*Compactness Properties of
Weakly p -Harmonic Maps into
Homogeneous Spaces*

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ABSTRACT. In this paper we prove that the space of weakly p -harmonic maps from an open set in $\mathbf{R}^n$ into a homogeneous space with a left invariant metric is compact. We give a regularity criteria for weakly p -harmonic maps. As an application we prove that stationary p -harmonic maps into homogeneous spaces are regular, except possibly for a closed singular set of $(n - p)$ -dimensional Hausdorff measure zero.

1. Introduction. In this paper we present some compactness results for weakly p -harmonic maps from an open subset of $\mathbf{R}^n$ into a compact homogeneous space, with a left invariant metric. In this context we prove that the weak limit of a sequence of weakly p -harmonic maps is again a weakly p -harmonic map. We also show that stationary p -harmonic maps into homogeneous spaces are regular, except possibly for a closed singular set of $(n - p)$ -dimensional Hausdorff measure zero. The proof crucially depends upon several observations of Hélein's and Shatah's, see [H1] and [Sh].

To state the results precisely, assume that Ω is a smooth open subset of $\mathbf{R}^n$, that M is a smooth compact manifold and that (N, h) is an m -dimensional smooth compact homogeneous space with a left invariant metric h . We also assume that M and N are isometrically embedded in some Euclidean space $\mathbf{R}^k$. A function u in the Sobolev space $W^{1,p}(\Omega, \mathbf{R}^k)$ belongs to $W^{1,p}(\Omega, M)$ provided that $u(x) \in M$ a.e. in Ω . Here $1 < p \leq n$.

Definition. A function $u \in W^{1,p}(\Omega, M)$ is a weakly p -harmonic map of Ω into M provided

(1.1)
$$-\operatorname{div}(|Du|^{p-2}Du) = |Du|^{p-2}A(u)(Du, Du) \quad \text{weakly in } \Omega,$$

where A denotes the second fundamental form of the embedding $M \subset \mathbf{R}^k$. This system of partial differential equations is to hold in the weak sense, that is,

$$(1.2) \quad \int_{\Omega} |Du|^{p-2} Du : Dw \, dx = \int_{\Omega} |Du|^{p-2} A(u)(Du, Du)w \, dx$$

for each test function $w \in W_0^{1,p}(\Omega, \mathbf{R}^k) \cap L^\infty(\Omega, \mathbf{R}^k)$. Here

$$Du = \left(\frac{\partial u^i}{\partial x_j} \right)_{\substack{1 \leq j \leq n \\ 1 \leq i \leq k}}, \quad Du : Dw = \frac{\partial u^i}{\partial x_j} \frac{\partial w^i}{\partial x_j}.$$

Let $g : \partial\Omega \rightarrow M$ be a smooth function, then (1.1) is the Euler-Lagrange equation for the variational problem of minimizing the p -energy

$$E_p[w] = \int_{\Omega} |Dw|^p \, dx$$

among functions $w \in W^{1,p}(\Omega, M)$ with $w = g$ on $\partial\Omega$ in the trace sense.

Under the previous assumptions we have:

Theorem 1. *Let (N, h) be a smooth compact homogeneous space, and h a left invariant metric on N . Let $u_k \in W^{1,p}(\Omega, N)$ be a sequence of weakly p -harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, N)$. Then u is a weakly p -harmonic map from Ω into N .*

In general, very little can be said about the regularity of weakly p -harmonic maps. Helein [H2] proved that a weakly harmonic (i.e., $p = 2$) map from $\Omega \subset \mathbf{R}^2$ into a smooth compact manifold is always smooth. On the other hand Rivière [R] constructed weakly harmonic maps from B^3 into the sphere S^2 which are singular in the whole B^3 . Schoen and Uhlenbeck [SU] proved that that energy minimizing harmonic maps from $\Omega \subset \mathbf{R}^n$ into a smooth compact manifold are smooth, except possibly for a closed set of Hausdorff dimension at most $(n - 3)$. Hardt and Lin [HL] proved a similar result for general p . Namely they showed that energy minimizing p -harmonic maps from $\Omega \subset \mathbf{R}^n$ into a smooth compact manifold are $C^{1,\alpha}$, for some $\alpha \in (0, 1)$, except possibly for a closed set of Hausdorff dimension at most $(n - [p] - 1)$.

If u is a minimizer of $E_p[\cdot]$, then, in addition to (1.2), u satisfies

$$(1.3) \quad \left. \frac{d}{dt} E_p(u_t) \right|_{t=0} = 0,$$

where $u_t(x) = u(x + t\zeta(x))$, and $\zeta \in C^1(\Omega, \mathbf{R}^n)$ any vector field with compact support in Ω .

Definition. A function $u \in W^{1,p}(\Omega, M)$ is a stationary p -harmonic map from Ω into M if u satisfies the identities (1.2), and (1.3) for all test functions w and ζ .

In particular, a stationary p -harmonic map satisfies the monotonicity inequalities

$$(1.4) \quad \frac{1}{\sigma^{n-p}} \int_{B_\sigma(x)} |Du|^p dy \leq \frac{1}{\rho^{n-p}} \int_{B_\rho(x)} |Du|^p dy,$$

for all concentric balls $B_\sigma(x) \subset B_\rho(x) \subset \Omega$ (see [P]).

Evans [E1] (for $M = S^m$), and Bethuel [B1] (for M a compact smooth manifold) proved that a stationary harmonic map u from Ω into M is smooth, except possibly for a closed set of $(n-2)$ -dimensional Hausdorff measure zero. For stationary p -harmonic maps we prove the following theorem. We recently learned that this result was independently proved by Takeuchi [T1], in the case $M = S^m$.

Theorem 2. *Let (N, h) be a smooth compact homogeneous space, and h a left invariant metric on N . Assume that $u \in W^{1,p}(\Omega, N)$ is a stationary p -harmonic map. Then there exists a closed subset $\mathcal{S} \subset \Omega$ such that*

$$\mathcal{H}^{n-p}(\mathcal{S}) = 0, \quad u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N),$$

for some $\alpha \in (0, 1)$. Here $\mathcal{H}^{n-p}$ denotes the $(n-p)$ -dimensional Hausdorff measure.

A natural question in this setting is whether the regularity is preserved when taking weak limits. Luckhaus [L1] proved that if $u_k \in W^{1,p}(\Omega, M)$ is a sequence of p -energy minimizing maps, where M is a smooth compact manifold, and $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega, M)$. Then u is a p -energy minimizing map and hence the partial regularity results from either [SU] or [HL] apply.

Theorem 3. *Let (N, h) be a smooth compact homogeneous space, and h a left invariant metric on N . Let $u_k \in W^{1,p}(\Omega, N)$ be a sequence of stationary p -harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, N)$. Then there exists a closed subset $\mathcal{S} \subset \Omega$ such that*

$$u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N),$$

for some $\alpha \in (0, 1)$, and for any compact set $K \subset \Omega$

$$\mathcal{H}^{n-p}(\mathcal{S} \cap K) < \infty.$$

In the case $p = 2$ Theorem 3 holds for any smooth compact target M . Moreover in this case standard elliptic regularity estimates (see [M]) guarantee that the limiting map $u \in C^\infty(\Omega \setminus \mathcal{S}, N)$.

Theorem 4. *Let M be a smooth compact manifold. Let $u_k \in W^{1,2}(\Omega, M)$ be a sequence of stationary harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega, M)$. Then u is a weakly harmonic map, and there exists a closed subset $\Sigma \subset \Omega$ such that*

$$u \in C^\infty(\Omega \setminus \Sigma, M),$$

and for any compact set $K \subset \Omega$

$$\mathcal{H}^{n-2}(\Sigma \cap K) < \infty.$$

Our proofs of Theorems 1, 2 and 3 depend crucially upon the structure of the target manifold. The fact that N is a homogeneous space with a left invariant metric guarantees that the right hand side of equation (1.1) belongs to the Hardy space $\mathcal{H}_{\text{loc}}^1(\Omega)$. We prove Theorem 1 in Section 2. In Sections 3 and 4 we establish a regularity criteria for weakly p -harmonic maps into a homogeneous space with a left invariant metric. The main idea is to show that u is continuous in a region where some analogue of the scaled energy is sufficiently small. In Section 5 we prove Theorems 2, 3 and 4. Theorem 2 is a simple corollary of the Energy Decay Lemma proved in Section 3, it can also be deduced from Theorem 2.6 in [MY]. In order to prove Theorem 3 we show that if u is the limit of stationary p -harmonic maps into a homogeneous space, then the regularity criteria described in Section 3 holds on a large region in Ω . We would like to remark that the convergence statement from Theorem 4 was proved by Bethuel [B2], in the case $\Omega \subset \mathbf{R}^2$. For further results on weakly p -harmonic maps we refer the reader to [L2], [Fu1] and [Fu2].

