Compactness of Spaces of Properly Embedded Minimal Surfaces with Finite Total Curvature

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Introduction. The study of the moduli space of embedded complete minimal surfaces in $\mathbb{R}^3$ (up to homotheties) with finite total curvature and fixed topology is one of the fundamental open problems in the classical theory of minimal surfaces. If the surface has either genus zero or at most two ends, then it is known, see the papers by Lopez, Perez, Ros and Schoen [LR], [PR], [S], that it must be a plane or a Catenoid. If the surface has genus one and three ends, then Costa [C] gives a complete description of the moduli space (it is parametrized by a half-line). The remaining of the known results concern the existence, see for instance Hoffman and Meeks [HM2], and the nonexistence, see Ros [R], of some kinds of symmetric surfaces. A conjecture in [HM2] asserts that if the number of ends is large with respect to the genus of the surface, then the moduli space is empty.

On the other hand, Choi and Schoen, [CS], have proved that the space of embedded compact minimal surfaces of fixed topology in the compact 3-manifold with a Riemannian metric of positive Ricci curvature is compact. This result does not extend to our situation as the moduli space of surfaces of genus one and three ends is noncompact. In this paper we will show that the compactness holds for certain topologies. More precisely, we will prove that: *If the genus is one and the number of ends is five or more, then the moduli space above is compact (in the strong sense).*

In fact, our arguments work in any genus g and number of ends bigger than $2g + 2$, if we previously know a nonexistence result for smaller genus. See Theorem 4.

The first ingredient in the proof of the result above is a structure theorem, Theorem 2, which describes the weak compactness of the moduli space for any given topology. We will prove that, given a sequence of embedded complete minimal in $\mathbb{R}^3$ with finite total curvature, fixed topology and horizontal ends, $\{M_n\}$, then, taking a subsequence if necessary and for n large enough, there are disjoint Euclidean balls, $B_{1,n}, \dots, B_{k,n}$ (k independent of n), such that after

suitable homotheties $\{M_n \cap B_{i,n}\}_n$ converges smoothly to $M_{i,\infty} \cap B(\rho)$ (with multiplicity one), where $M_{i,\infty}$ is a nonflat embedded complete minimal surface with finite total curvature and horizontal ends, $i = 1, \dots, k$, and $B(\rho)$ is a large Euclidean ball. Moreover, $C(M_n) = C(M_{1,\infty}) + \dots + C(M_{k,\infty})$ and M_n intersects $\mathbb{R}^3 - [B_{1,n} \cup \dots \cup B_{k,n}]$ in a finite number of graphs over unbounded domains in an horizontal plane. Each one of these graphs contains exactly one end of M_n , see Figure 1. To obtain this theorem, we need to use an improvement of the ideas of Choi and Schoen [CS] or White [W], see Theorem 1.

The second ingredient of the proof of our main result is a nonexistence theorem, given in Ros [R], for properly embedded minimal surfaces with one finite total curvature end, genus zero and two boundary components which are convex Jordan curves lying in parallel planes, see Theorem 3.

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1. Preliminaries. Given $p \in \mathbb{R}^3$ and $\rho \geq 0$, we will denote the Euclidean ball $\{q \in \mathbb{R}^3 / |q - p| \leq \rho\}$ by $B(p; \rho)$. We will also use the notation $B(\rho) = B(0; \rho)$ and $B(X; \rho) = \{q \in \mathbb{R}^3 / \text{dist}(q, X) \leq \rho\}$, where X is a subset of $\mathbb{R}^3$.

If $\psi : M \rightarrow \mathbb{R}^3$ is an immersed oriented minimal surface and C is a smooth one cycle on M , then the force of ψ along C is defined by

$$\text{Force}(C) = \int_C n(s) ds,$$

where n is the (oriented) unit conormal vector along C and s is the arclength parameter of C . This notion depends only on the homology class of C and, so it defines a linear map $\text{Force} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^3$. We denote by $A(M)$ the area of M and by $C(M)$ the total absolute curvature of M , i.e., if K is the Gauss curvature of M ,

$$C(M) = \int_M |K| dA.$$

Now we recall some global properties of embedded minimal surfaces. More details can be found in Osserman [O] and Hoffman and Meeks [HM2]. If $M \subset \mathbb{R}^3$ is a nonflat properly embedded minimal surface with finite total curvature (in this family properness is equivalent to completeness), then we know that there is a compact Riemann surface $\bar{M}$ and a finite number of points $p_1, \dots, p_r \in \bar{M}$, such that M is conformally equivalent to $\bar{M} - \{p_1, \dots, p_r\}$. We will identify the ends of M with the points p_i . The Gauss map of M extends to an antiholomorphic map $\nu : \bar{M} \rightarrow S^2(1)$. Moreover, after a rotation of M , we can assume that the limit normal directions $\nu(p_i)$, $i = 1, \dots, r$, are vertical (in this case we will say that the ends of M are horizontal). Around each p_i the surface is asymptotic to

a vertical Catenoid (if the Gauss map ν is unbranched at p_i) or to an horizontal plane (if p_i is a branch point of ν). We say that p_i is an end of Catenoid type or of planar type depending on the case. The force along a cycle around an end of M is a nonzero vertical vector in the Catenoid case and vanishes in the planar one. The total curvature of M (or equivalently the degree of the Gauss map ν) depends only on the topology of M : If g denotes the genus of M , then $C(M) = 4\pi(g + r - 1)$. Moreover, M satisfies a uniform area bound depending only on the numbers of ends of the surface: If $\rho > 0$ and $p \in \mathbb{R}^3$, then $A(M \cap B(p; \rho)) \leq \pi r \rho^2$.

