Minimal Discs with Free Boundaries in a Lagrangian Submanifold of $\mathbf{C}^n$

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ABSTRACT. The question when an energy functional stationary disc p with free boundary in a Lagrangian submanifold of $\mathbf{C}^n$ is holomorphic or antiholomorphic is considered. A partial answer is given in terms of its partial indices. It is proved that if all its partial indices are greater or equal to -1 , then the stationary disc p is holomorphic, and if all its partial indices are less or equal to 1 , the disc p is antiholomorphic.

1. Introduction. Let $D = \{z \in \mathbf{C}; |z| < 1\}$ and let ∂D be the unit circle in $\mathbf{C}$. For every nonnegative integer m and for every real number α , $0 < \alpha < 1$, we denote by $C^{m,\alpha}(D)$ the Banach algebra of m -times differentiable complex-valued functions on the closed unit disc $\bar{D}$, whose m derivatives are Hölder continuous with finite Lipschitz norm of the exponent α . The subalgebra of functions on $\bar{D}$ which are holomorphic on D we denote by $A^{m,\alpha}(D)$.

Let $M \subseteq \mathbf{C}^n$ be a C^2 manifold and let $p : \bar{D} \rightarrow \mathbf{C}^n$ be a disc of class $C^{1,\alpha}$ with boundary in M , i.e., $p : \partial D \rightarrow M$. By the energy of the disc p we mean the Dirichlet integral of p

$$E(p) := \frac{1}{2} \int_D |\nabla p|^2 dx dy .$$

The maps which minimize the energy functional are of a special interest in the Riemannian geometry, namely, any map with boundary in a submanifold M which minimizes energy E in a conjugacy invariant subset $\emptyset \neq G \subseteq \pi_1(M)$, also minimizes the area functional

$$A(p) := \int_D \left| \frac{\partial p}{\partial x} \wedge \frac{\partial p}{\partial y} \right| dx dy$$

in G —see [Nit] and [Ye2] for more details.

While the existence of the energy functional minimizers with free boundaries in a submanifold M is known in quite general settings, [Ye2], the existence of nontrivial holomorphic discs attached to a certain submanifold of $\mathbf{C}^n$ is a much harder problem. It is a well-known and not at all elementary theorem by Gromov, [Gro], that such a disc always exists when the submanifold $M \subseteq \mathbf{C}^n$ is a closed Lagrangian submanifold of $\mathbf{C}^n$. It is also a well known fact that in this case, i.e., in the case the submanifold $M \subseteq \mathbf{C}^n$ is Lagrangian, every holomorphic or antiholomorphic disc attached to M is also an energy functional minimizer with free boundary in M . It was Y.-G. Oh's idea, [Oh3], to investigate the reverse implication.

In the present paper we find sufficient conditions on an energy functional stationary disc p with free boundary in a Lagrangian submanifold M of $\mathbf{C}^n$ to be holomorphic or antiholomorphic. The main result of the paper is the following theorem.

Theorem 1.1.

- (1) *If all partial indices of the pull-back bundle $p^*(TM)$ are greater or equal to -1 , then p is a holomorphic disc.*
- (2) *If all partial indices of the pull-back bundle $p^*(TM)$ are less or equal to 1 , then p is an antiholomorphic disc.*

2. Partial indices. Let $Gl(n, \mathbf{C})$ be the Lie group of invertible $n \times n$ complex matrices. It is known, see [Vek1], that for every closed path $B : \partial D \rightarrow Gl(n, \mathbf{C})$ of class $C^{0,\alpha}$ one can find matrix functions

$$F^+ : \bar{D} \longrightarrow Gl(n, \mathbf{C}) , \quad F^- : \bar{C} \setminus D \longrightarrow Gl(n, \mathbf{C})$$

of class $C^{0,\alpha}$, holomorphic in the interiors of their domains, and n integers $k_1 \geq k_2 \geq \dots \geq k_n$ such that

$$B = F^+(\xi)\Lambda(\xi)F^-(\xi) \quad (\xi \in \partial D),$$

where

$$\Lambda(\xi) := \begin{pmatrix} \xi^{k_1} & 0 & \dots & \dots & 0 \\ 0 & \xi^{k_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \xi^{k_n} \end{pmatrix}.$$

The matrix Λ is called the *characteristic matrix* of the path B . One can prove that under the condition $k_1 \geq \dots \geq k_n$, the characteristic matrix Λ does not depend on the factorization of the matrix function B of the form above—see

[Vek1], [Glo1], [Cla-Goh] for more details. The integers $k_1, \dots, k_n$ are called the *partial indices* of the path B and their sum

$$k := k_1 + \dots + k_n$$

is called the *total index* of the matrix function B .