2. Compactness Results. We start by recalling some relevant facts about homogeneous spaces with left invariant metrics, for more information see [CE]. Let (N^m, h) be a smooth compact homogeneous space with a left invariant metric h . Let X be a Killing tangent vector field on N , and let $u \in W^{1,p}(\Omega, N)$ be a weakly p -harmonic map. Then Noether's Theorem guarantees that the tangent vector field $|Du|^{p-2} \langle Du, X(u) \rangle$ defined on Ω is divergence free in the distribution sense. In fact since u satisfies equation (1.1), for $\zeta \in C^\infty$ with compact support in Ω , we have

$$\int_{\Omega} \langle D_{\alpha}(\zeta X(u)), |Du|^{p-2} D_{\alpha} u \rangle dx = 0.$$

We expand this equation to obtain

$$\begin{aligned} & \int_{\Omega} \zeta \langle D_{\alpha} X(u), |Du|^{p-2} D_{\alpha} u \rangle dx + \int_{\Omega} D_{\alpha} \zeta \langle X(u), |Du|^{p-2} D_{\alpha} u \rangle dx \\ &= \int_{\Omega} \zeta |Du|^{p-2} \langle D_{D_{\alpha} u} X(u), D_{\alpha} u \rangle dx + \int_{\Omega} D_{\alpha} \zeta \langle X(u), |Du|^{p-2} D_{\alpha} u \rangle dx = 0. \end{aligned}$$

But $\langle D_{D_\alpha u} X(u), D_\alpha u \rangle$ equals zero for any Killing vector field X . Therefore

$$\int_{\Omega} D_\alpha \zeta \langle X(u), |Du|^{p-2} D_\alpha u \rangle dx = 0,$$

for any $\zeta \in C^\infty$ with compact support in Ω , i.e.,

$$(2.1) \quad \operatorname{div}(|Du|^{p-2} \langle X(u), Du \rangle) = 0 \quad \text{weakly in } \Omega.$$

Helein [H1] proved that under the previous assumptions there exist q smooth tangent vector fields $Y_1, \dots, Y_q$ and q smooth Killing tangent vector fields $X_1, \dots, X_q$ on N such that for any $y \in N$ and for any tangent vector field V in $T_y N$, we have:

$$V = \sum_{i=1}^q \langle X_i, V \rangle Y_i.$$

A homogeneous space N can always be identified with a quotient space G/H , where G is a connected Lie group and H is a closed subgroup. q above represents the dimension of the Lie algebra of G . In particular, for each $\alpha = 1, \dots, n$,

$$|Du|^{p-2} D_\alpha u = \sum_{i=1}^q |Du|^{p-2} \langle D_\alpha u, X_i \rangle Y_i.$$

Computing the divergence on both sides and recalling that $|Du|^{p-2} \langle Du, X_i \rangle$ is a divergence free vector field, for each $i = 1, \dots, q$ we have

$$(2.2) \quad \operatorname{div}(|Du|^{p-2} Du) = \sum_{\alpha=1}^n \sum_{i=1}^q |Du|^{p-2} \langle D_\alpha u, X_i(u) \rangle D_\alpha Y_i(u) \quad \text{weakly in } \Omega.$$

It is easy to check that a weak solution of (2.2) is also a weak solution of (1.1). Note that the right hand side of (2.2) is of the general form, the inner product of a gradient and a divergence free vector field.

We now recall some of the definitions and properties of the spaces BMO and $\mathcal{H}^1$, see [FS], [CLMS] and [S] for more information. If $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is in $L^1_{\text{loc}}(\mathbf{R}^n)$, we set

$$\|f\|_* = \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - (f)_{x,r}| dy : x \in \mathbf{R}^n, r > 0 \right\},$$

where

$$(f)_{x,r} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f dy.$$

We say that f has bounded mean oscillation provided $\|f\|_* < \infty$, and refer to $\|f\|_*$ as the BMO norm of f . Assume now that $g \in L^1(\mathbf{R}^n)$. Define a class $\mathcal{T}$ of normalized test functions in $\mathbf{R}^n$ by

$$\mathcal{T} = \{\varphi \in C^\infty(\mathbf{R}^n) : \text{supp } \varphi \subset B_1(0) \text{ and } \|D\varphi\|_{L^\infty} \leq 1\}.$$

We set

$$g^*(x) = \sup_{t>0} \sup_{\varphi \in \mathcal{T}} |\varphi_t * g(x)|, \quad \text{where } \varphi_t(x) = t^{-n} \varphi\left(\frac{x}{t}\right),$$

and we say that g belongs to the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ if $g^* \in L^1(\mathbf{R}^n)$. We write

$$\|g\|_{\mathcal{H}^1(\mathbf{R}^n)} = \|g^*\|_{L^1(\mathbf{R}^n)}.$$

Note that $g \in \mathcal{H}^1(\mathbf{R}^n)$ implies $\int_{\mathbf{R}^n} g \, dx = 0$.

A fundamental theorem of Fefferman [F] asserts that $(\mathcal{H}^1)^* = \text{BMO}$ and establishes the inequality

$$(2.3) \quad \left| \int_{\mathbf{R}^n} f g \, dx \right| \leq C \|f\|_* \|g\|_{\mathcal{H}^1(\mathbf{R}^n)}$$

for $f \in L^\infty$ and $g \in \mathcal{H}^1(\mathbf{R}^n)$.

We now mention some results about $\mathcal{H}_{\text{loc}}^1(\Omega)$ which we use in the proof of Theorem 1, and in Section 4; for more information see [S]. Let $\Omega \subset \mathbf{R}^n$ be an open set, we say that $g \in L_{\text{loc}}^1(\Omega)$ lies in $\mathcal{H}_{\text{loc}}^1(\Omega)$ if for each compact set $K \subset \Omega$ there is an $\varepsilon > 0$ so that

$$(2.4) \quad \|g\|_{\mathcal{H}^1(K)} = \int_K \left(\sup_{0 < t < \varepsilon} \sup_{\varphi \in \mathcal{T}} |\varphi_t * g(x)| \, dx \right) < \infty.$$

Note that $\varphi_t * g(x)$ is defined for $\varphi \in \mathcal{T}$ as long as $t < \text{dist}(x, \partial\Omega)$.

The following proposition relates $\mathcal{H}_{\text{loc}}^1(\Omega)$ to $\mathcal{H}^1(\mathbf{R}^n)$, allowing us to use the duality $(\mathcal{H}^1)^* = \text{BMO}$ “locally”. For a proof see [S].

Proposition ([S]). *Let $\Omega \subset \mathbf{R}^n$ be an open set, and let $g \in L_{\text{loc}}^1(\Omega)$. Then $g \in \mathcal{H}_{\text{loc}}^1(\Omega)$ if and only if for each C^∞ function ϑ with compact support contained in Ω , and $\int \vartheta \neq 0$ there is a constant μ such that*

$$\vartheta(g - \mu) \in \mathcal{H}^1(\mathbf{R}^n).$$

Furthermore,

$$\mu = \left(\int \vartheta \right)^{-1} \left(\int \vartheta g \right),$$

and if $K = \text{supp } \vartheta$,

$$(2.5) \quad \|\vartheta(g - \mu)\|_{\mathcal{H}^1(\mathbf{R}^n)} \leq C_K (1 + \|\vartheta(g - \mu)\|_{L^1(\mathbf{R}^n)} + \|g\|_{\mathcal{H}^1(K)}).$$

As remarked above, the right hand side of (2.2) is of the general form the inner product of a gradient and a divergence free vector field. A result of Coifman, Lions, Meyer and Semmes [CLMS] guarantees then that this expression is in the Hardy space $\mathcal{H}_{\text{loc}}^1(\Omega)$.