Now we explain, adapted to our context, some of the results of Choi and Schoen [CS] and White in [W]. Let $\{M_n\}_n$ be a sequence of properly embedded minimal surfaces in $\mathbb{R}^3$ without boundary. Assume that $\{M_n\}$ admits uniform local bounds for the area and for the total curvature, i.e., for each ball $B \subset \mathbb{R}^3$, there exist positive constants c_1 and c_2 , independent of n , such that

$$A(M_n \cap B) \leq c_1, \quad C(M_n \cap B) \leq c_2.$$

In this situation, if the sequence $\{M_n\}$ has an accumulation point in $\mathbb{R}^3$, then there exists a properly embedded minimal surface $M \subset \mathbb{R}^3$, a discrete subset $X \subset M$ and a subsequence of $\{M_n\}$, which we will denote again by $\{M_n\}$, verifying the following properties:

- (1) $\{M_n\}$ converges to M (with finite multiplicity) in $\mathbb{R}^3 - X$. The convergence above is the C^k -convergence on compact subsets of $\mathbb{R}^3 - X$ for any $k \geq 0$ (briefly we will say that $\{M_n\}$ converges smoothly). It means that M is the set of accumulation points of $\{M_n\}$ and that, in some cylindrical neighbourhood B of each point in $M - X$ and for n large enough, $M_n \cap B$ consists of a finite number of graphs (this number gives the multiplicity of the limit) over the disk in the tangent plane of M at p , $M \cap B$ is a single graph over the disk above and each graph in $M_n \cap B$ converges smoothly to $M \cap B$. In general, this multiplicity is well defined only over the connected components of M . (However, in our family, the only nonconnected M is given by a union of parallel planes, see Hoffman and Meeks [HM1].)
- (2) For each $p \in X$ and $\rho > 0$, the limit of $C(M_n \cap B(p; \rho))$, when n goes to infinity, exists and is greater than or equal to 4π . We will say that X is the *bubble set* of the limit $M_n \rightarrow M$. Thus, if we know that for some $p \in M$ the absolute total curvature of M_n in a neighbourhood of p is strictly smaller than 4π , $n \in \mathbb{N}$, then we have that $\{M_n\}$ converges to M in this neighbourhood.

2. Weak compactness results.

Theorem 1. *Let $\{M_n\}_n$ be a sequence of properly embedded minimal surfaces in $\mathbb{R}^3$ which admits uniform local bounds for the area and the absolute total curvature. If $\{M_n\}$ has an accumulation point in $\mathbb{R}^3$ then, either*

- (i) *there exists a discrete union of parallel planes $M \subset \mathbb{R}^3$ and a discrete subset $X \subset M$ such that a subsequence of $\{M_n\}$ converges to M (with finite multiplicity) uniformly on compact subsets of $\mathbb{R}^3 - X$, or*
- (ii) *there exists a properly embedded nonflat minimal surface $M \subset \mathbb{R}^3$ and a subsequence of $\{M_n\}$ which converges to M (with multiplicity one) uniformly on compact subsets of $\mathbb{R}^3$.*

Proof. From the results stated in Section 1, taking a subsequence if necessary, we can suppose that $\{M_n\}$ converges to a properly embedded minimal surface $M \subset \mathbb{R}^3$ uniformly on compact subsets of $\mathbb{R}^3 - X$, where X is a discrete subset of M . Assume that M is nonflat and, so, connected.

Let Ω be a compact, connected and smooth subdomain of $M - X$. Let U be an embedded a -tubular neighbourhood of Ω , $a > 0$, and denote by $\pi : U \rightarrow \Omega$ and $d : U \rightarrow (-a, a)$ the orthogonal projection of U onto Ω and the oriented distance to Ω respectively. Thus the product map $\pi \times d : U \rightarrow \Omega \times (-a, a)$ is a global diffeomorphism. For n large enough, $\Omega_n = U \cap M_n$ is a compact subsurface of M_n and the projection $\pi : \Omega_n \rightarrow \Omega$ is a proper local diffeomorphism and, hence, a finite-sheeted unbranched covering with possibly disconnected total space. Moreover, Ω_n converges to Ω with multiplicity given by the number of sheets of this covering. Of course, the multiplicities of the limits $M_n \rightarrow M$ and $\Omega_n \rightarrow \Omega$ coincide. Denote by $\Omega_{n,1}, \dots, \Omega_{n,k}$ the components of Ω_n , where k is independent of n . As M_n is embedded, it follows that the continuous function d restricted to $\Omega_{n,i}$ separates the points in the fibers of the covering $\pi : \Omega_{n,i} \rightarrow \Omega$ and, so, it must have only one sheet for any $i = 1, \dots, k$.

