q \in \bigcap_{j \geq 0} \Omega_j$.

Then in [P] the same uniqueness result was proved for operator-valued functions by using thematic factorizations for infinite matrix functions. However in the case of infinite matrix functions the situation is more complicated since the factorization reduces the problem to finding a superoptimal approximation to another infinite matrix function.

In [PY1] with a thematic factorization a sequence of indices $\{k_j\}$ was associated. A simple example (see [PY1]) shows that the indices are not determined by the function itself: they depend on the choice of a thematic factorization. However it was shown in [PY2] that in the case of finite matrix functions the sums of indices which correspond to the same superoptimal singular value do not depend on the choice of a thematic factorization. In [P] it was conjectured that the same should be true in the operator-valued case. In this paper we prove that conjecture. To prove it it is important to establish a connection between the original approach to superoptimal approximations as in [PY1] and the geometrical approach presented in [T2] (see Theorem 3.2).

In [P] the uniqueness of the superoptimal approximant was obtained by establishing an inequality between the superoptimal singular values and the singular values of the corresponding Hankel operator. This inequality was obtained in

[PY2] for finite matrix functions (earlier in [T2] a weaker inequality was obtained both for finite and infinite matrix functions which is sufficient to establish the uniqueness). The proof of the inequality given in [P] is based on certain approximations and reduces the problem to the case of finite matrices.

In this paper we give a more natural proof of the same inequality which is based on the solution of an operator-valued corona problem and on the construction of a family of functions that have certain important properties. The same method was used in [PY2] for finite matrix functions. However the case of infinite matrix functions is considerably more complicated. The approach given in this paper gives a new proof of the uniqueness of a superoptimal approximation in the operator-valued case.

1. Thematic factorizations. It is easy to see that without loss of generality we can consider the case of $\mathcal{B}(\mathcal{H})$ -valued functions, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. We can certainly identify $\mathcal{H}$ with ℓ^2 and interpret operator-valued functions as infinite matrix functions. In the sequel “matrix” can mean “infinite matrix”.

Recall the following notions (see [H], [SNF]). A matrix function Φ in $H^\infty(\mathcal{B})$ is called *inner* if $\Phi(\zeta)$ is an isometry for almost all $\zeta \in \mathbb{T}$. (Note that this definition differs from the one used by engineers: they call a function Φ inner if $\Phi(\zeta)$ is a unitary operator a.e. on $\mathbb{T}$. The notation we use was established much earlier, and it is accepted by most mathematicians.) A function Φ in $H^\infty(\mathcal{B})$ is called *outer* if multiplication by Φ on $H^2(\ell^2)$ has dense range. Finally, a function $\Phi \in H^\infty(\mathcal{B})$ is called *co-outer* if its transposed matrix function Φ^t is outer.

We also need the definitions of Hankel and Toeplitz operators.

Let $\Phi \in L^\infty(\mathcal{B})$. The *Hankel operator* $H_\Phi : H^2(\ell^2) \rightarrow H_-^2(\ell^2) \stackrel{\text{def}}{=} L^2(\ell^2) \ominus H^2(\ell^2)$ is defined by

$$H_\Phi f = \mathbb{P}_- \Phi f,$$

where $\mathbb{P}_-$ is the orthogonal projection from $L^2(\ell^2)$ onto $H_-^2(\ell^2)$. It is well known that $\|H_\Phi\| = \text{dist}_{L^\infty}\{\Phi, H^\infty(\mathcal{B})\}$ and H_Φ is compact if and only if $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$ (see [Pa]). The *Toeplitz operator* T_Φ on $H^2(\ell^2)$ is defined by

$$T_\Phi f = \mathbb{P}_+ \Phi f, \quad f \in H^2(\ell^2),$$

where $\mathbb{P}_+$ is the orthogonal projection from $L^2(\ell^2)$ onto $H^2(\ell^2)$.