Proposition ([CLMS]). *Let $\Omega \subset \mathbf{R}^n$. Assume $B \in W^{1,p}(\Omega)$, $E \in L^{p'}(\Omega, \mathbf{R}^n)$, where $1/p + 1/p' = 1$. Suppose also that $\operatorname{div} E = 0$ in the distribution sense in Ω . Then $\langle DB, E \rangle \in \mathcal{H}_{\text{loc}}^1(\Omega)$, and for any $K \subset \Omega$ compact,*

$$(2.6) \quad \|\langle DB, E \rangle\|_{\mathcal{H}^1(K)} \leq C(\|DB\|_{L^p(\Omega, \mathbf{R}^n)}^p + \|E\|_{L^{p'}(\Omega, \mathbf{R}^n)}^{p'}).$$

Proof of Theorem 1. Let $\{u_k\}_{k \geq 1} \subset W^{1,p}(\Omega, N)$ be a sequence of weakly p -harmonic maps into a smooth compact homogeneous space with a left invariant metric. Assume that $u_k \rightharpoonup u$ weakly, $u \in W^{1,p}(\Omega, N)$. Since for $k = 1, 2, \dots$, u_k is a weak solution of

$$(2.7) \quad \operatorname{div}(|Du_k|^{p-2} Du_k) = g_k$$

where $g_k = \sum_{i=1}^q |Du_k|^{p-2} \langle Du_k, X_i(u_k) \rangle DY_i(u_k)$, there exists a subsequence (which we reindex as necessary and denote also by $\{u_k\}_{k \geq 1}$) such that $u_k \rightarrow u$ strongly in $W^{1,p-1}(\Omega, N)$, so, in particular, $Du_k \rightarrow Du$ a.e. in Ω (see [HLM]), and $|Du_k|^{p-2} Du_k \rightharpoonup |Du|^{p-2} Du$ weakly in $L^{p'}(\Omega, M^{k \times n})$. Hence,

$$(2.8) \quad g_k \rightarrow g = \sum_{i=1}^q |Du|^{p-2} \langle Du, X_i(u) \rangle DY_i(u) \text{ a.e. in } \Omega,$$

with $g \in L^1(\Omega)$. Moreover, by the proposition above $g_k \in \mathcal{H}_{\text{loc}}^1(\Omega)$. Combining inequality (2.6) with the fact that N is smooth and compact, we obtain the following estimate for any compact set K in Ω

$$(2.9) \quad \begin{aligned} \|g_k\|_{\mathcal{H}^1(K)} &\leq C \sum_{i=1}^q (\|DY_i(u_k)\|_{L^p(\Omega, \mathbf{R}^n)}^p \\ &\quad + \|Du_k\|^{p-2} \|\langle Du_k, X_i(u_k) \rangle\|_{L^{p'}(\Omega, \mathbf{R}^n)}^{p'}) \\ &\leq C \int_{\Omega} |Du_k|^p dx. \end{aligned}$$

Let ϑ be a C^∞ function with compact support in Ω , and assume that $\int \vartheta \neq 0$. Then if $\mu_k = (\int \vartheta)^{-1} (\int \vartheta g_k) \in \mathbf{R}^k$, $\vartheta(g_k - \mu_k) \in \mathcal{H}^1(\mathbf{R}^n)$. Since

$$(2.10) \quad |Du_k|^{p-2} Du_k \rightharpoonup |Du|^{p-2} Du \text{ weakly in } L^{p'}(\Omega, M^{k \times n}),$$

then using (2.6) and (2.9),

$$(2.11) \quad \mu_k = \frac{\int \vartheta g_k}{\int \vartheta} = - \frac{\int |Du_k|^{p-2} \langle Du_k, D\vartheta \rangle}{\int \vartheta} \rightarrow - \frac{\int |Du|^{p-2} \langle Du, D\vartheta \rangle}{\int \vartheta},$$

$$(2.12) \quad G_k = \vartheta(g_k - \mu_k) \rightarrow G = \vartheta(g - \mu) \text{ a.e. in } \mathbf{R}^n,$$

with

$$(2.13) \quad \mu = -\frac{\int |Du|^{p-2} \langle Du, D\vartheta \rangle}{\int \vartheta},$$

and $G = \vartheta(g - \mu) \in L^1(\mathbf{R}^n)$. Furthermore, if $K = \text{supp } \vartheta$ for some $\varepsilon > 0$, using (2.8)

$$\begin{aligned} \|G_k\|_{\mathcal{H}^1(\mathbf{R}^n)} &\leq C_K(1 + \|\vartheta(g_k - \mu_k)\|_{L^1(\mathbf{R}^n)} + \|g_k\|_{\mathcal{H}^1(K)}) \\ &\leq C\left(1 + \int_{\Omega} |Du_k|^p dx\right), \end{aligned}$$

where $C = C(\vartheta, N, n)$. Since $\{u_k\}_{k \geq 1}$ is uniformly bounded in $W^{1,p}(\Omega, N)$, then $\{G_k\}_{k \geq 1}$ is uniformly bounded in $\mathcal{H}^1(\mathbf{R}^n)$. We can apply the following theorem by Jones and Journé [JJ].

Theorem ([JJ]). *Let $\{G_k\}_{k \geq 1}$ be bounded in $\mathcal{H}^1(\mathbf{R}^n)$. Assume that $\{G_k\}_{k \geq 1}$ converges a.e. to some $G \in L^1(\mathbf{R}^n)$. Then $G \in \mathcal{H}^1(\mathbf{R}^n)$ and G_k converges to G in the weak-* topology of $\mathcal{H}^1(\mathbf{R}^n)$.*

In particular, for $G_k = \vartheta(g_k - \mu_k)$ as above, if $w \in L^\infty(\Omega, \mathbf{R}^k)$ with compact support in Ω , then

$$(2.14) \quad \int G_k w dx = \int \vartheta(g_k - \mu_k) w dx \rightarrow \int G w dx = \int \vartheta(g - \mu) w dx.$$

We finally have all the necessary ingredients to finish the proof of Theorem 1. Let $w \in W^{1,p}(\Omega, \mathbf{R}^k) \cap L^\infty(\Omega, \mathbf{R}^k)$ be a test function with compact support $K \subset \Omega$. Let ϑ be a smooth function with compact support in Ω , assume $\vartheta = 1$ on K , and $0 \leq \vartheta \leq 1$. Since each u_k satisfies (2.2) we have, using the notation above,

$$\begin{aligned} -\int_{\Omega} |Du_k|^{p-2} Du_k : Dw dx &= \int_{\Omega} g_k w dx \\ &= \int_{\Omega} \vartheta(g_k - \mu_k) w dx + \int_{\Omega} \vartheta \mu_k w dx \\ &= \int_{\Omega} G_k w dx + \int_{\Omega} \vartheta \mu_k w dx. \end{aligned}$$

Letting k go to infinity and applying (2.8), (2.10), (2.11) and (2.12), we conclude that

$$\begin{aligned}
 - \int_{\Omega} |Du|^{p-2} Du : Dw \, dx &= \int_{\Omega} Gw \, dx + \int_{\Omega} \vartheta \mu w \, dx \\
 &= \int_{\Omega} \vartheta (g - \mu) w \, dx + \int_{\Omega} \vartheta \mu w \, dx \\
 &= \int_{\Omega} gw \, dx \\
 &= \int_{\Omega} \left(\sum_{i=1}^q |Du|^{p-2} \langle Du, X_i(u) \rangle DY_i(u) \right) w \, dx. \quad \square
 \end{aligned}$$

3. Energy Decay. Let N be a smooth compact homogeneous space with a left invariant metric. Letting $u \in W^{1,p}(\Omega, N)$ for $B_{\rho}(x) \subset \Omega$, we define

$$(3.1) \quad N(x, \rho, u) = \sup \left\{ \frac{1}{r^{n-p}} \int_{B_r(y)} |Du|^p : B_r(y) \subset B_{\rho}(x) \right\}.$$

Note that for $0 < \sigma \leq \rho$, $N(x, \sigma, u) \leq N(x, \rho, u)$, and so in particular

$$N(x, u) = \lim_{\rho \rightarrow 0} N(x, \rho, u) \text{ exists.}$$

The key to the proof of Theorem 2 is the following decay lemma for the quantity defined above, which should be interpreted as the analogue of the scaled energy for stationary p -harmonic maps.

Decay Lemma 3.1. *There exist constants $\varepsilon_0 > 0$ and $\tau \in (0, \frac{1}{16})$ such that if $u \in W^{1,p}(\Omega, N)$ is a weakly p -harmonic map, then*

$$(3.2) \quad N(x, \rho, u) \leq \varepsilon_0$$

implies

$$(3.3) \quad N(x, \tau \rho, u) \leq \frac{1}{2} N(x, \rho, u)$$

for $x \in \Omega$, $B_{\rho}(x) \subset \Omega$.