Now we assume that $k \geq 2$ and we will prove that Ω is stable. If we denote by L the Jacobi operator of M , i.e., $L = \Delta + |\sigma|^2$ where Δ is the Laplacian of M and $|\sigma|$ denotes the length of the second fundamental form of the embedding $M \subset \mathbb{R}^3$, we can suppose, after a small perturbation of Ω if necessary, that the only solution of the equation $Lv = 0$ on Ω and $v = 0$ on $\partial\Omega$ is the function $v \equiv 0$. So, there exists a function $u \in C^\infty(\Omega)$ such that $Lu = 1$ on Ω and $u = 0$ on $\partial\Omega$. It is well known, see [N], that the mean curvature $H(t)$, $|t| < \varepsilon$, of the immersions $\psi_t : \Omega \rightarrow \mathbb{R}^3$, $\psi_t(p) = p + tu(p)\nu(p)$ for all $p \in \Omega$, where ν is the unit normal vector field of M , has derivative at $t = 0$ given by $2H'(0) = Lu = 1$ on Ω ($H(t)$ is determined by the choice of the unit normal ν_t of ψ_t which is continuous with respect to t and satisfies $\nu_0 = \nu$). Thus, from the compactness of Ω , it follows that if ε small enough, $H(t) > 0$ on Ω , $0 < t < \varepsilon$. Using the diffeomorphism $\pi : \Omega_{n,i} \rightarrow$

Ω , we consider on $\Omega_{n,i}$ the function $u_n = u \circ \pi$ and we can construct immersions $\psi_{n,t}$ from $\Omega_{n,i}$ into $\mathbb{R}^3$ given by $\psi_{n,t}(p) = p + tu_n(p)\nu_n(p)$, $p \in \Omega_{n,i}$ and ν_n is the unit normal vector of $\Omega_{n,i}$ which converges to ν . Moreover, taking n large enough, we will have that the mean curvature $H_n(t)$ of the immersions above is positive, $0 < t < \varepsilon$. If u is positive somewhere, the immersions $\psi_{n,t}$, $0 < t < \varepsilon$, cut the component of $\mathbb{R}^3 - M_n$ where ν_n points to. The maximum principle for surfaces of nonnegative mean curvature implies that $M_n \cap \psi_{n,t}(\{u_n > 0\}) = \emptyset$, $0 < t < \varepsilon$, and changing ν_n by $-\nu_n$ we also obtain $M_n \cap \psi_{n,t}(\{u_n > 0\}) = \emptyset$, $-\varepsilon < t < 0$. As these facts are clearly impossible if $k \geq 2$, because for n large $\Omega_{n,j}$, $j \neq i$, necessarily cuts some of the surfaces $\psi_{n,t}(\{u_n > 0\})$, $|t| < \varepsilon$, $t \neq 0$, we conclude that $u_n \leq 0$. So, $u \leq 0$ in Ω using the equation $Lu = 1$, we have that $u < 0$ in $\text{int}(\Omega)$. Finally, we have that $w = -u$ is a positive function which verifies $Lw \leq 0$. The existence of this w is equivalent to the stability of Ω , see Fisher-Colbrie and Schoen [FS]. As Ω is an arbitrary compact domain in $M - X$, we obtain that $M - X$ is stable and, then, it follows that M is stable, see Gulliver and Lawson [GL]. As M is complete and orientable, again by the results of [FS] or Do Carmo and Peng [DP], we have that M is a plane. This contradicts our assumption and so $k = 1$.

We have proved that if the surface M is nonflat, then the limit $M_n \rightarrow M$ has multiplicity one. Now we will prove that $X = \emptyset$. In fact, if $p \in X$, we can consider a right circular cylinder S of radius $\delta > 0$ and axis given by the affine normal line to M at the point $p \in M$. For $\rho > 0$ small and $\delta > 0$ small with respect to ρ , $M \cap S \cap B(p; \rho)$ is a Jordan curve C which projects diffeomorphically onto a circle in the tangent plane of M at p and, as we are assuming that the limit has multiplicity one, we have that, for n large enough, the surface $M_n \cap B(p; \rho)$ cuts the solid cylinder bounded by S in a compact minimal surface S_n where ∂S_n is a Jordan curve in S which is near C in the smooth topology. So ∂S_n projects injectively onto the circle above and a standard application of the maximum principle for minimal surfaces, see for instance [N], shows that S_n is a graph over the disk bounded by this circle. As the total curvature of this graph is small if δ is small, it follows from item (2) in Section 1 that any bubble point $p \in X$ can be removed and, thus, M_n converges to M in $\mathbb{R}^3$. This concludes the proof of the theorem. $\square$

From the results in Sections 1, if $\{M_n\}$ is a sequence of embedded complete minimal surfaces with fixed topology and finite total curvature, then it has uniform local bounds for the area and the total curvature (in fact $C(M_n)$ is constant). Thus Theorem 1 can be applied to this sequence. In this case, in the option (i), the subset X is finite and in (ii), $C(M)$ is finite and not bigger than $C(M_n)$.