Let L be a maximal real subspace of $\mathbf{C}^n$, i.e., its real dimension is n and $L \oplus iL = \mathbf{C}^n$. To every maximal real subspace L one can associate an $\mathbf{R}$ -linear map R_L on $\mathbf{C}^n$, called the reflection about L , given by

$$z = x + i\tilde{x} \longmapsto x - i\tilde{x} \quad (x, \tilde{x} \in L),$$

where $z = x + i\tilde{x}$ is the unique decomposition of z into the sum of vectors from L and iL . The mapping

$$R_L : \mathbf{C}^n \longrightarrow \mathbf{C}^n$$

is an $\mathbf{R}$ -linear automorphism of $\mathbf{C}^n$ that is also $\mathbf{C}$ -antilinear, i.e., $R_L(iv) = -iR_L(v)$ for every $v \in \mathbf{C}^n$. The reflection about the maximal real subspace $\mathbf{R}^n \subseteq \mathbf{C}^n$ will be denoted by R_o . Note that in the standard notation, R_o is just the ordinary conjugation on $\mathbf{C}^n$ and that for every matrix A the following identity holds:

$$\bar{A} = R_o A R_o .$$

Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product on $\mathbf{C}^n$. By iL we denote the maximal real subspace of vectors of the form iv , $v \in L$, and by $L^\perp$ the maximal real subspace of vectors perpendicular to L , i.e., a vector $u \in \mathbf{C}^n$ is perpendicular to L if and only if $\operatorname{Re}\langle u, v \rangle = 0$ for every $v \in L$. The proof of the following simple lemma is omitted; see [Čer1], [Čer2], also [Glo1].

Lemma 2.1. *Let L be a maximal real subspace of $\mathbf{C}^n$ and let $x_1, \dots, x_n$ be any set of vectors spanning L . Let $A := [x_1, \dots, x_n]$ be the matrix whose columns are the given vectors x_j , $j = 1, \dots, n$, and let $B_L := \overline{AA^{-1}}$. Then*

$$B_L = R_L R_o .$$

Remark. In the lemma above $n \times n$ matrices A and B_L are identified with $\mathbf{C}$ -linear automorphisms of $\mathbf{C}^n$ in the standard basis.

Corollary 2.1. (a) *The matrix B_L does not depend on a basis of L ,*
 (b) $\overline{B_L} = B_L^{-1}$, (c) $B_{iL} = -B_L$, (d) $B_{L^\perp} = -B_L^t$.

The following definition is taken from [Glo1]; see also [For1].

Definition 2.1. Let $L = \{L_\xi; \xi \in \partial D\}$ be a real rank n subbundle of the product bundle $\partial D \times \mathbf{C}^n$ of class $C^{0,\alpha}$. If for each $\xi \in \partial D$ the fiber L_ξ is a maximal real subspace of $\mathbf{C}^n$, the bundle L is called a maximal real bundle of rank n over ∂D and the partial indices of the $Gl(n, \mathbf{C})$ closed path

$$B_L : \xi \longmapsto R_{L_\xi} R_o$$

of class $C^{0,\alpha}$, are called the partial indices of the bundle L , and their sum is called the total index of L .

Example. Here is a very important example of a maximal real bundle over ∂D . Let M be a C^1 maximal real submanifold of $\mathbf{C}^n$ and let $p : \partial D \rightarrow M$ be a $C^{0,\alpha}$ closed curve in M . Then the pull-back bundle $p^*(TM)$, where TM is the tangent bundle of the submanifold M , is a maximal real bundle over ∂D of rank n .

Remark. The total index of a closed path p on a maximal real submanifold $M \subseteq \mathbf{C}^n$ is also called the Maslov index of p .