Proof. We argue by contradiction, for $\tau > 0$ selected as below. Were the lemma false there would exist balls $B_{\rho_m}(x_m) \subset \Omega$ such that

$$(3.4) \quad N(x_m, \rho_m, u) = \lambda_m^p \rightarrow 0,$$

whereas

$$(3.5) \quad N(x_m, \tau \rho_m, u) > \frac{1}{2} \lambda_m^p.$$

Thus there exists $B_{\sigma_m}(y_m) \subset B_{\tau \rho_m}(x_m)$ such that

$$\frac{1}{\sigma_m^{n-p}} \int_{B_{\sigma_m}(y_m)} |Du|^p \geq \frac{1}{4} \lambda_m^p.$$

Note that $B_{\sigma_m/4\tau}(y_m) \subset B_{\rho_m/2}(x_m)$. For $r_m = \sigma_m/4\tau$ and $\eta = 4\tau \in (0, \frac{1}{4})$, we have

$$(3.6) \quad \frac{1}{r_m^{n-p}} \int_{B_{r_m}(y_m)} |Du|^p \leq \lambda_m^p,$$

and

$$(3.7) \quad \frac{1}{(\eta r_m)^{n-p}} \int_{B_{\eta r_m}(y_m)} |Du|^p \geq \frac{1}{4} \lambda_m^p.$$

We rescale our variables to the unit ball $B_1(0) \subset \mathbf{R}^n$, as follows. For $z \in B_1(0)$,

$$(3.8) \quad v_m(z) = \frac{u(r_m z + y_m) - a_m}{\lambda_m},$$

where

$$a_m = (u)_{y_m, r_m} = \frac{1}{|B_{r_m}(y_m)|} \int_{B_{r_m}(y_m)} u \, dx$$

denotes the average of u over $B_{r_m}(y_m)$. Using (3.6), (3.7) and (3.8) we verify that

$$(3.9) \quad \sup_m \int_{B_1(0)} |v_m|^p \, dz < \infty, \quad \sup_m \int_{B_1(0)} |Dv_m|^p \, dz \leq 1,$$

but

$$(3.10) \quad \sup_m \frac{1}{\eta^{n-p}} \int_{B_\eta(0)} |Dv_m|^p \, dz \geq \frac{1}{4} \quad (m = 1, 2, \dots).$$

The sequence $\{v_m\}_{m=1}^\infty$ is bounded in $W^{1,p}(B_1(0), \mathbf{R}^k)$, and hence there exists a subsequence (which we reindex as necessary and denote also by $\{v_m\}_{m=1}^\infty$) such that

$$(3.11) \quad v_m \rightarrow v \text{ strongly in } L^p(B_1(0), \mathbf{R}^k),$$

$$(3.12) \quad Dv_m \rightharpoonup Dv \text{ weakly in } L^p(B_1(0), M^{k \times n}),$$

$M^{k \times n}$ being the space of the real $k \times n$ matrices.

Next select any smooth function $w : B_1(0) \rightarrow \mathbf{R}^k$ with compact support. Define

$$w_m(y) = w\left(\frac{y - y_m}{r_m}\right) \quad \text{for } y \in B_{r_m}(y_m).$$

Since u is weakly p -harmonic, (1.2) gives

$$\int_{B_{r_m}(y_m)} |Du|^{p-2} Du : Dw_m dy = \int_{B_{r_m}(y_m)} |Du|^{p-2} A(u)(Du, Du) w_m dy.$$

We rescale this identity to the unit ball, and use the fact that the second fundamental form is a bilinear form in order to obtain the equality

$$\begin{aligned} & \int_{B_1(0)} |Dv_m|^{p-2} Dv_m : Dw dz \\ &= \lambda_m \int_{B_1(0)} |Dv_m|^{p-2} A(\lambda_m v_m + a_m)(Dv_m, Dv_m) w dz. \end{aligned}$$

Thus since N is a smooth compact manifold using (3.9) we have

$$(3.13) \quad \left| \int_{B_1(0)} |Dv_m|^{p-2} Dv_m : Dw dz \right| \leq C \lambda_m \int_{B_1(0)} |Dv_m|^p dz \leq C \lambda_m.$$

Note that for each $m = 1, 2, \dots$, v_m satisfies the equation

$$-\operatorname{div}(|Dv_m|^{p-2} Dv_m) = f_m \text{ weakly in } B_1(0),$$

where

$$f_m = \lambda_m |Dv_m|^{p-2} A(\lambda_m v_m + a_m)(Dv_m, Dv_m),$$

and

$$\|f_m\|_{L^1(B_1(0), \mathbf{R}^k)} \leq C \lambda_m \int_{B_1(0)} |Dv_m|^p dz.$$

Hence there exists a further subsequence (which we reindex as necessary and denote also by $\{v_m\}_{m=1}^\infty$) such that $Dv_m \rightarrow Dv$ a.e. in Ω (see [HLM]) and

$$(3.14) \quad |Dv_m|^{p-2} Dv_m \rightharpoonup |Dv|^{p-2} Dv \text{ weakly in } L^{p'}(B_1(0), M^{k \times n}).$$

Letting m go to infinity in (3.13) from (3.14), we deduce

$$\int_{B_1(0)} |Dv|^{p-2} Dv : Dw \, dz = 0.$$

This equality is obtained for all w as above. Thus

$$\operatorname{div}(|Dv|^{p-2} Dv) = 0 \quad \text{weakly in } B_1(0).$$

Hence $v \in C_{\text{loc}}^{1,\alpha}(\Omega)$ (see [U], and [E2] for $p \geq 2$; see [T] for $1 < p < 2$) and

$$(3.15) \quad \|Dv\|_{L^\infty(B_{1/2}(0), M^{k \times n})} \leq C(p, n, \|v\|_{W^{1,p}(\Omega, \mathbf{R}^k)}) < \infty.$$

In particular,

$$(3.16) \quad \frac{1}{\eta^{n-p}} \int_{B_\eta(0)} |Dv|^p \, dz \leq C\eta^p < \frac{1}{4},$$

provided that $\eta \in (0, \frac{1}{4})$ is chosen small enough.

Next we use Lemma 4.2, to be proved in Section 4. This asserts that

$$(3.17) \quad Dv_m \rightarrow Dv \text{ strongly in } L^p(B_{1/4}(0), M^{k \times n}).$$

But then (3.10) forces

$$\frac{1}{\eta^{n-p}} \int_{B_\eta(0)} |Dv|^p \, dz \geq \frac{1}{4},$$

a contradiction to (3.16). □

So far it is not clear where the hypothesis that N is a homogeneous space with a left invariant metric is needed. In fact the proof of the strong convergence (3.17) depends heavily on the structure of the target manifold. We finish this section by giving a characterization of the regular set for a weakly p -harmonic map.

Corollary 3.2. *Let $u \in W^{1,p}(\Omega, N)$ be a weakly p -harmonic map, and assume that $N(x_0, u) < \varepsilon_0$, $x_0 \in \Omega$ with $\varepsilon_0 > 0$ as in the decay lemma. Then u is $C^{1,\alpha}$ in a neighborhood of x_0 , for some $\alpha \in (0, 1)$.*

Proof. Since $N(x_0, u) < \varepsilon_0$ there exists $\rho_0 > 0$ such that $B_{2\rho_0}(x_0) \subset \Omega$, and $N(x_0, 2\rho_0, u) \leq \varepsilon_0$. Then for $x \in B_{\rho_0}(x_0)$, $N(x, \rho_0, u) \leq N(x_0, 2\rho_0, u) \leq \varepsilon_0$, and for $k \geq 1$

$$N(x, \tau^k \rho_0, u) \leq \frac{1}{2^k} N(x, \rho_0, u).$$

We conclude that for $0 < \rho \leq \tau \rho_0$,

$$\frac{1}{\rho^{n-p}} \int_{B_\rho(x)} |Du|^p \leq N(x, \rho, u) \leq C\rho^\gamma,$$

for some $\gamma > 0$ and $C > 0$. It follows that u is uniformly Hölder continuous on $B_{\rho_0}(x_0)$ (see [GT]), and therefore $u \in C^{1,\alpha}(B_{\rho_0}(x_0), N)$ for some $\alpha \in (0, 1)$ (see [D]). $\square$

Corollary 3.3. *Let $u \in W^{1,p}(\Omega, N)$ be a weakly p -harmonic map, then $u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N)$, where $\mathcal{S}$ is the closed set defined by*

$$\mathcal{S} = \{x \in \Omega : N(x, u) \geq \varepsilon_0\},$$

where $\varepsilon_0 > 0$ is as in the decay lemma.