For the most part, the arguments in the proof of Theorem 2 work in general Riemannian 3-manifolds. For instance the next result follows directly from these arguments: *Let $\{M_n\}$ be a sequence of compact connected orientable minimal*

surfaces, embedded in a compact orientable Riemannian 3-manifold and such that $\{A(M_n)\}$ and $\{\text{genus}(M_n)\}$ are bounded. Then there exist an embedded compact connected minimal $M \subset N$, a finite subset $X \subset M$ and a subsequence $\{M_n\}$, of $\{M_n\}$ such that $\{M_n\}$ converges to M (with finite multiplicity) on $N - X$ (until here the result has been proved by Choi and Schoen [CS]). Moreover, if M is orientable and unstable, then the limit has multiplicity one.

Now we describe the structure of weak limits for sequences of surfaces with fixed finite total curvature. Although it is more or less clear that this space satisfies some kind of weak compactness, our result contains important restrictions for the weak limits: they are given by a finite number of embedded complete minimal surfaces of multiplicity one and without bubble points. Moreover, the normal direction at the ends is the same for all the surfaces, see Figure 1.

FIGURE 1

Theorem 2 (Weak Compactness Theorem). *Let $\{M_n\}$ be a sequence of embedded complete minimal surfaces in $\mathbb{R}^3$ with fixed topology, nonzero finite total curvature and horizontal ends. Then there exists a subsequence of $\{M_n\}$, denoted again by $\{M_n\}$, an integer number $k > 0$, and nonflat embedded complete*

minimal surfaces $M_{1,\infty}, \dots, M_{k,\infty} \subset \mathbb{R}^3$ with finite total curvature, horizontal ends and $C(M_{1,\infty}) + \dots + C(M_{k,\infty}) = C(M_m)$, such that, $\{M_n\}$ converges to $(M_{1,\infty}, \dots, M_{k,\infty})$. This convergence means that, for any $\varepsilon > 0$ small enough, there exists a $\rho > 0$ and sequences of closed Euclidean balls $\{B_{1,n}\}, \dots, \{B_{k,n}\}$ with $B_{i,n} \cap B_{j,n} = \emptyset$ for $i \neq j$, verifying the following properties:

- (i) The image of the surfaces M_n , under the homotheties that transform $B_{i,n}$ into $B(\rho)$, converges smoothly to $M_{i,\infty}$ (with multiplicity one) uniformly on compact subsets of $\mathbb{R}^3$, $i = 1, \dots, k$.
- (ii) $C(M_n - [B_{1,n} \cup \dots \cup B_{k,n}]) < \varepsilon$, for n large enough.
- (iii) Each component of $M_n - [B_{1,n} \cup \dots \cup B_{k,n}]$, n large enough, is a graph over an unbounded domain in an horizontal plane whose boundary consists of a finite number of pairwise disjoint convex Jordan curves. In particular, the number of components of $M_n - [B_{1,n} \cup \dots \cup B_{k,n}]$ coincides with the number of ends of M_n .

Proof. For each $n \in \mathbb{N}$ consider the closed Euclidean balls B in $\mathbb{R}^3$ such that $C(M_n \cap B) = 2\pi$. Denote by $B'_{1,n}$ a ball in this family of minimum radius. This ball exists because the family is nonempty and if the center of a ball in this family goes to infinity, then also its radius must diverge to infinity. Denote by $M_{1,n}$ the image of M_n under the homothetic transformation in $\mathbb{R}^3$ which maps the ball $B'_{1,n}$ into the unit ball $B(1)$. By the results in Section 1, there exist an embedded complete minimal surface $M_{1,\infty} \subset \mathbb{R}^3$ and a finite subset $X \subset M_1$, such that a subsequence of $\{M_{1,n}\}$, which we will denote with the same symbol, converges to $M_{1,\infty}$ uniformly on compact subsets of $\mathbb{R}^3 - X$. If one of the points in X is not removable, then, for large n , a small neighborhood of this point in $\mathbb{R}^3$ will meet $M_{1,n}$ in a piece whose absolute total curvature is at least 3π . As this contradicts our choice of $B'_{1,n}$, we obtain that X is empty and, so, $M_{1,n}$ converges to $M_{1,\infty}$ on compact subsets on $\mathbb{R}^3$ and, as $C(M_{1,\infty} \cap B(1)) = 2\pi$, the limit surface is nonflat. Thus, using Theorem 1, we have that $M_{1,n}$ converges to $M_{1,\infty}$ with multiplicity one. Finally, consider a large positive number ρ_1 , such that $|C(M_{1,\infty}) - C(M_{1,n} \cap B(\rho_1))| < \varepsilon$, for n large, and denote by $B_{i,n}$ the ball which transforms into $B(\rho_1)$ under the homothety which applies $B'_{i,n}$ into $B(1)$.