Let us describe briefly the factorization method which was developed in [PY1] in the case of finite matrix functions. Suppose that $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$. Let v be a maximizing vector for H_Φ and let $H_\Phi v = \|H_\Phi\| w = t_0 w$, $w \in H_-^2(\ell^2)$. Then (see [PY1], [P]) the functions v and $\bar{z}\bar{w}$ in $H^2(\ell^2)$ admit the following factorizations

$$v = v_{(i)} \vartheta_1 h, \quad \bar{z}\bar{w} = w_{(i)} \vartheta_2 h,$$

where h is an outer scalar function, ϑ_1 and ϑ_2 are inner scalar functions, and $v_{(i)}$ and $w_{(i)}$ are columns which are inner and co-outer. It can be shown (see [P]) that the Hankel operators $H_{\overline{v_{(i)}}}$ and $H_{\overline{w_{(i)}}}$ are compact.

It was shown in [P] that if φ is a column which is inner and co-outer, then it can be completed to a unitary-valued matrix of the form $\begin{pmatrix} \varphi & \overline{\Theta} \end{pmatrix}$, where Θ is inner and co-outer. Such matrices $\begin{pmatrix} \varphi & \overline{\Theta} \end{pmatrix}$ are called *thematic matrices*.

Consider thematic completions $V \stackrel{\text{def}}{=} \begin{pmatrix} v_{(i)} & \bar{\alpha} \end{pmatrix}$ and $W^t \stackrel{\text{def}}{=} \begin{pmatrix} w_{(i)} & \bar{\beta} \end{pmatrix}$ of $v_{(i)}$ and $w_{(i)}$. It was shown in [P] that if Q_0 is an arbitrary best approximation to Φ by functions in H^∞ , then

$$W(\Phi - Q_0)V = \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_1 \end{pmatrix},$$

where $u_0 = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$ and Φ_1 is a function in $H^\infty(\mathcal{B}) + C(\mathcal{K})$ such that $\|\Phi_1\| \leq t_0$. Note that $u_0 \in QC \stackrel{\text{def}}{=} H^\infty + C \cap \overline{H^\infty + C}$ (see [PY1], [P]). Moreover if $Q \in H^\infty$, then Q is a best approximation to Φ if and only if there exists a function Q_1 in $H^\infty(\mathcal{B})$ such that

$$W(\Phi - Q)V = \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_1 - Q_1 \end{pmatrix},$$

and $\|\Phi_1 - Q_1\| \leq t_0$. The facts for finite matrix functions above were proved earlier in [PY1].

The construction above reduces the problem to finding a superoptimal approximation to Φ_1 : Q is a superoptimal approximation to Φ if and only if Q_1 is a superoptimal approximation to Φ_1 . If $\Phi_1 \in H^\infty(\mathcal{B})$, then $Q_1 = \Phi_1$ is the unique superoptimal approximation to Φ_1 and the process terminates.

Otherwise we can continue the process. If the process does not terminate at the m th step (i.e., $t_{m-1} \neq 0$), we can obtain the following factorization for $\Phi - Q$ where Q is a function in Ω_{m-1} :

$$\begin{aligned} (1.\Phi) - Q &= W_0^* \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & W_1^* \end{pmatrix} \cdots \begin{pmatrix} I_{m-1} & \mathbb{O} \\ \mathbb{O} & W_{m-1}^* \end{pmatrix} D \begin{pmatrix} I_{m-1} & \mathbb{O} \\ \mathbb{O} & V_{m-1}^* \end{pmatrix} \\ &\quad \cdots \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & V_1^* \end{pmatrix} V_0^*, \end{aligned}$$

where

$$D = \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{m-1} u_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & \Phi_m \end{pmatrix},$$

$\Phi_m \in H^\infty(\mathcal{B}) + C(\mathcal{K})$ and $\|\Phi_m\|_\infty \leq t_{m-1}$.

We can now define the *indices* of the factorization above. If $t_j \neq 0$, we define the j th index by $k_j = \text{wind } u_j$ (since $u_j \in QC$, the winding number is well defined as $\text{wind } u_j = -\text{ind } T_{u_j}$; it is negative because u_j has the form $u_j = \bar{z}\bar{\vartheta}\bar{h}/h$, where ϑ is inner and h is outer). The indices in general are not uniquely determined by the function Φ —they may depend on the choice of maximizing vectors and thematic completions (see [PY1]). However, it was proved in [PY2] that in the case of finite matrix functions the sums of the indices that correspond to equal superoptimal singular values are uniquely determined by Φ . In [P] it was conjectured that the same is true for infinite matrix functions. In Section 4 we shall prove this conjecture.