Proof. The regularity is a straight forward consequence of Corollary 3.2. In order to prove that $\mathcal{S}$ is closed it is enough to prove that $N(x, u)$ is an upper semi-continuous function of x . Let $\{x_i\} \subset \Omega$ be a sequence converging to $x \in \Omega$. Since $B_\rho(x_i) \subset B_{\rho+|x-x_i|}(x)$, as long as $B_{\rho+|x-x_i|}(x) \subset \Omega$ we have

$$N(x_i, \rho, u) \leq N(x, \rho + |x - x_i|, u).$$

For $\varepsilon > 0$ there exist $\rho > 0$ and $i_0 \geq 1$ such that

$$N(x, 2\rho, u) < N(x, u) + \varepsilon,$$

and for all $i \geq i_0$, $|x_i - x| < \rho$. Hence for $i \geq i_0$,

$$N(x_i, u) \leq N(x_i, \rho, u) \leq N(x, 2\rho, u) < N(x, u) + \varepsilon.$$

We conclude that

$$\limsup_{i \rightarrow \infty} N(x_i, u) \leq N(x, u). \quad \square$$

Remark 3.4. Unfortunately, in the general case it is impossible to estimate the size of $\mathcal{R} = \Omega \setminus \mathcal{S}$. As shown by Rivière [R], $\mathcal{R}$ may be empty, at least in the case $p = 2$. In Section 5 we estimate the Hausdorff dimension of $\mathcal{S}$ for stationary p -harmonic maps, and for the weak limit of a sequence of stationary p -harmonic maps.

Corollary 3.5. *Suppose that N is as above. Let $u \in W^{1,n}(\Omega, N)$ be a weakly p -harmonic map. Then $u \in C^{1,\alpha}(\Omega, N)$ for some $\alpha \in (0, 1)$.*

Proof. For $B_r(y) \subset B_\rho(x) \subset \Omega$,

$$\frac{1}{r^{n-p}} \int_{B_r(y)} |Du|^p \leq C \left(\int_{B_r(y)} |Du|^n \right)^{p/n} \leq C \left(\int_{B_\rho(x)} |Du|^n \right)^{p/n},$$

and for any $x \in \Omega$

$$N(x, u) \leq C \lim_{\rho \rightarrow 0} \left(\int_{B_\rho(x)} |Du|^n \right)^{p/n} = 0.$$

Therefore $\mathcal{S} = \emptyset$ and $u \in C^{1,\alpha}(\Omega, N)$. □

4. Strong Convergence. All the calculations and assertions in Section 3 are routine, except for the compactness assertion (3.17), which depends crucially on the structure of the target manifold. We proceed to establish that result now.

Let $\vartheta : \mathbf{R}^n \rightarrow \mathbf{R}$ be a smooth cutoff function satisfying

$$0 \leq \vartheta \leq 1, \quad \vartheta = 1 \text{ on } B_{1/4}(0), \quad \vartheta = 0 \text{ on } \mathbf{R}^n \setminus B_{3/8}(0).$$

Lemma 4.1. *The sequence $\{\vartheta v_m\}_{m \geq 1}$ is bounded in $\text{BMO}(\mathbf{R}^n, \mathbf{R}^k)$.*

Proof. We claim that for each $m = 1, 2, \dots$,

$$\begin{aligned} \|\vartheta v_m\|_* &\leq C \sup \left\{ \frac{1}{r^{n-1}} \int_{B_r(x)} |Dv_m| dx : B_r(x) \subset B_{1/2}(0) \right\} \\ &\leq CN \left(0, \frac{1}{2}, v_m \right)^{1/p} \leq CN(0, 1, v_m)^{1/p}. \end{aligned}$$

For a proof of this statment see either [E1] or [B1]. Note that if $B_r(x) \subset B_1(0)$, then $B_{rr_m}(r_mx + y_m) \subset B_{r_m}(y_m)$. Since

$$N(0, 1, v_m) = \sup \left\{ \frac{1}{r^{n-p}} \int_{B_r(x)} |Dv_m|^p : B_r(x) \subset B_1(0) \right\},$$

and

$$\frac{1}{r^{n-p}} \int_{B_r(x)} |Dv_m|^p = \frac{1}{\lambda_m^p} \cdot \frac{1}{(r_m r)^{n-p}} \int_{B_{rr_m}(r_mx + y_m)} |Du|^p,$$

we conclude that

$$N(0, 1, v_m) \leq \frac{1}{\lambda_m^p} N(y_m, r_m, u).$$

Recall that $B_{r_m}(y_m) \subset B_{\rho_m/2}(x_m) \subset B_{\rho_m}(x_m)$, and therefore

$$N(0, 1, v_m) \leq \frac{1}{\lambda_m^p} N(x_m, \rho_m, u) = 1. \quad \square$$

Lemma 4.2. *The rescaled functions $\{Dv_m\}_{m \geq 1}$ converge strongly to Dv in $L^p(B_{1/4}(0), M^{k \times n})$.*

Proof. Since u satisfies equation (2.2), then

$$(4.1) \quad \begin{aligned} & \operatorname{div}(|Dv_m|^{p-2} Dv_m) \\ &= \lambda_m \sum_{\beta=1}^n \sum_{i=1}^q |Dv_m|^{p-2} \langle D_\beta v_m, X_i(\lambda_m v_m + a_m) \rangle \left(\frac{1}{\lambda_m} D_\beta Y_i(\lambda_m v_m + a_m) \right) \end{aligned}$$

weakly in $B_1(0)$ for each $m = 1, 2, \dots$. Since

$$\operatorname{div} \left(\sum_i^q |Du|^{p-2} \langle Du, X_i(u) \rangle \right) = \sum_\alpha^n \sum_i^q D_\alpha (|Du|^{p-2} \langle D_\alpha u, X_i(u) \rangle) = 0$$

weakly in Ω , a simple change of variables shows that

$$\begin{aligned} & \operatorname{div} \left(\sum_{i=1}^q |Dv_m|^{p-2} \langle Dv_m, X_i(\lambda_m v_m + a_m) \rangle \right) \\ &= \sum_{\beta=1}^n \sum_{i=1}^q D_\beta (|Dv_m|^{p-2} \langle D_\beta v_m, X_i(\lambda_m v_m + a_m) \rangle) = 0 \end{aligned}$$

weakly in $B_1(0)$. Therefore, applying the Proposition from [CLMS] we conclude that

$$\begin{aligned} & \sum_{\beta=1}^n \sum_{i=1}^q \int_{B_1(0)} |Dv_m|^{p-2} \langle D_\beta v_m(y), X_i(\lambda_m v_m(y) + a_m) \rangle \\ & \quad \cdot \left(\frac{1}{\lambda_m} D_\beta Y_i(\lambda_m v_m + a_m) \right) \end{aligned}$$

belongs to $\mathcal{H}_{\text{loc}}^1(B_1(0))$, and

$$(4.2) \quad \sup_{m \geq 1} \|h_m\|_{L^1(B_1(0))} \leq C \sup_{m \geq 1} \int_{B_1(0)} |Dv_m|^p dx \leq C.$$

For a compact set $K \subset B_1(0)$ we estimate $\sup_{m \geq 1} \|h_m\|_{\mathcal{H}^1(K)}$ using (2.6) and (3.9),

$$\begin{aligned}
 (4.3) \quad & \sup_{m \geq 1} \|h_m\|_{\mathcal{H}^1(K)} \\
 & \leq C \left\| \sum_{i=1}^q |Dv_m|^{p-2} \langle Dv_m(y), X_i(\lambda_m v_m(y) + a_m) \rangle \right\|_{L^{p'}(B_1(0), \mathbf{R}^n)}^{p'} \\
 & \quad + C \left\| \sum_i^q \left(\frac{1}{\lambda_m} DY_i(\lambda_m v_m + a_m) \right) \right\|_{L^p(B_1(0), \mathbf{R}^n)}^p, \\
 & \sup_{m \geq 1} \|h_m\|_{\mathcal{H}^1(K)} \leq C \|Dv_m\|_{L^p(B_1(0), \mathbf{R}^n)}^p \leq C.
 \end{aligned}$$

Note that $\int \vartheta \neq 0$. Hence for each $m = 1, 2, \dots$, $\vartheta(h_m - \mu_m) \in \mathcal{H}^1(\mathbf{R}^n)$, where

$$(4.4) \quad \mu_m = \frac{\int h_m \vartheta}{\int \vartheta} \quad \text{and} \quad \sup_{m \geq 1} \|\vartheta(h_m - \mu_m)\|_{\mathcal{H}^1(\mathbf{R}^n)} \leq C.$$