The special form of the $Gl(n, \mathbf{C})$ closed path B_L implies the following result; see [Glo1], also [Oh1], [Čer1].

Lemma 2.2. *The $C^{0,\alpha}$ closed path in $Gl(n, \mathbf{C})$*

$$B_L : \xi \longmapsto R_{L_\xi} R_o \quad (\xi \in \partial D)$$

can be decomposed in the form

$$B_L(\xi) = \Theta(\xi) \Lambda(\xi) \overline{\Theta(\xi)^{-1}} \quad (\xi \in \partial D),$$

where the map $\Theta : \bar{D} \longrightarrow Gl(n, \mathbf{C})$ is of class $C^{0,\alpha}$ and holomorphic on D , i.e., the $n \times n$ matrix Θ is in $A^{0,\alpha}(D)^{n \times n}$.

Let $r(\xi)$, $\xi \in \partial D \setminus \{1\}$, denote the principal branch of the square root, i.e., the complex plane is cut along the positive real line and $r(-1) = i$. The characteristic matrix Λ can be decomposed further as

$$\Lambda = \Lambda_o^2 = \Lambda_o \overline{\Lambda_o^{-1}},$$

where

$$\Lambda_o(\xi) := \begin{pmatrix} \xi^{k_1/2} & 0 & \dots & \dots & 0 \\ 0 & \xi^{k_2/2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \xi^{k_n/2} \end{pmatrix}.$$

Here $\xi^{\frac{k}{2}}$ stands for ξ^m if $k = 2m$ and for $\xi^m r(\xi)$ if $k = 2m + 1$. We will refer to Λ_o as the *square root* of the characteristic matrix Λ and we say that the matrix function

$$\xi \longmapsto \Theta(\xi)\Lambda_o(\xi) \quad (\xi \in \partial D)$$

represents a *normal form* of the bundle L .

3. Minimal discs with free boundaries. Let (r, ϑ) denote the polar coordinates on the unit disc D . It is well known, see e.g., [Jäg], [Lew], [Ye2], that any $C^{1,\alpha}$ solution p of the Euler-Lagrange equations for the energy functional E with free boundary in a C^2 manifold M must satisfy the following conditions:

- (a) All component functions of the mapping p are harmonic in D , and
- (b) $(\partial p / \partial r)(\xi) \perp T_{p(\xi)}M$ for every $\xi \in \partial D$.

Since p maps ∂D into M , we also have

$$\frac{\partial p}{\partial \vartheta}(\xi) \in T_{p(\xi)}M$$

on ∂D . Combining both conditions, we conclude

$$\frac{\partial p}{\partial r}(\xi) \perp \frac{\partial p}{\partial \vartheta}(\xi) \quad (\xi \in \partial D).$$

Since p is harmonic on the unit disk D , it can be written in the form

$$p = f + \bar{g}$$

for some unique holomorphic vector functions f and g , ($g(0) = 0$), from $(A^{1,\alpha}(D))^n$. The condition on p to be minimal can be written as

$$\operatorname{Re} \left\langle \frac{\partial p}{\partial r}(\xi), \frac{\partial p}{\partial \vartheta}(\xi) \right\rangle = 0$$

and so

$$\operatorname{Re} \left\langle \xi f'(\xi) + \overline{\xi g'(\xi)}, i\xi f'(\xi) - i\overline{\xi g'(\xi)} \right\rangle = 0 \quad (\xi \in \partial D).$$

A short calculation shows

$$\operatorname{Im} \xi^2 \langle f'(\xi), \overline{g'(\xi)} \rangle = 0$$

for every $\xi \in \partial D$. But the function

$$F(\xi) := \xi^2 \langle f'(\xi), \overline{g'(\xi)} \rangle$$

extends holomorphically to D and so is a constant function. Since $F(0) = 0$, one concludes that

$$\langle f'(z), \overline{g'(z)} \rangle = \sum_{j=1}^n f'_j(z) \overline{g'_j(z)} = 0$$

for every $z \in \bar{D}$.