Another way to state this conjecture is to consider the *extended t -sequence*

$$t_0, \dots, t_0, t_1, \dots, t_1, t_2, \dots$$

where t_j is repeated k_j times. We denote the terms of the extended sequence by $\{\tilde{t}_j\}_{j \geq 0}$ (i.e. $\tilde{t}_0 = \cdots = \tilde{t}_{k_0-1} = t_0$, $\tilde{t}_{k_0-1} = \cdots = \tilde{t}_{k_0+k_1-1} = t_1$, etc). The conjecture on the invariance of the indices is equivalent to the assertion that the extended t -sequence is uniquely determined by the function Φ .

The uniqueness of the superoptimal approximation for a function $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$ was derived in [P] from the following important inequality between the Hankel singular values and the superoptimal singular values:

$$(1.2) \quad \tilde{t}_j \leq s_j(H_\Phi), \quad j \geq 0.$$

Indeed it follows easily from the factorization above that if $Q_1, Q_2 \in \Omega_m$, then $\|Q_1 - Q_2\| \leq 2t_{m-1} \leq 2s_{m-1}(H_\Phi) \rightarrow 0$ since H_Φ is compact, which implies the uniqueness. The proof of inequality (1.2) is rather complicated and uses an approximation process to reduce the situation to the case of finite matrices for which (1.2) was proved earlier in [PY2]. In Section 4 we give a more direct proof of (1.2).

Note that in [P] it is deduced from (1.2) that if Q is the unique superoptimal approximation to Φ , then $\Phi - Q$ admits the following factorization

$$\Phi - Q = W_0^* \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & W_1^* \end{pmatrix} \begin{pmatrix} I_2 & \mathbb{O} \\ \mathbb{O} & W_2^* \end{pmatrix} \cdots D \cdots \begin{pmatrix} I_2 & \mathbb{O} \\ \mathbb{O} & V_2^* \end{pmatrix} \begin{pmatrix} 1 & \mathbb{O} \\ \mathbb{O} & V_1^* \end{pmatrix} V_0^*,$$

where

$$D = \begin{pmatrix} t_0 u_0 & 0 & 0 & \cdots \\ 0 & t_1 u_1 & 0 & \cdots \\ 0 & 0 & t_2 u_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and both infinite products converge in the strong operator topology.

2. The main construction. In this section we consider thematic completions of inner co-outer columns of the form $\begin{pmatrix} \varphi & \overline{\Omega} \end{pmatrix}$, and we prove that if the Hankel operator $H_{\overline{\varphi}}$ is compact, then the function Ω is left invertible in the space of bounded analytic operator functions. In the case of finite matrices, this fact was proved in [PY1] where it was derived from the matrix version of the corona theorem. However, in the case of infinite matrices the analog of the corona theorem is false (see [T1]). Fortunately in our case, to prove that Ω is left invertible we can use a theorem of Sz.-Nagy and Foias [SNF], which states that if F is a function in $H^\infty(\mathcal{B})$ such that $\|T_{\overline{F}}f\| \geq \text{const}\|f\|$, $f \in H^2(\ell^2)$, then there exists a function G in $H^\infty(\mathcal{B})$ such that $G(\zeta)F(\zeta) = I$, $\zeta \in \mathbb{D}$.

Then, using this invertibility result, we construct a family of functions in $H^2(\ell^2)$ which has certain important properties. This family will be used in the next section to prove the main results.

Theorem 2.1. *Let φ be an inner co-outer column such that the Hankel operator $H_{\overline{\varphi}}$ is compact and let $\begin{pmatrix} \varphi & \overline{\Omega} \end{pmatrix}$ be a thematic matrix function. Then there exists a function Ξ in $H^\infty(\mathcal{B})$ such that $\Xi(\zeta)\Omega(\zeta) = I$, $\zeta \in \mathbb{D}$.*