Since

$$\operatorname{div}(|Dv_m|^{p-2} Dv_m) = \lambda_m h_m \quad \text{weakly in } B_1(0),$$

and

$$\operatorname{div}(|Dv|^{p-2} Dv) = 0 \quad \text{weakly in } B_1(0),$$

then for any $w \in W_0^{1,p}(B_1(0), \mathbf{R}^k) \cap L^\infty(B_1(0), \mathbf{R}^k)$ we have

$$(4.5) \quad \int_{B_1(0)} (|Dv|^{p-2} Dv - |Dv_m|^{p-2} Dv_m) : Dw \, dx = \lambda_m \int_{B_1(0)} h_m w \, dx.$$

We now insert $w = \vartheta^2(v - v_m)$ into (4.5),

$$\begin{aligned}
 (4.6) \quad & \int_{B_1(0)} \vartheta^2(|Dv|^{p-2} Dv - |Dv_m|^{p-2} Dv_m) : (Dv - Dv_m) \, dx \\
 & + 2 \int_{B_1(0)} \vartheta(|Dv|^{p-2} Dv - |Dv_m|^{p-2} Dv_m) : (D\vartheta \otimes (v - v_m)) \, dx \\
 & = \lambda_m \int_{B_1(0)} h_m \vartheta^2(v - v_m) \, dx.
 \end{aligned}$$

For $a, b \in \mathbf{R}^\ell$ and $p \geq 2$,

$$(4.7) \quad |a - b|^p \leq 2^{p-2}(|a|^{p-2} + |b|^{p-2})|a - b|^2 \leq 2^{p-1}(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b).$$

Applying (4.7)

$$(4.8) \quad \begin{aligned} & \int_{B_1(0)} \vartheta^2 |Dv - Dv_m|^p dx \\ & \leq C \int_{B_1(0)} \vartheta (|Dv|^{p-2}Dv - |Dv_m|^{p-2}Dv_m) |D\vartheta| |v - v_m| dx \\ & \quad + C\lambda_m \left| \int_{B_1(0)} h_m \vartheta^2 (v - v_m) dx \right|. \end{aligned}$$

For $a, b \in \mathbf{R}^\ell$ and $1 < p < 2$ (see [HLM])

$$(4.9) \quad (|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq (p-1)(|a| + |b|)^{p-2}|a - b|^2.$$

Applying (4.9) as in Corollary 2 in [HLM], (3.9), (3.15) and (4.6) we have

$$(4.10) \quad \begin{aligned} & \int_{B_1(0)} \vartheta^2 |Dv - Dv_m|^p dx \\ & \leq \left(\int_{B_1(0)} \vartheta^2 (|Dv| + |Dv_m|)^p dx \right)^{(2-p)/2} \\ & \quad \cdot \left(\frac{1}{p-1} \int_{B_1(0)} \vartheta^2 (|Dv|^{p-2}Dv - |Dv_m|^{p-2}Dv_m) : D(v - v_m) dx \right)^{p/2} \\ & \leq C \left(\int_{B_1(0)} \vartheta (|Dv|^{p-2}Dv - |Dv_m|^{p-2}Dv_m) |D\vartheta| |v - v_m| dx \right)^{p/2} \\ & \quad + C \left(\lambda_m \left| \int_{B_1(0)} h_m \vartheta^2 (v - v_m) dx \right| \right)^{p/2}. \end{aligned}$$

In both cases we have similar terms, we consider each of them separately. Applying (3.9) and Hölder's inequality, we estimate the first term as follows

$$(4.11) \quad \int_{B_1(0)} \vartheta (|Dv|^{p-2}Dv - |Dv_m|^{p-2}Dv_m) |D\vartheta| |v - v_m| dx$$

$$\begin{aligned}
&\leq \left(\int_{B_1(0)} \vartheta^{p'} (|Dv|^{p-1} + |Dv_m|^{p-1})^{p'} dx \right)^{1/p'} \left(\int_{B_1(0)} |D\vartheta|^p |v - v_m|^p dx \right)^{1/p} \\
&\leq C \left(\int_{B_1(0)} (|Dv|^p + |Dv_m|^p) dx \right)^{1/p'} \left(\int_{B_1(0)} |v - v_m|^p dx \right)^{1/p} \\
&\leq C \left(\int_{B_1(0)} |v - v_m|^p dx \right)^{1/p}.
\end{aligned}$$

Note that

$$\begin{aligned}
(4.12) \quad \int_{B_1(0)} h_m \vartheta^2 (v - v_m) dx &= \int_{B_1(0)} \vartheta (h_m - \mu_m) \vartheta (v - v_m) dx \\
&\quad + \int_{B_1(0)} \mu_m \vartheta^2 (v - v_m) dx,
\end{aligned}$$

and

$$\begin{aligned}
(4.13) \quad \left| \int_{B_1(0)} \mu_m \vartheta^2 (v - v_m) dx \right| &\leq |\mu_m| \left(\int_{\mathbf{R}^n} \vartheta \right) \int_{B_1(0)} |v - v_m| dx \\
&\leq C \left(\int_{B_1(0)} |h_m| dx \right) \left(\int_{B_1(0)} |v - v_m| dx \right) \\
&\leq C \left(\int_{B_1(0)} |Dv_m|^p dx \right) \left(\int_{B_1(0)} |v - v_m| dx \right), \\
\left| \int_{B_1(0)} \mu_m \vartheta^2 (v - v_m) dx \right| &\leq C \int_{B_1(0)} |v - v_m| dx.
\end{aligned}$$

In order to estimate $\int_{B_1(0)} \vartheta (h_m - \mu_m) \vartheta (v - v_m) dx$, we use the duality between $\mathcal{H}^1(\mathbf{R}^n)$ and BMO. Note that since v is weakly p -harmonic in $B_1(0)$, $\|Dv\|_{L^\infty(B_{1/2}(0), M^{k \times n})}$ is bounded, see (3.15). Thus

$$\|\vartheta v\|_* \leq CN \left(0, \frac{1}{2}, v \right)^{1/p} \leq C.$$

Using Lemma 4.1, and inequality (2.3) we have

$$\begin{aligned}
(4.14) \quad &\left| \int_{B_1(0)} \vartheta (h_m - \mu_m) \vartheta (v - v_m) dx \right| \\
&\leq \|\vartheta (h_m - \mu_m)\|_{\mathcal{H}^1(\mathbf{R}^n)} \|\vartheta (v - v_m)\|_* \\
&\leq C (\|\vartheta v\|_* + \|\vartheta v_m\|_*) \leq C.
\end{aligned}$$

Putting together (4.8), (4.10), (4.11), (4.12), (4.13) and (4.14) we have for $2 \leq p \leq n$

$$\begin{aligned}
 (4.15) \quad & \int_{B_{1/4}(0)} |Dv - Dv_m|^p dx \\
 & \leq \int_{B_1(0)} \vartheta^2 |Dv - Dv_m|^p dx \\
 & \leq C \left(\int_{B_1(0)} |v - v_m|^p dx \right)^{1/p} + C\lambda_m \left(1 + \int_{B_1(0)} |v - v_m| dx \right),
 \end{aligned}$$

and for $1 < p < 2$,

$$\begin{aligned}
 (4.16) \quad & \int_{B_{1/4}(0)} |Dv - Dv_m|^p dx \\
 & \leq \int_{B_1(0)} \vartheta^2 |Dv - Dv_m|^p dx \\
 & \leq C \left(\int_{B_1(0)} |v - v_m|^p dx \right)^{1/2} + C\lambda_m^{p/2} \left(1 + \int_{B_1(0)} |v - v_m| dx \right)^{p/2}.
 \end{aligned}$$

Since $v_m \rightarrow v$ strongly in $L^p(B_1(0), \mathbf{R}^k)$, and $\lambda_m \rightarrow 0$, letting m go to infinity in (4.15) and (4.16) we conclude that for $1 < p \leq n$,

$$\int_{B_{1/4}(0)} |Dv - Dv_m|^p dx \rightarrow 0,$$

i.e., $Dv_m \rightarrow Dv$ strongly in $L^p(B_{1/4}(0), M^{k \times n})$. □

5. Regularity Properties for some Weakly p -Harmonic Maps. Now we show that under the hypothesis of Theorems 2 and 3, it is possible to estimate the size of the set $\mathcal{S}$ defined in Corollary 3.3.

Theorem 2. *Let (N, h) be a smooth compact homogeneous space, and h a left invariant metric on N . Assume that $u \in W^{1,p}(\Omega, N)$ is a stationary p -harmonic map. Then there exists a closed subset $\mathcal{S} \subset \Omega$ such that*

$$\mathcal{H}^{n-p}(\mathcal{S}) = 0, \quad u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N),$$

for some $\alpha \in (0, 1)$. Here $\mathcal{H}^{n-p}$ denotes the $(n-p)$ -dimensional Hausdorff measure.