Proposition 3.1. *Let p be a complex function $p: \bar{D} \rightarrow \mathbf{C}$ of class $C^{1,\alpha}$. If either of the discs*

$$F, G: \bar{D} \longrightarrow \mathbf{C}^2,$$

where

$$\begin{aligned} F(z) &= (z, p(z)), \\ G(z) &= (\bar{z}, p(z)), \end{aligned}$$

is minimal for some C^2 submanifold M of $\mathbf{C}^2$, then the function p is either holomorphic or antiholomorphic.

Proof. Let us assume that the disc $F(z) = (z, p(z))$ is minimal for some submanifold M . Then the corresponding holomorphic discs f and g are of the form

$$f(z) = (z, a(z)) \quad \text{and} \quad g(z) = (0, b(z))$$

for some holomorphic functions a and b from $A^{1,\alpha}(D)$. The argument above then implies

$$a'(z)b'(z) = 0$$

for every $z \in \bar{D}$. Thus at least one of the functions a and b has to be a constant function. In the case the disc $G(z) = (\bar{z}, p(z))$ is minimal, the proof is similar. $\square$

We would like to use the special normal form of the pull-back bundle to investigate the energy functional minimizers with boundaries in a C^2 Lagrangian submanifold $M \subseteq \mathbf{C}^n$. We recall that a submanifold $M \subseteq \mathbf{C}^n$ is called Lagrangian if for each $z \in M$ the tangent space $T_z M$ to the submanifold M at the point z is a Lagrangian subspace of $\mathbf{C}^n$, i.e.,

$$(T_z M)^\perp = iT_z M.$$

Proposition 3.2. *Let L be a $C^{0,\alpha}$ maximal real vector bundle of rank n over ∂D such that every fiber L_ξ , $\xi \in \partial D$, is a Lagrangian subspace of $\mathbf{C}^n$, and let B_L be the $C^{0,\alpha}$ closed path in $Gl(n, \mathbf{C})$ which represents L . Then:*

- (a) $B_L^* = B_L^{-1}$, i.e., the matrix $B_L(\xi)$ is unitary for every $\xi \in \partial D$.
- (b) $B_L^t = B_L$, and conversely, if for a maximal real bundle L over ∂D one has $B_L^t = B_L$, then the bundle L is Lagrangian, i.e., each fiber is a Lagrangian subspace of $\mathbf{C}^n$.
- (c) Let n_o denote the number of partial indices of the bundle L that are greater or equal to -1 . Then there are $n \times n_o$ and $n \times (n - n_o)$ dimensional matrix functions X and Y , respectively, with entries from the space $A^{0,\alpha}(D)$ such that the columns of the matrix function

$$[X, \bar{Y}] \Lambda_o$$

span the fiber L_ξ for each $\xi \in \partial D$. Moreover, $X^t Y = 0$ on $\bar{D}$. Here Λ_o is the square root of the corresponding characteristic matrix Λ .

Remark. In the case n_o is n (resp. 0) the matrix Y (resp. X) does not exist.

Proof. Since the fibers of the bundle L are Lagrangian subspaces of $\mathbf{C}^n$, the bundles iL and $L^\perp$ are the same. Corollary 2.1 implies $B_L^t = B_L$ and $B_L^* = B_L^{-1}$. This proves (a) and the first part of (b). To prove the reverse implication of part (b), one should observe that if A is any $n \times n$ matrix whose columns span L_ξ for some $\xi \in \partial D$, then

$$A \overline{A^{-1}} = B_L(\xi) = B_L^t(\xi) = \overline{A^{-1}}^t A^t = (A^*)^{-1} A^t.$$

So

$$A^* A = A^t \bar{A} = \overline{A^* A}$$

and the matrix $A^* A$ is a real $n \times n$ matrix. Hence

$$\operatorname{Re}((iA)^* A) = 0$$

and hence the columns of the matrix iA are perpendicular to the columns of the matrix A . Part (b) is proved.