Suppose that in $k - 1$ steps we have obtained sequences of closed balls $\{B_{1,n}\}, \dots, \{B_{k-1,n}\}$ and $\{B'_{1,n}\}, \dots, \{B'_{k-1,n}\}$, with $B_{i,n}$ disjoint from $B_{j,n}$ for any $i \neq j$, a subsequence of $\{M_n\}$, denoted again by $\{M_n\}$, such that, if we denote by $M_{i,n}$ the image of M_n under the dilation of $\mathbb{R}^3$ which transforms the $B'_{i,n}$ into $B(1)$, the surfaces $M_{i,n}$ converge to a nonflat embedded complete minimal surface with finite total curvature $M_{1,\infty}$ uniformly on compact subsets of $\mathbb{R}^3$ and with multiplicity one. Assume also that the dilations above transform $B_{i,n}$ into $B(\rho_i)$, for some large positive number ρ_i , and that, for n large enough

and $i = 1, \dots, k-1$, we have $|C(M_{i,\infty}) - C(M_{i,n} \cap B_{i,n})| < \varepsilon$. Moreover, assume that the balls $B'_{i,n}$ verify the following minimizing property:

$B'_{i,n}$ has the smallest radius among balls $B \subset \mathbb{R}^3$ such that

$$(*) \quad C\left(B \cap M_n - \bigcup_{j < i} B_{j,n}\right) = 2\pi.$$

From the property above, if $i < j$, then $\text{radius}(B'_{i,n}) \leq \text{radius}(B'_{j,n})$.

Now, assuming that $C(M_{1,\infty}) + \dots + C(M_{k-1,\infty}) < C(M_n)$, we will prove that we can apply once more the argument. As the two numbers above are integer multiples of 4π , we can consider, for small $\varepsilon > 0$, a ball $B'_{k,n}$, $n \in \mathbb{N}$, verifying the property $(*)$ with $i = k$. Using suitable homotheties, we will suppose that $B'_{k,n} = B(1)$, $n \in \mathbb{N}$. Thus, $\text{radius}(B'_{i,n}) \leq 1$, $n \in \mathbb{N}$, and then $\text{radius}(B_{i,n}) \leq \rho_i$, $i = 1, \dots, k-1$. Following the notation above, we will denote the surfaces M_n by $M_{k,n}$. We can also assume, taking a subsequence if necessary, that the surfaces $M_{k,n}$ converge to an embedded limit $M_{k,\infty}$ (with finite multiplicity). Denote by $X \subset M_{k,\infty}$ the bubble set of this limit.

If for some $i < k$ a subsequence of $\{B_{i,n}\}_n$ converges to a closed ball B_∞ of positive radius, then the surfaces $M_{i,n}$ and $M_{k,n}$ differ in a homothety whose center and ratio are controlled independently of n . So, the limits $M_{i,n} \rightarrow M_{i,\infty}$ and $M_{k,n} \rightarrow M_{k,\infty}$ are essentially the same. In particular, $X = \emptyset$ and, thus, $C(M_{k,n} \cap (B(1) - B_{i,n})) < \varepsilon$ for large n , which contradicts our choice of $B'_{k,n} = B(1)$. So, if one of the sequences $\{B_{i,n}\}_n$, $i < k$, has a limit, this limit is necessarily a point.

Because of the minimizing property of $B'_{k,n} = B(1)$, the bubble set X of $\{M_{k,n}\}$ contains only the limit points of the sequences $\{B_{i,n}\}_n$, $i < k$. If we suppose that X is not empty, then from the alternative in Theorem 1, $M_{k,\infty}$ must be a disjoint union of planes. Take a point p in X and let $J \subset \{1, \dots, k-1\}$ be the subset of integers i such that $p = \lim_{n \rightarrow \infty} B_{i,n}$. Consider a small positive number $\delta < 1$ such that $\{p\} = X \cap B(p; \delta)$ and n large such that $B_{i,n} \cap B(p; \delta) = \emptyset$, $i \notin J$. Let S_n be a component of

$$(**) \quad M_{k,n} \cap \left[B(p; \delta) - \bigcup_{i \in J} B_{i,n} \right].$$