Proof. Let $V = \begin{pmatrix} \varphi & \overline{\Omega} \end{pmatrix}$. It was proved in [P], Corollary 1.5, that the Toeplitz operator T_V is invertible. Let $f \in H^2(\ell^2)$. We have

$$T_V \begin{pmatrix} \Omega \\ f \end{pmatrix} = T_{\overline{\Omega}}f$$

and so

$$\|T_{\overline{\Omega}}f\|_2 \geq \text{const}\|f\|_2, \quad f \in H^2(\ell^2).$$

The result follows now from the theorem of Sz.-Nagy and Foias. $\square$

To prove the invariance of indices an inequality (1.2) we construct a certain family of vectors similarly to the case of finite matrices [PY2]. Let $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$ and $Q \in \Omega_{m-1}$. Consider the factorization (1.1) with indices $\{k_j\}_{0 \leq j \leq m-1}$.

Lemma 2.2. *There exist linearly independent functions*

$$(2.1) \quad f_1^{(0)}, \dots, f_{k_0}^{(0)}, f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_1^{(m-1)}, \dots, f_{k_{m-1}}^{(m-1)}$$

in $H^2(\ell^2)$ such that

$$(\Phi - Q)f_r^{(\ell)} \in H_-^2(\ell^2), \quad 0 \leq \ell \leq m-1, \quad 1 \leq r \leq k_\ell,$$

and only the first $\ell + 1$ components of $V_{m-1}^* V_{m-2}^* \cdots V_0^* f_r^{(\ell)}$ can be nonzero.

Proof. Let the thematic matrices V_0 and W_0^t in (1.1) have the form $V_0 = \begin{pmatrix} v_{(i)} & \bar{\alpha} \end{pmatrix}$ and $W_0^t = \begin{pmatrix} w_{(i)} & \bar{\beta} \end{pmatrix}$. By Theorem 1.1 there are bounded analytic functions A and B such that $A\alpha = I$ and $B\beta = I$. Now we can construct the family $\{f_r^{(\ell)}\}$ in exactly the same way as was done in [PY2] in the case of finite matrix functions.

Let us recall the construction for completeness. We proceed by induction on m . Let $x_1, \dots, x_{k_0}$ be a basis in $\text{Ker } T_{u_0}$. If $m = 1$, we just put $f_r^{(0)} = x_r v_{(i)}$ and we are done.

Now suppose that $m > 1$. We have

$$\Phi - Q = W_0^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi - Q_1 \end{pmatrix} V_0^*,$$

where Q_1 is in the set Ω_{m-1} for the function Ψ . By the induction hypothesis we can construct linearly independent functions

$$g_1^{(1)}, \dots, g_{k_1}^{(1)}, \dots, g_1^{(m-1)}, \dots, g_{k_{m-1}}^{(m-1)}$$

such that

$$(\Psi - Q_1)g_r^{(\ell)} \in H_-^2(\ell^2), \quad 1 \leq \ell \leq m-1, \quad 1 \leq r \leq k_\ell,$$

and only the first ℓ components of $V_{m-1}^* V_{m-2}^* \cdots V_1^*$ can be nonzero.

We can define the sequence (2.1) as follows. Put

$$f_r^{(0)} = x_r v_{(i)}, \quad 0 \leq r \leq k_0.$$

For $\ell \geq 1$ we put

$$f_r^{(\ell)} = A^t g_r^{(\ell)} + q_r^{(\ell)} v_{(i)}, \quad 1 \leq \ell \leq m-1, \quad 1 \leq r \leq k_\ell,$$

where $q_r^{(\ell)}$ is a scalar function in H^2 which satisfies the equation

$$t_0 T_{u_0} q_r^{(\ell)} = -\mathbb{P}_+(t_0 u_0 v_{(i)}^* A^t g_r^{(\ell)} - w_{(i)}^t B^*(\Psi - Q_1) g_r^{(\ell)}).$$

The equation is solvable since T_{u_0} has dense range in H^2 and T_{u_0} is Fredholm, and so $\text{Range } T_{u_0} = H^2$.

The proof that the family $\{f_r^{(\ell)}\}$ has the desired properties is exactly the same as in Lemma 1.2 of [PY2]. $\square$

3. The main results. In this section we shall prove that the sums of the indices which correspond to equal superoptimal singular values are uniquely determined by the function Φ itself and do not depend on the choice of maximizing vectors and thematic completions. The proof is similar to the proof of the corresponding finite dimensional fact given in [PY2] (which in turn was inspired by the approach of [T] to the uniqueness of superoptimal approximations). Then we shall prove inequality (1.2) in a more natural way that it was proved in [P].