To prove part (c) we use the factorization of the matrix function B_L ,

$$B_L = \Phi \Lambda \overline{\Phi^{-1}},$$

for some map $\Phi : \bar{D} \rightarrow Gl(n, \mathbf{C})$ from $A^{0,\alpha}(D)^{n \times n}$. We define $\Psi := (\Phi^{-1})^*$ and by part (b) we also have

$$B_L = \Psi \Lambda \overline{\Psi^{-1}}.$$

Let

$$\Phi = [\Phi_1, \Phi_2] \quad \text{and} \quad \Psi = [\Psi_1, \Psi_2]$$

be the block notation of the matrix functions Φ and Ψ such that the matrices Φ_1 and Ψ_1 have dimensions $n \times n_o$. Since

$$\Phi^t \bar{\Psi} = \Phi^t (\Phi^{-1})^t = I_n ,$$

we get

$$\begin{bmatrix} \Phi_1^t \\ \Phi_2^t \end{bmatrix} [\bar{\Psi}_1, \bar{\Psi}_2] = I_n$$

and so

$$\Phi_1^t \bar{\Psi}_2 = 0 .$$

This means that the columns of the matrix Φ_1 are orthogonal to the columns of the matrix Ψ_2 , and thus the matrix

$$[\Phi_1, \Psi_2]$$

is invertible. We set $X := \Phi_1$ and $Y := \bar{\Psi}_2$ and the proof is finished. $\square$

The proof of part (c) of the proposition above also implies the following result.

Corollary 3.2. *Let L be a Lagrangian vector bundle over ∂D of class $C^{0,\alpha}$, let B_L be the corresponding $C^{0,\alpha}$ closed path in $Gl(n, \mathbf{C})$ and let Λ be its characteristic matrix. Then the characteristic matrix of the $Gl(n, \mathbf{C})$ closed path*

$$\xi \longmapsto B_L(\bar{\xi})$$

is $\bar{\Lambda}$.

Remark. Statement (c) of the proposition above does not hold for an arbitrary maximal real vector bundle over the ∂D . As a counterexample, one can take the bundle L whose matrix B_L is given by

$$\begin{pmatrix} -\xi^2 & 1 \\ 0 & \bar{\xi}^2 \end{pmatrix} .$$

It is easy to check that the partial indices of B_L are 2 and -2 and that there are no nontrivial holomorphic functions $a, b \in A^{0,\alpha}(D)$ that satisfy the equation

$$\begin{pmatrix} -\xi^2 & 1 \\ 0 & \bar{\xi}^2 \end{pmatrix} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} = \bar{\xi}^2 \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} \quad (\xi \in \partial D).$$

Henceforth M will denote a C^2 Lagrangian submanifold in $\mathbf{C}^n$ and $p : \bar{D} \rightarrow \mathbf{C}^n$ will be an energy functional stationary disc of class $C^{1,\alpha}$ with boundary in M , i.e., the first variation of the energy functional E at p is 0. We recall that the pair (r, ϑ) represents the polar coordinates on D . Then one has

$$\xi \frac{\partial p}{\partial z}(\xi) = \frac{1}{2} \left(\frac{\partial p}{\partial r}(\xi) - i \frac{\partial p}{\partial \vartheta}(\xi) \right)$$

and

$$\bar{\xi} \frac{\bar{\partial} p}{\partial \bar{z}}(\xi) = \frac{1}{2} \left(\frac{\partial p}{\partial r}(\xi) + i \frac{\partial p}{\partial \vartheta}(\xi) \right)$$

for every $\xi \in \partial D$. The disc p is attached to the manifold M ; hence, the vector $(\partial p / \partial \vartheta)(\xi)$ is tangent to M at $p(\xi)$ for each $\xi \in \partial D$. Since p is also minimal, the vector $(\partial p / \partial r)(\xi)$ is perpendicular to M at $p(\xi)$ for every $\xi \in \partial D$.

Let A_o denote the matrix constructed in part (c) of Proposition 3.2. Since M is a Lagrangian submanifold of $\mathbf{C}^n$, we get

$$\operatorname{Re} \left(i A_o^* \frac{\partial p}{\partial \vartheta}(\xi) \right) = 0 \quad (\xi \in \partial D),$$

and since the first variation of the energy functional E at p vanishes, we have

$$\operatorname{Re} \left(A_o^* \frac{\partial p}{\partial r}(\xi) \right) = 0 \quad (\xi \in \partial D).$$