The Gauss map of $M_{k,n}$ converges to a constant value on each component of the part of the boundary of S_n which lies in $\partial B(p; \delta)$ because outside the points in X the limit of $M_{k,\infty}$ is flat. It is also almost constant, for small ε , on each component of $\partial S_n \cap \partial B_{i,n}$, $i \in J$, because of the existence of the limit surface $M_{i,\infty}$. As the Gauss map of a nonflat minimal surface is an open map,

if the constants above are different, then the Gauss map image of S_n will cover almost all the sphere $S^2(1)$. In particular, $C(S_n)$ is near a positive multiple of 4π . This contradicts the minimizing property of $B'_{k,n} = B(1)$ and so we conclude that the Gauss map image of S_n is contained in a small neighbourhood of some vector $a \in S^2(1)$. As the degree of the Gauss map of the surfaces M_n is fixed, $C(S_n)$ is small if ε and δ are small. As the number of boundary components of the surface in $(**)$ is finite and independent of n , it follows that the number of components S_n is also uniformly bounded. Thus, the absolute total curvature of the surface $(**)$ can be taken, for suitable values of ε and δ , uniformly small. If we apply the arguments above to each point p in $X \cap B(1)$, we obtain that $C(M_{k,n} \cap B(1) - B(X; \delta)) \geq \pi$ and, so, $M_{k,\infty}$ is nonflat. This contradiction shows that $X = \emptyset$.

As a consequence, the condition $C(M_{k,n} \cap B(1)) = 2\pi$ guarantees that $M_{k,\infty}$ is nonflat and from Theorem 1 we obtain that $M_{k,n}$ converges to $M_{k,\infty}$ with multiplicity one. Moreover, the sequences $\{B_{i,n}\}_n$, $i < k$, are divergent in $\mathbb{R}^3$ and, in particular, we can assume that $B_{i,n}$ is disjoint from $B_{k,n}$ for $i < k$ and $n \in \mathbb{N}$. Now it is easy to choose ρ_k large enough such that $B_{k,n} = B(\rho_k)$ verifies $|C(M_{k,\infty}) - C(M_{k,n} \cap B_{k,n})| < \varepsilon$. Moreover, it is clear that $C(M_{i,\infty}) + \dots + C(M_{k,\infty}) \leq C(M_n)$. If the equality does not hold, then we can reproduce the arguments above and in a finite number of steps we will obtain the equality. Assuming we have obtained the equality in the k^{th} step, we take $\rho = \max\{\rho_1, \dots, \rho_k\}$ and we consider, for $n \in \mathbb{N}$ and $i = 1, \dots, k$, the ball concentric with $B_{i,n}$ whose radius is ρ/ρ_i times the radius of $B_{i,n}$. If we denote these balls again by $B_{i,n}$, we have the assertions (i), (ii) in the statement of the theorem (after changing ε by $\varepsilon/2k$ in the arguments above). To conclude the proof of the theorem, it remains to prove (iii) and the fact that the surfaces $M_{i,\infty}$ have horizontal ends.

Let Ω_n be the closure of a component of $M_n - [B_{1,n} \cup \dots \cup B_{k,n}]$. Then Ω_n is properly embedded in $\mathbb{R}^3 - [B_{1,n} \cup \dots \cup B_{k,n}]$ and its boundary consists in a finite number of Jordan curves lying on the boundary of the balls above. Moreover as, up to scaling and translations, the surface $M_n \cap B_{i,n}$ is uniformly near in the C^k topology to the intersection of an embedded complete minimal surface of finite total curvature, $M_{i,\infty} \subset \mathbb{R}^3$ with a large Euclidean ball, it follows that any component of $\partial\Omega_n$ looks like a round circle and the Gauss map of M_n along this curve is uniformly near a constant value (the value of the Gauss map of $M_{i,\infty}$ at the corresponding end). Again from the openness of the Gauss map of $\bar{M}_n$, and because the total curvature of Ω_n is small, it follows that the Gauss map ν_n of M_n converges over all the components of the boundary of Ω_n to the same constant value $a \in S^2(1)$ and that $\nu_n(\Omega_n)$ is contained in a small neighbourhood of a . Along each component of its boundary we can glue smoothly the domain Ω_n with a compact surface inside the ball $B_{i,n}$ which is a graph over

the plane orthogonal to the vector a and looks like a planar disk parallel to this plane. In this way we obtain a properly embedded surface Ω'_n without boundary whose Gauss map image is contained in a small neighbourhood of a . Then its orthogonal projection over the plane $\langle a \rangle^\perp$ is a proper local diffeomorphism (the properness follows in a standard way because the pull-back of the planar metric is a complete metric on Ω'_n). Thus the projection is a covering map and then necessarily a global diffeomorphism. As a consequence, the domain Ω_n is a graph over the noncompact region in $\langle a \rangle^\perp$ bounded by a finite number of disjoint convex curves. In particular, Ω_n contains exactly one end of M_n and a is the value of the Gauss map of M_n at this end, i.e., $a = \pm(0, 0, 1)$. This proves (iii). As a is also the normal vector at the ends of $M_{i,\infty}$ and if we consider all the components of $M_n - [B_{i,n} \cup \dots \cup B_{k,n}]$, then all the surfaces $M_{i,\infty}$ will appear in the arguments above, we conclude that $M_{i,\infty}$, $i = 1, \dots, k$ has horizontal ends. Thus the theorem is proved. $\square$

Remark. If in the statement of the theorem above we have $k = 1$, then $\{M_n\}$ converges, up to homotheties, uniformly on compact subsets of $\mathbb{R}^3$ to an embedded surface, M_∞ (with multiplicity one), with loss of neither total curvature nor topology. In this case, we say that the sequence is *strongly convergent*.