Let m be a positive integer such that $t_{m-1} > t_m$. Clearly, it is sufficient to show that if $j \leq m-1$, then the sum of the indices that correspond to the superoptimal singular values equal to t_j is uniquely determined by Φ . Let $Q \in \Omega_{m+1} \subset \Omega_m$. Put $\mathcal{W} = (\Phi - Q)^*(\Phi - Q)$. It is obvious from (1.1) that

$$\|H_\Phi f\|^2 \leq (\mathcal{W}f, f), \quad f \in H^2(\ell^2).$$

Clearly $\mathcal{W}(\zeta)$ has eigenvalues t_j^2 , $j \leq m-1$, which do not depend on $\zeta \in \mathbb{T}$. Let

$$\lambda_0 > \lambda_1 > \cdots > \lambda_d$$

be all distinct eigenvalues of $\mathcal{W}(\zeta)$ which are greater than or equal to t_{m-1}^2 . We define the functions ρ_j , $0 \leq j \leq d$, on $\mathbb{R}$ as follows.

$$\rho_j(\tau) = \begin{cases} \tau, & \tau \geq \lambda_j \\ \lambda_j, & \tau < \lambda_j \end{cases}.$$

Put

$$\mathcal{W}_j(\zeta) = \rho_j(\mathcal{W}(\zeta)), \quad 0 \leq j \leq d, \quad \zeta \in \mathbb{T}.$$

Clearly $\mathcal{W}_j \geq \mathcal{W}_{j+1} \geq \mathcal{W}$, $0 \leq j \leq d-1$. We have

$$H_\Phi^* H_\Phi \leq T_{\mathcal{W}} \leq T_{\mathcal{W}_j}, \quad 0 \leq j \leq d.$$

As in [PY2] we define

$$\mathcal{L}_j = \{f \in H^2(\ell^2) : \|H_G f\|^2 = (\mathcal{W}_j f, f)\}.$$

It is easy to see that

$$\mathcal{L}_j = \text{Ker}(T_{\mathcal{W}_j} - H_{\Phi}^* H_{\Phi}),$$

and so $\mathcal{L}_j$ is a subspace of $H^2(\ell^2)$. Clearly,

$$\mathbb{O} \leq T_{\mathcal{W}_{j+1}} - H_{\Phi}^* H_{\Phi} \leq T_{\mathcal{W}_j} - H_{\Phi}^* H_{\Phi},$$

which implies that $\mathcal{L}_j \subset \mathcal{L}_{j+1}$, $0 \leq j \leq d-1$.

Put

$$\nu_r = \sum_{t_j^2 = \lambda_r} k_j, \quad 0 \leq r \leq d.$$

Theorem 3.1. *The numbers ν_r , $0 \leq r \leq d$, are uniquely determined by Φ .*

Clearly, Theorem 3.1 follows immediately from the next theorem.

Theorem 3.2.

$$\nu_0 = \dim \mathcal{L}_0, \quad \nu_j = \dim \mathcal{L}_j \ominus \mathcal{L}_{j-1} \quad 1 \leq j \leq d.$$

Remark 3.3. Note that the weights $\mathcal{W}_j$ are exactly the weights W_j that were used in [T2] to construct the superoptimal weights. The spaces $\mathcal{L}_j$ are exactly the spaces $\mathcal{E}_j$ from the construction in [T2]. Note also that if Q is a superoptimal approximation then the weight $\mathcal{W}$ defined above is just the superoptimal weight constructed in [T2].

It is easy to see that to prove Theorem 3.2 it is sufficient to prove the following lemma where the family $\{f_r^{(\ell)}\}$ is constructed in Section 2.

Lemma 3.4.

$$(3.1) \quad \mathcal{L}_j = \text{span} \{f_r^{(\ell)} : t_l^2 \geq \lambda_j, \quad 1 \leq r \leq k_l\}, \quad 0 \leq j \leq d.$$

Note that the fact that the right-hand side of (3.1) is contained in the left-hand side can be proved in the same way as in [PY2] in the case of finite matrix functions, while the proof of the opposite inclusion given in [PY2] is purely finite-dimensional.