Thus on ∂D the following identities hold:

$$\operatorname{Re} \left(\bar{\xi} A_o^*(\xi) \frac{\partial p}{\partial \bar{z}}(\xi) \right) = 0 \quad \text{and} \quad \operatorname{Re} \left(\xi A_o^*(\xi) \frac{\partial p}{\partial z}(\xi) \right) = 0.$$

We know that the disc p can be decomposed into a sum $p = f + \bar{g}$ for some unique vector functions f and g , ($g(0) = 0$), from $A^{1,\alpha}(D)^n$. Therefore, we have

$$(1) \quad \operatorname{Re}(\xi A_o^t(\xi) g'(\xi)) = 0 \quad \text{and} \quad \operatorname{Re}(\xi A_o^*(\xi) f'(\xi)) = 0$$

for every $\xi \in \partial D \setminus \{1\}$. Since also $A_o = [X, \bar{Y}] \Lambda_o$, where the matrices X and Y have holomorphic extensions into D and are “orthogonal” to each other, i.e., $Y^t X = 0$ on $\bar{D}$, we first conclude that

$$X^t g' = 0 \quad \text{and} \quad Y^t f' = 0 ,$$

and finally

$$f' = Xh \quad \text{and} \quad g' = Yk$$

for some vector functions $h \in (A^{0,\alpha}(D))^{n_o}$ and $k \in (A^{0,\alpha}(D))^{n-n_o}$. For these results we used only one part of each equation in (1). The rest, together with the equalities above, implies

$$\operatorname{Re}(\xi \overline{\Lambda_o^+} X^* X h) = 0 \quad \text{and} \quad \operatorname{Re}(\xi \overline{\Lambda_o^-} Y^* Y k) = 0 ,$$

where Λ_o^+ and Λ_o^- are the $n_o \times n_o$ and the $(n - n_o) \times (n - n_o)$ dimensional matrices, respectively, such that

$$\Lambda_o = \begin{pmatrix} \Lambda_o^+ & 0 \\ 0 & \Lambda_o^- \end{pmatrix} .$$

The manifold M is Lagrangian, thus

$$\operatorname{Re}(-i A_o^* A_o) = 0$$

on ∂D , and the matrices

$$\overline{\Lambda_o^+} X^* X \Lambda_o^+ \quad \text{and} \quad \overline{\Lambda_o^-} Y^* Y \Lambda_o^-$$

are real and invertible. Hence

$$\operatorname{Re}(\xi \overline{\Lambda_o^+(\xi)} h(\xi)) = 0 \quad \text{and} \quad \operatorname{Re}(\xi \overline{\Lambda_o^-(\xi)} k(\xi)) = 0$$

for every $\xi \in \partial D$. This proves the following theorem.

Theorem 3.1. *Let M be a C^2 Lagrangian submanifold of $\mathbf{C}^n$ and let $p = f + \bar{g}$, $f, g \in (A^{1,\alpha}(D))^n$, be an energy functional stationary disc of class $C^{1,\alpha}$ with boundary in M . Let n_o be the number of partial indices of the path $p(\partial D) \subseteq M$ which are greater or equal to -1 , and let X and Y be the $n \times n_o$ and $n \times (n - n_o)$ dimensional holomorphic matrices given by part (c) of Proposition 3.2. Then there exist vector functions $h \in (A^{0,\alpha}(D))^{n_o}$ and $k \in (A^{0,\alpha}(D))^{n-n_o}$ such that*

$$\operatorname{Re}(\xi \overline{\Lambda_o^+(\xi)} h(\xi)) = 0 \quad \text{and} \quad \operatorname{Re}(\xi \overline{\Lambda_o^-(\xi)} k(\xi)) = 0 \quad (\xi \in \partial D)$$

and

$$f' = Xh \quad \text{and} \quad g' = Yk .$$

Remark. In the case n_o equals n (resp. 0), one part of the conclusion above is empty, namely, the matrix Y (resp. X) does not exist.

As a simple consequence, one has the following result:

Corollary 3.3. (Hypothesis as above.)

- (a) *If all partial indices of the pull-back bundle $p^*(TM)$ are greater or equal to -1 , then p is a holomorphic disc.*
- (b) *If all partial indices of the pull-back bundle $p^*(TM)$ are less or equal to 1, then p is an antiholomorphic disc.*

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