Proof. Recall that a stationary p -harmonic map u satisfies the monotonicity inequalities, namely

$$(5.1) \quad \frac{1}{r^{n-p}} \int_{B_r(y)} |Du|^p dy \leq \frac{1}{\rho^{n-p}} \int_{B_\rho(y)} |Du|^p dy,$$

for all concentric balls $B_r(y) \subset B_\rho(y) \subset \Omega$, and

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^{n-p}} \int_{B_\rho(y)} |Du|^p dy \text{ exists.}$$

Moreover if $B_r(y) \subset B_\rho(x) \subset \Omega$, then

$$(5.2) \quad \frac{1}{\rho^{n-p}} \int_{B_\rho(y)} |Du|^p dy \leq 2^{n-p} \frac{1}{(2\rho)^{n-p}} \int_{B_{2\rho}(x)} |Du|^p dy.$$

Putting together (5.1) and (5.2), and taking the supremum over all balls $B_r(y) \subset B_\rho(x)$, we obtain

$$(5.3) \quad N(x, \rho, u) \leq 2^{n-p} \frac{1}{(2\rho)^{n-p}} \int_{B_{2\rho}(x)} |Du|^p dy.$$

Letting ρ go to zero in (5.3)

$$N(x, u) \leq 2^{n-p} \lim_{\rho \rightarrow 0} \frac{1}{\rho^{n-p}} \int_{B_\rho(x)} |Du|^p dy.$$

Hence for $\mathcal{S}$ defined as in Corollary 3.3,

$$\mathcal{S} = \{x \in \Omega : N(x, u) \geq \varepsilon_0\} \subset \left\{x \in \Omega : \lim_{\rho \rightarrow 0} \frac{1}{\rho^{n-p}} \int_{B_\rho(x)} |Du|^p dy \geq \frac{\varepsilon_0}{2^{n-p}}\right\},$$

but

$$\mathcal{H}^{n-p} \left(\left\{x \in \Omega : \lim_{\rho \rightarrow 0} \frac{1}{\rho^{n-p}} \int_{B_\rho(x)} |Du|^p dy > 0\right\} \right) = 0,$$

see [EG], Section 2.4.3. Then

$$\mathcal{H}^{n-p}(\mathcal{S}) = 0,$$

and $u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N)$, for some $\alpha \in (0, 1)$. □

Theorem 3. *Let (N, h) be a smooth compact homogeneous space, and h a left invariant metric on N . Let $u_k \in W^{1,p}(\Omega, N)$ be a sequence of stationary p -harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, N)$. Then there exists a closed subset $\mathcal{S} \subset \Omega$ such that*

$$u \in C^{1,\alpha}(\Omega \setminus \mathcal{S}, N),$$

for some $\alpha \in (0, 1)$, and for any compact set $K \subset \Omega$

$$\mathcal{H}^{n-p}(\mathcal{S} \cap K) < \infty.$$

Proof. Let $\mathcal{S} = \{x \in \Omega : N(x, u) \geq \varepsilon_0\}$ be as in Corollary 3.3, and let K be a compact set in Ω . Since $\mathcal{S}$ is closed then $\mathcal{S} \cap K$ is compact. Let $\delta > 0$, cover $\mathcal{S} \cap K$ by ball $B_{\rho_i}(x_i)$ centered on $\mathcal{S}$, and so that $\rho_i \leq \delta$. Since $\mathcal{S} \cap K$ is compact, there is a disjoint finite subcollection $\{B_{\rho_i}(x_i)\}_{1 \leq i \leq N}$ such that $\mathcal{S} \cap K \subset \bigcup_{i=1}^N B_{5\rho_i}(x_i)$. But since $x_i \in \mathcal{S}$, $N(x_i, u) \geq \varepsilon_0$, there exist $y_i \in B_{\rho_i/2}(x_i)$ and $r_i \in (0, \rho_i/2)$ such that

$$\frac{\varepsilon_0}{2} \leq \frac{1}{r_i^{n-p}} \int_{B_{r_i}(y_i)} |Du|^p \leq \frac{1}{r_i^{n-p}} \liminf_{k \rightarrow \infty} \int_{B_{r_i}(y_i)} |Du_k|^p,$$

where in order to obtain the second inequality we have used the lower semi-continuity of the L^p norm. There exists $k_i \in \mathbf{N}$ such that $\forall k \geq k_i$,

$$\frac{\varepsilon_0}{4} \leq \frac{1}{2} \cdot \frac{1}{r_i^{n-p}} \int_{B_{r_i}(y_i)} |Du|^p \leq \frac{1}{r_i^{n-p}} \int_{B_{r_i}(y_i)} |Du_k|^p,$$

and by applying the monotonicity inequality to each u_k , we have

$$\begin{aligned} \frac{\varepsilon_0}{4} &\leq \frac{1}{r_i^{n-p}} \int_{B_{r_i}(y_i)} |Du_k|^p \leq \left(\frac{2}{\rho_i}\right)^{n-p} \int_{B_{\rho_i/2}(y_i)} |Du_k|^p \\ &\leq \frac{2^{n-p}}{\rho_i^{n-p}} \int_{B_{\rho_i}(x_i)} |Du_k|^p. \end{aligned}$$

Thus, since the sequence u_k is uniformly bounded in $W^{1,p}(\Omega, N)$, for $k \geq \max\{k_i : 1 \leq i \leq N\}$,

$$\sum_i^N \rho_i^{n-p} \leq C(n, \varepsilon_0) \int_{\bigcup_{i=1}^N B_{\rho_i}(x_i)} |Du_k|^p \leq C \int_{\Omega} |Du_k|^p \leq C,$$

and for $\delta > 0$,

$$\mathcal{H}_{5\delta}^{n-p}(\mathcal{S} \cap K) \leq C \sum_i^N (5\rho_i)^{n-p} \leq C$$

which implies $\mathcal{H}^{n-p}(\mathcal{S} \cap K) \leq C(n, \varepsilon_0)$. □

Theorem 4. *Let M be a smooth compact manifold. Let Ω be an open set in $\mathbf{R}^n$ for $n \geq 3$, let $u_k \in W^{1,2}(\Omega, M)$ be a sequence of stationary harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega, M)$. Then u is a harmonic map, and there exists a closed subset $\Sigma \subset \Omega$ such that*

$$u \in C^\infty(\Omega \setminus \Sigma, M),$$

and for any compact set $K \subset \Omega$

$$\mathcal{H}^{n-2}(\Sigma \cap K) < \infty.$$

Let us recall a result of Bethuel's [B1, Theorem I.4],

Theorem ([B1]). *There exists $\varepsilon_1 > 0$ such that if $v \in W^{1,2}(\Omega, M)$ is a stationary harmonic map, and for $B_{r_0}(x_0) \subset \Omega$, v satisfies*

$$\frac{1}{r_0^{n-2}} \int_{B_{r_0}(x_0)} |Dv|^2 < \varepsilon_1^2,$$

then v is smooth on $B_{r_0/2}(x_0)$. Moreover, for $\ell \geq 1$,

$$\|v\|_{C^\ell(B_{r_0/2}(x_0), M)} \leq C(\varepsilon_1, n, k, \ell).$$

Proof of Theorem 4. We define Σ , as in [Sch], by

$$\Sigma = \bigcap_{r>0} \left\{ x \in \Omega : \liminf_{k \rightarrow \infty} \frac{1}{r^{n-2}} \int_{B_r(x)} |Du_k|^2 \geq \varepsilon_1^2 \right\},$$

where $\varepsilon_1 > 0$ is as in Bethuel's Theorem. We claim that Σ is a closed subset of Ω with locally finite $(n-2)$ -dimensional Hausdorff measure, and that $u \in C^\infty(\Omega \setminus \Sigma, M)$.