Proof. Let us first prove that $f_r^{(\ell)} \in \mathcal{L}_j$ whenever $t_l^2 \geq \lambda_j$. Consider the matrix function

$$G_j = W_0^* W_1^* \cdots W_{m-1}^* D_j V_{m-1}^* V_{m-2}^* \cdots V_0^*,$$

where

$$D_j = \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_p u_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_j & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda_j I \end{pmatrix}$$

(here $t_p^2 = \lambda_j > t_{p+1}^2$).

It is easy to see that $\mathcal{W}_j = G_j^* G_j$. It follows easily from Lemma 2.2 that if $t_j^2 \geq \lambda_j$, then $H_\Phi f_r^{(\ell)} = \Phi f_r^{(\ell)} = G_j f_r^{(\ell)}$ and so

$$\|H_\Phi f_r^{(\ell)}\|^2 = \|G_j f_r^{(\ell)}\|^2 = (G_j^* G_j f_r^{(\ell)}, f_r^{(\ell)}) = (\mathcal{W}_j f_r^{(\ell)}, f_r^{(\ell)}),$$

which implies that $f_r^{(\ell)} \in \mathcal{L}_j$ which proves one of the inclusions in (3.1).

Let us establish the opposite inclusion. Consider the matrix

$$G_j^\# = W_0^* W_1^* \cdots W_{m-1}^* D_j^\# V_{m-1}^* V_{m-2}^* \cdots V_0^*,$$

where

$$D_j^\# = \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_p u_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \epsilon & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \epsilon & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \epsilon I \end{pmatrix}$$

and $0 < \epsilon < \lambda_j$ (recall that $t_p^2 = \lambda_j > t_{p+1}^2$).

Let us show that both sides of (3.1) coincide with $\text{Ker } T_{G_j^\#}$. Suppose that $f \in \mathcal{L}_j$. We have $\|H_\Phi f\|^2 \leq \|(\Phi - Q)f\|^2 \leq (\mathcal{W}_j f, f)$ and by the definition of $\mathcal{L}_j$ all inequalities are in fact equalities. This means that $(\Phi - Q)f \in H_-^2(\ell^2)$ and only the first $p+1$ coordinates of $V_{m-1}^* \cdots V_0^* f$ can be nonzero. Therefore $(\Phi - Q)f = G_j^\# f$ and so $f \in \text{Ker } T_{G_j^\#}$. To complete the proof it remains to show that

$$\dim \text{Ker } T_{G_j^\#} = \dim \text{span} \{f_r^{(\ell)} : t_l^2 \geq \lambda_j, 1 \leq r \leq k_l\}.$$

Clearly, the latter is equal to $k_0 + k_1 + \cdots + k_p$.

Since the Hankel operators H_{V_s} , $H_{V_s^*}$, H_{W_s} , and $H_{W_s^*}$ are compact (see [P]), the Toeplitz operator $T_{D_j^\#}$ is Fredholm (this is obvious), the Toeplitz operators $T_{V_s^*}$ and $T_{W_s^*}$ are invertible (see [P]), it follows that $T_{G_j^\#}$ is Fredholm and $\text{ind } T_{G_j^\#} = \text{ind } T_{D_j^\#}$. Obviously, $\text{ind } T_{D_j^\#} = \text{ind } T_{u_0} + \text{ind } T_{u_1} + \cdots + \text{ind } T_{u_p} = k_0 + k_1 + \cdots + k_p$. Consequently, the proof will be completed if we prove that the Toeplitz operator $T_{G_j^\#}$ has dense range in $H^2(\ell^2)$. This is a consequence of the following fact.

Theorem 3.5. *Let*

$$\Phi = W^* \begin{pmatrix} u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where u is a scalar function such that T_u has dense range in H^2 , Φ_1 is a bounded infinite matrix function such that T_Ψ has dense range in $H^2(\ell^2)$, and V and W^t are thematic matrix functions. Then T_Φ has dense range in $H^2(\ell^2)$.