3. Strong compactness results. Given integers $g \geq 0$ and $r \geq 1$, we will denote by $\mathbf{M}(g, r)$ the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite topology, genus g and r horizontal ends. Moreover, two surfaces in this space which differ by a dilation of $\mathbb{R}^3$ will be considered as the same surface. It is an elementary fact that $\mathbf{M}(g, 1)$ contains only the horizontal plane if $g = 0$ and it is empty if $g \geq 1$. We also have that $\mathbf{M}(0, 2)$ contains only the vertical Catenoid. Results of Schoen [S] and Lopez and Ros [LR] assert that the spaces $\mathbf{M}(g, 2)$, $g \geq 1$, and $\mathbf{M}(0, r)$, $r \geq 3$, are empty. It has been conjectured by Hoffman and Meeks [HM2] that for each $g \geq 1$, there exists an integer $r(g)$ such that $\mathbf{M}(g, r)$ is empty if $r > r(g)$ (more precisely they conjecture that $r(g) = g + 2$). In this section we prove that the spaces $\mathbf{M}(1, r)$, $r \geq 5$, are (strongly) compact. This result may be viewed as a first step in the proof of the Hoffman-Meeks conjecture for the genus one case. Moreover, assuming that a version of their conjecture is true for surfaces with genus smaller than g , we can obtain strong compactness theorems for the spaces $\mathbf{M}(g, r)$. We will need the following nonexistence result proved by the author; see Corollary 12 in [R].

Theorem 3 ([R]). *There are no properly embedded minimal surfaces $M \subset \mathbb{R}^3$ with compact boundary verifying the following properties:*

- (i) *M has genus zero, two boundary components and one end.*
- (ii) *The end has finite total curvature and its limit normal direction is vertical.*
- (iii) *Each component C_i , $i = 1, 2$, of ∂M is a convex Jordan curve lying in an horizontal plane and M is transversal to this plane along C_i .*
- (iv) *$\text{Force}(C_i)$ is a vertical vector, $i = 1, 2$.*

Theorem 4. *Let g and r be two integers with $g \geq 1$ and $r > 2g + 2$. Assume that for each pair (g', r') with $0 \leq g' < g$ and $r' > 2g' + 2$, the space $\mathbf{M}(g', r')$ is empty. Then $\mathbf{M}(g, r)$ is strongly compact.*

Corollary 5. *The spaces $\mathbf{M}(1, r)$, $r \geq 5$, are strongly compact.*

Proof. We know from [LR], see also Perez and Ros [PR], that $\mathbf{M}(0, r')$, $r' > 2$, is empty. Thus, the corollary follows directly from the theorem. $\square$

Proof of the Theorem. Let $\{M_n\}_n \subset \mathbf{M}(g, r)$ be a weakly convergent sequence. Reasoning by contradiction, we assume that its limit is given by $M_{1,\infty}, \dots, M_{k,\infty}$, $k \geq 2$, verifying the properties stated in Theorem 2. Denote by g_i and r_i the genus and the number of ends of $M_{i,\infty}$. For n large enough there exist pairwise disjoint Euclidean balls $B_{1,n}, \dots, B_{k,n}$ such that $M_{i,n} = B_{i,n} \cap M_n$ is a compact surface with genus g_i and r_i boundary components. Moreover, $M_n - [M_{1,n} \cup \dots \cup M_{k,n}]$ has r components $\Omega_1, \dots, \Omega_r$ which are genus zero surfaces with one end and the total number of boundary components of $\Omega_1 \cup \dots \cup \Omega_r$ is $r_1 + \dots + r_k$. So $\Sigma r_i \geq r$. Another clear relation is $\Sigma g_i \leq g$. The Euler characteristics of the surfaces above are related by the formula $\chi(M_n) = \chi(M_{1,n}) + \dots + \chi(M_{k,n}) + \chi(\Omega_1) + \dots + \chi(\Omega_r)$ and, grouping suitably, we have that

$$(1) \quad (k-1) + (g - \Sigma g_i) - (\Sigma r_i - r) = 0.$$

Both the pieces Ω_j and the boundary components of each $M_{i,n}$ are naturally ordered by their heights with respect to the vertical direction. We say that a component of C of $\partial\Omega_j$ is a top (or a bottom) boundary component, if C is the top (or the bottom) boundary component of the compact surface $M_{i,n}$ which contains C . The top (resp. bottom) piece has only top (resp. bottom) boundary components. Denote by $\mathbf{P}$ the set of pieces Ω_j , $j = 1, \dots, r$, and observe that $\mathbf{P}$ is the disjoint union of the following sets:

$\mathbf{A} = \{\text{the top and bottom pieces}\},$

$\mathbf{B} = \left\{ \begin{array}{l} \text{pieces, others than the top and the bottom pieces, which} \\ \text{have only top or bottom boundary components} \end{array} \right\},$