In fact, let $\{x_i\}$ be a sequence in Σ and assume $x_i \rightarrow x \in \Omega$. Suppose $x \notin \Sigma$, there exists $r > 0$ such that

$$\liminf_{k \rightarrow \infty} \frac{1}{r^{n-2}} \int_{B_r(x)} |Du_k|^2 < \varepsilon_1^2.$$

There exist $\delta > 0$, and a subsequence $\{u_{n_k}\}$ (depending on x , r and δ) such that for each n_k

$$\frac{1}{r^{n-2}} \int_{B_r(x)} |Du_{n_k}|^2 < \varepsilon_1^2 - \delta.$$

Taking i large enough in order to guarantee that $|x - x_i| < r$ and applying the monotonicity inequality (5.1) to each u_{n_k} we have

$$\begin{aligned}
\left(\frac{r}{2}\right)^{2-n} \int_{B_{r/2}(x_i)} |Du_{n_k}|^2 &\leq (r - |x_i - x|)^{2-n} \int_{B_{(r-|x_i-x|)}(x_i)} |Du_{n_k}|^2 \\
&\leq (r - |x_i - x|)^{2-n} \int_{B_r(x)} |Du_{n_k}|^2 \\
&\leq \left(\frac{r}{r - |x - x_i|}\right)^{n-2} \cdot \frac{1}{r^{n-2}} \int_{B_r(x)} |Du_{n_k}|^2 \\
&\leq \left(\frac{r}{r - |x - x_i|}\right)^{n-2} (\varepsilon_1^2 - \delta).
\end{aligned}$$

Choosing i large enough we can insure that for each n_k

$$\left(\frac{r}{2}\right)^{2-n} \int_{B_{r/2}(x_i)} |Du_{n_k}|^2 \leq \left(\frac{r}{r - |x - x_i|}\right)^{n-2} (\varepsilon_1^2 - \delta) < \varepsilon_1^2,$$

which contradicts the fact that $x_i \in \Sigma$. Therefore $x \in \Sigma$ and Σ is a closed subset of Ω .

Let $K \subset \Omega$ be a compact set. For $\delta > 0$ there exists a finite collection of disjoint balls $\{B_{r_i}(x_i)\}_{1 \leq i \leq N}$ centered on Σ , so that $\Sigma \cap K \subset \bigcup_{i=1}^N B_{5r_i}(x_i)$, and

$$\liminf_{k \rightarrow \infty} \frac{1}{r_i^{n-2}} \int_{B_{r_i}(x_i)} |Du_k|^2 \geq \varepsilon_1^2.$$

Hence for each $i = 1, \dots, N$, there exists a subsequence $\{u_{i_k}\}$ (depending on x_i , r_i and ε_1) such that for each i_k

$$\frac{1}{r_i^{n-2}} \int_{B_{r_i}(x_i)} |Du_{i_k}|^2 \geq \frac{\varepsilon_1^2}{2}.$$

It is possible by taking successive subsequences to construct a subsequence $\{u_{n_k}\}$ so that for each $i = 1, \dots, N$,

$$\frac{1}{r_i^{n-2}} \int_{B_{r_i}(x_i)} |Du_{n_k}|^2 \geq \frac{\varepsilon_1^2}{2}.$$

Then

$$\begin{aligned}
\mathcal{H}_{5\delta}^{n-2}(\Sigma \cap K) &\leq C \sum_{i=1}^N (5r_i)^{n-2} \leq C(n, \varepsilon_1) \sum_{i=1}^N \int_{B_{r_i}(x_i)} |Du_{n_k}|^2 \\
&\leq C(n, \varepsilon_1) \int_{\bigcup_{i=1}^N B_{r_i}(x_i)} |Du_{n_k}|^2 \leq C(n, \varepsilon_1) \int_{\Omega} |Du_{n_k}|^2 \leq C,
\end{aligned}$$

and therefore $\mathcal{H}^{n-2}(\Sigma \cap K) \leq C(n, \varepsilon_1)$.

In order to prove that $u \in C^\infty(\Omega \setminus \Sigma, M)$, let $x \notin \Sigma$. There exists $r > 0$ such that

$$\liminf_{k \rightarrow \infty} \frac{1}{r^{n-2}} \int_{B_r(x)} |Du_k|^2 < \varepsilon_1^2.$$

There is subsequence $\{u_{n_k}\}$ (depending on x , r and ε_1) such that for each n_k

$$\frac{1}{r^{n-2}} \int_{B_r(x)} |Du_{n_k}|^2 < \varepsilon_1^2 - \delta.$$

Bethuel's result guarantees that $u_{n_k} \in C^\infty(B_{r/2}(x), M)$, and

$$\|u_{n_k}\|_{C^\ell(B_{r/2}(x), M)} \leq C(\varepsilon_1, n, \ell).$$

Then $u_{n_k} \rightarrow u$ in $C^2(B_{r/2}(x), M)$, and u is a classical harmonic map from $B_{r/2}(x)$ into M . Elliptic regularity theory guarantees that $u \in C^\infty(B_{r/2}(x), M)$.

In order to prove that u is a harmonic map from Ω into M , we use the fact that a set of finite $(n-2)$ -dimensional Hausdorff measure in $\mathbf{R}^n$ has 2-capacity zero (see [EG] Section 4.7.2). Let $w \in W^{1,2}(\Omega, \mathbf{R}^k) \cap L^\infty(\Omega, \mathbf{R}^k)$ be a test function with compact support. Denote by K the support of w , since $\mathcal{H}^{n-2}(\Sigma \cap K) < \infty$, then $\text{Cap}_2(\Sigma \cap K) = 0$. There exists a sequence $\{\eta_i\}$ of functions in $\mathbf{R}^n$ such that $0 \leq \eta_i \leq 1$,

$$(5.4) \quad \Sigma \cap K \subset \text{int} \{\eta_i = 1\},$$

$$(5.5) \quad \int_{\mathbf{R}^n} |D\eta_i|^2 \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

in particular,

$$(5.6) \quad \int_{\mathbf{R}^n} \eta_i^{2n/(n-2)} \rightarrow 0 \quad \text{and} \quad \eta_i \rightarrow 0 \text{ almost everywhere } \mathbf{R}^n \text{ as } i \rightarrow \infty.$$

Then

$$\begin{aligned} \int_{\Omega} Du : Dw &= \int_{\Omega} Du : D((1 - \eta_i)w + \eta_i w) \\ &= \int_{\Omega} Du : D((1 - \eta_i)w) + \int_{\Omega} \eta_i Du : Dw + \int_{\Omega} Du : (D\eta_i \otimes w). \end{aligned}$$

From (5.4) we conclude that $(1 - \eta_i)w$ is compactly supported in $\Omega \setminus \Sigma$, and therefore,

$$\int_{\Omega} Du : D((1 - \eta_i)w) = \int_{\Omega} A(u)(Du, Du)(1 - \eta_i)w.$$

By the Dominated Convergence Theorem and (5.6), we have

$$\int_{\Omega} A(u)(Du, Du)\eta_i w \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

$$\int_{\Omega} \eta_i Du : Dw \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Applying (5.5) we have

$$\left| \int_{\Omega} Du : (D\eta_i \otimes w) \right| \leq \|w\|_{L^\infty(\Omega, \mathbf{R}^k)} \left(\int_{\Omega} |Du|^2 \right)^{1/2} \left(\int_{\mathbf{R}^n} |D\eta_i|^2 \right)^{1/2} \rightarrow 0$$

as $i \rightarrow \infty$; therefore,

$$\int_{\Omega} Du : Dw = \int_{\Omega} A(u)(Du, Du)w. \quad \square$$

Remark 5.1. Let N be a smooth compact homogenous space with a left invariant metric. Let $u_k \in W^{1,2}(\Omega, N)$ be a sequence of stationary harmonic maps, and assume that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega, N)$. Then u is a harmonic map. From Theorem 3 we know that $u \in C^\infty(\Omega \setminus \mathcal{S}, N)$, where

$$\mathcal{S} = \{x \in \Omega : N(x, u) \geq \varepsilon_0\},$$

and from Theorem 4 we know that $u \in C^\infty(\Omega \setminus \Sigma, N)$, where

$$\Sigma = \bigcap_{r>0} \left\{ x \in \Omega : \liminf_{k \rightarrow \infty} \frac{1}{r^{n-2}} \int_{B_r(x)} |Du_k|^2 \geq \varepsilon_1^2 \right\}.$$

Note that $\mathcal{S} \subset \Sigma$. In general these two sets are not equal. There is an example, due to R. Hardt, of a sequence of stationary harmonic maps from $B_1(0) \subset \mathbf{R}^3$ into S^2 obtained by rescaling a harmonic map with a singularity at the origin. The limit map is constant and therefore $\mathcal{S} = \emptyset$. On the other hand $\Sigma = \{0\}$.

In some sense Σ measures the failure of the strong convergence of u_k to u in $W^{1,2}(\Omega, M)$. In fact, if $\Omega' \Subset \Omega$ and $\mathcal{H}^{n-2}(\Sigma \cap \bar{\Omega}') = 0$, then $u_k \rightarrow u$ strongly in $W^{1,2}(\Omega', N)$. In particular, we can then assert that u is stationary in Ω' .

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Received: May 24th, 1994.