Indeed, it follows from Theorem 3.4 by induction that $T_{G_j^\#}$ has dense range in $H^2(\ell^2)$. $\square$

3.1. Proof of Theorem 3.4. We have to prove that $\text{Ker } T_\Phi^* = \mathbb{O}$. Let $V = \begin{pmatrix} v & \bar{\alpha} \end{pmatrix}$, $W^t = \begin{pmatrix} w & \bar{\beta} \end{pmatrix}$. Suppose that $f \in \text{Ker } T_\Phi^*$. We have

$$v^* \Phi^* f = v^* V \begin{pmatrix} \bar{u} & \mathbb{O} \\ \mathbb{O} & \Psi^* \end{pmatrix} W f = \begin{pmatrix} \bar{u} & \mathbb{O} \end{pmatrix} W f = \bar{u} w^t f \in H_-^2.$$

Hence $w^t f \in \text{Ker } T_{\bar{u}}$ and by the assumption, $w^t f = \mathbb{O}$. In other words, f is pointwise orthogonal to $\bar{w}$. Therefore $f(\zeta)$ belongs to the closed linear span of the columns of β , and so $f = \beta h$ for some ℓ^2 -valued function h . Since $h = \beta^* f$, it follows that $h \in L^2(\ell^2)$.

Let us show that $h \in H^2(\ell^2)$. Indeed, let x be an arbitrary vector in ℓ^2 . Since β^t is outer, one can find a sequence $\{u_j\}_{j \geq 0}$ in $H^2(\ell^2)$ such that $\beta^t u_j \rightarrow x$. Then $x^t h = \lim_{j \rightarrow \infty} u_j^t(\beta h) \in H^2$, which proves that $h \in H^2(\ell^2)$.

We have

$$\Phi^* f = V \begin{pmatrix} \bar{u} & \mathbb{O} \\ \mathbb{O} & \Psi^* \end{pmatrix} W \beta h = \bar{\alpha} \Psi^* h,$$

hence

$$\bar{\alpha} \Psi^* h \in H_-^2(\ell^2).$$

Since α is co-outer, the same reasoning as above shows that $\Psi^*h \in H_-^2(\ell^2)$, i.e., $h \in \text{Ker } T_\Psi$. By the assumption $h = \mathbb{O}$ and so $f = \beta h = \mathbb{O}$. $\square$

Note that the proof of Theorem 3.4 is similar to the proof of Theorem 3.1 in [PY1].

We complete the paper by giving a proof of inequality (1.2) which is more natural than the proof given in [P]. Recall that $\{\tilde{t}_j\}_{j \geq 0}$ is the extended t -sequence associated with $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$.

Theorem 3.6. *Let $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$. Then*

$$\tilde{t}_j \leq s_j(H_\Phi), \quad j \geq 0.$$

Proof. Clearly, it is sufficient to show that for each j there exists a subspace $\mathcal{L}$ of $H^2(\ell^2)$ such that $\dim \mathcal{L} \geq j + 1$ and $\|H_\Phi g\| \geq \tilde{t}_j \|g\|$ for any $g \in \mathcal{L}$. Let $\tilde{t}_j = t_d$. Let $m \geq d + 1$. Define $\mathcal{L}$ by

$$\mathcal{L} = \text{span} \{f_r^{(\ell)} : 0 \leq \ell \leq d, 1 \leq r \leq k_l\},$$

where $\{f_r^{(\ell)}\}$ is the family constructed in Lemma 2.2. Then $\dim \mathcal{L} = k_0 + \cdots + k_d \geq j + 1$. Clearly, it follows from Lemma 2.2 that $\|H_\Phi g\| \geq t_d \|g\|$. $\square$

4. Added in proof. We have recently improved Theorem 3.6. Namely, we have proved that if $\Phi \in H^\infty(\mathcal{B}) + C(\mathcal{K})$ and

$$W(\Phi - Q_0)V = \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_1 \end{pmatrix},$$

where $Q_0 \in H^\infty(\mathcal{B})$ is a best approximant to Φ , and V and W^t are thematic matrix functions, then

$$s_j(H_{\Phi_1}) \leq s_{j+k_0}(H_\Phi).$$

The proof will appear soon for finite matrix functions. For infinite matrix functions the same proof works.

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