$\mathbf{C} = \left\{ \begin{array}{l} \text{pieces with some boundary component which is neither a} \\ \text{top nor a bottom boundary component} \end{array} \right\}.$

If we denote by b and c the numbers of elements of $\mathbf{B}$ and $\mathbf{C}$ respectively, then $r = 2 + b + c$ and $c \leq \Sigma(r_i - 2) = \Sigma r_i - 2k$. Hence we obtain that

$$(2) \quad 2(k-1) - (\Sigma r_i - r) \leq b.$$

Assertion. *There exists $\Omega \in \mathbf{B}$ with exactly two boundary components.*

To prove this assertion, first observe that $\partial\Omega \neq \emptyset$, for each Ω in $\mathbf{P}$, and if $\Omega \in \mathbf{A} \cup \mathbf{B}$ has only one boundary component, then we can consider the properly embedded topological plane $\Omega \cup D$, where D is a disk with $\partial D = \partial\Omega$ and $M_n \cap \text{int}(D) = \emptyset$. As M_n is contained in the closure of one of the components of $\mathbb{R}^3 - (\Omega \cup D)$, it follows that Ω is the top or the bottom piece. In particular $\Omega \notin \mathbf{B}$. Clearly the total number of boundary components of the pieces in $\mathbf{A} \cup \mathbf{B}$ is at most $2k$. Hence the total number of boundary components of the pieces in $\mathbf{B}$ is at most $2k - 2$. If we assume that each $\Omega \in \mathbf{B}$ has at least three boundary components, then we obtain that

$$(3) \quad b \leq \frac{2k-2}{3}.$$

Now we separate our argument in two cases.

▷ FIRST CASE: $g_i < g$, $i = 1, \dots, k$. Using the hypothesis $r_i \leq 2g_i + 2$ and $r > 2g + 2$, we transform (1) into $4(k-1) - 3(\sum r_i - r) > 0$, which joint with (2) give us $2(k-1) < 3b$, and this is impossible because of (3). So we have proved the assertion in this case.

▷ SECOND CASE: $g_i = g$, FOR SOME $i = 1, \dots, k$. To fix notation, suppose that $g_1 = g$. Then, from the definition of genus of a surface, we obtain that $g_2 = \dots = g_k = 0$ and so, using the uniqueness result in [LR], $r_2 = \dots = r_k = 2$. If we find the value of r_1 in (1) and we substitute it in (2), then we conclude that $k-1 \leq b$. Using again (3), we have that this is impossible if $k \geq 2$. Thus the assertion is proved.

If we take Ω in $\mathbf{B}$ with two boundary components, then, from Theorem 2 and for n large enough, near each one of the Jordan curves in $\partial\Omega$, Ω looks like a piece of a vertical Catenoid (recall that the top and the bottom ends of $M_{i,\infty}$, $i = 1, \dots, k$, are necessarily of Catenoid type) bounded by an horizontal round circle. Thus, cutting Ω with two suitable transversal horizontal planes, we obtain a subsurface $\Omega' \subset \Omega$ which is properly embedded in $\mathbb{R}^3$ with genus zero, one end and two boundary components. The end has finite total curvature and vertical limit normal direction, and the boundary components are planar convex curves in horizontal planes. If we can prove that the force of any component of $\partial\Omega'$ is a vertical vector, we will contradict Theorem 3 and our theorem will be proved. To prove that the forces of the two Jordan curves in $\partial\Omega'$ are vertical, it is enough to show that each one of these curves disconnects M_n (because in this case the curve will be homologous to a sum of curves around some ends of M_n). To show this last fact, denote by D_1 and D_2 the planar disks bounded by the convex curves in $\partial\Omega'$. Then, for large integer n , $M_n \cap \text{int}(D_i) = \emptyset$, $i = 1, 2$, and $\Omega' \cup D_1 \cup D_2$ is a properly embedded topological plane. If we denote by N_1 , and

N_2 the two components of $\mathbb{R}^3 - (\Omega' \cup D_1 \cup D_2)$, as Ω is neither the top nor the bottom piece, it follows that M_n meets N_1 and N_2 . Hence, as M_n is connected, a component of $\partial\Omega'$ is the topological boundary of $M_n \cap N_1$ and the other of $M_n \cap N_2$. So each one of these curves disconnects M_n as we have asserted. This completes the proof of the theorem. $\square$

The arguments in the proof above show that for any g and r and any weakly convergent sequence $\{M_n\} \subset \mathbf{M}(g, r)$, the pieces $\Omega \in \mathbf{B}$ have at least three boundary components. As in the second case of the proof of the assertion, we do not use the hypothesis of the theorem, we have that:

If the sequence $\{M_n\}$, like in Theorem 2, converges weakly to the surfaces $M_{1,\infty}, \dots, M_{k,\infty}$, with $k \geq 2$, then $\text{genus}(M_{i,\infty}) < \text{genus}(M_n)$, for each $i = 1, \dots, k$.

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