Characterization of the Besov Spaces via the Commutator Operator of Coifman, Rochberg and Weiss

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ABSTRACT. Let $C_f(g) = T(f \cdot g) - f \cdot T(g)$ where T is the Calderón-Zygmund singular integral operator. It is a well known result of Coifman, Rochberg and Weiss that the boundedness of this operator on $L^p(\mathbf{R}^n)$ is equivalent to f being in the space BMO. In this paper we introduce a version of this operator which allows f to be in the Besov-Lipschitz class $\dot{\Lambda}_\beta$, with $0 < \beta < n$.

0. Introduction. A well known result of Coifman, Rochberg and Weiss ([2], [6], [14]) states that the commutator operator

$$C_f(g) = T(f \cdot g) - f \cdot T(g)$$

(where T is a Calderón-Zygmund singular integral operator) is bounded on some $L^p(\mathbf{R}^n)$, $1 < p < \infty$, if and only if $f \in \text{BMO}$. There are other links between the boundedness properties of the operator C_f and the smoothness of f . A particular case of the result of Janson ([6]) states that $C_f : L^p(\mathbf{R}^n) \rightarrow L^q(\mathbf{R}^n)$ is bounded, $1 < p < q < \infty$, if and only if $f \in \text{Lip } \beta$, $\beta = n(1/p - 1/q)$. Here, $\text{Lip } \beta$ is the homogeneous Lipschitz space determined by the first difference operator. In [14] Uchiyama showed that C_f is a compact operator on some $L^p(\mathbf{R}^n)$ if and only if $f \in \text{CMO}$, a subclass of BMO.

A related type of commutator was introduced by Chanillo ([1]). Let I^α be the Riesz potential of order α , and define

$$C_f^\alpha(g) = I^\alpha(f \cdot g) - f \cdot I^\alpha(g).$$

The result in [1] states that $C_f^\alpha : L^p(\mathbf{R}^n) \rightarrow L^q(\mathbf{R}^n)$ is bounded, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, if and only if $f \in \text{BMO}$. For technical reasons, the “only if” part of the proof in [1] is only valid for α an even integer.

In this paper, we extend and generalize the results from [1] and [6]. Namely, if $1 < p < \infty$, $0 < \beta < 1$, then $f \in \dot{\Lambda}_\beta$ if and only if $C_f : L^p(\mathbf{R}^n) \rightarrow \dot{F}_p^{\beta, \infty}$ is bounded. Here, $\dot{\Lambda}_\beta$ is the homogeneous Lipschitz space, and $\dot{F}_p^{\beta, \infty}$ is the homogeneous Triebel-Lizorkin space. The corresponding result is true for C_f^α . If $1 < p < q < \infty$, $0 < \beta < 1$, $1/p - 1/q = \alpha/n$, then $f \in \dot{\Lambda}_\beta$ if and only if $C_f^\alpha : L^p(\mathbf{R}^n) \rightarrow \dot{F}_q^{\beta, \infty}$ is bounded. If $p < n(\alpha + \beta)$, then $f \in \dot{\Lambda}_\beta$ if and only if $C_f^\alpha : L^p(\mathbf{R}^n) \rightarrow L^r(\mathbf{R}^n)$ is bounded, $1/p - 1/r = (\alpha + \beta)/n$. We note that the methods we employ give a complete proof of the result of Chanillo in [1]. In Section 2 we give a definition of the generalized commutator operators $C_{f,k}$ and $C_{f,k}^\alpha$. The result in [6] implies that if $n(1/p - 1/q) > 1$, then $C_f : L^p(\mathbf{R}^n) \rightarrow L^q(\mathbf{R}^n)$ is bounded if and only if f is a constant. In this case, the characterization of the spaces $\dot{\Lambda}_\beta$ can be obtained using the operators $C_{f,k}$ and $C_{f,k}^\alpha$, $k > n(1/p - 1/q)$. Let $K(x)$ be the Calderón-Zygmund kernel, $K^\alpha(x) = \omega_\alpha |x|^{\alpha-n}$. We then define

$$C_{f,k}g(x) = \int_{\mathbf{R}^n} \Delta_h^k f(x) K(h)g(x+h) dh,$$

$$C_{f,k}^\alpha g(x) = \int_{\mathbf{R}^n} \Delta_h^k f(x) K^\alpha(h)g(x+h) dh,$$

where Δ_h^k is the k^{th} order difference operator with step size h . If $k = 1$, these operators coincide with the original commutator operators. We obtain the following characterization. Let $1 < p < q < \infty$, $1/p - 1/q = \beta/n$, $0 < \beta < k$, k an integer. Then, $C_{f,k} : L^p(\mathbf{R}^n) \rightarrow L^q(\mathbf{R}^n)$ is bounded if and only if $f = f_1 + P$, with $f_1 \in \dot{\Lambda}_\beta$ and P is a polynomial of degree less than k . The same is true for $C_{f,k}^\alpha$ provided $1/p - 1/q = (\alpha + \beta)/n$.

We denote by Δ_h^k the k^{th} difference operator. That is,

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x),$$

$$\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x), \quad k \geq 1,$$

$Q(x, t)$ denotes a cube centered at x , with side length t , sides parallel to the axes. For $c > 0$, cQ denotes the cube with the same center as Q , and the side length increased c times. We denote by x_Q the center of Q . For a cube Q we let

$$f_Q = \frac{1}{|Q|} \int_Q f.$$

For $1 < p < \infty$, p' is the dual, $1/p + 1/p' = 1$, and we use the notation

$$\|f\|_p = \|f\|_{L^p(\mathbf{R}^n)},$$

the usual norm in the Lebesgue space $L^p(\mathbf{R}^n)$. For $\beta > 0$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in \mathbf{R}^n \\ h \neq 0}} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty.$$

In this paper a Calderón-Zygmund singular integral kernel is a function K on $\mathbf{R}^n$ with the following properties:

$$K(x) = \frac{\Omega(x)}{|x|^n}, \quad \int_{\Sigma_{n-1}} \Omega = 0, \quad \Omega \in C^\infty(\Sigma_{n-1}),$$

where Ω is homogeneous of degree 0; that is,

$$\Omega(rx) = \Omega(x) \quad \text{for } r > 0,$$

and Σ_{n-1} denotes the unit sphere in $\mathbf{R}^n$. The Calderón-Zygmund singular integral operator is the operator T , given by the formula

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} K(x-y)f(y) dy.$$

The Riesz potential operator of the order α , $0 < \alpha < n$, denoted by I^α is the convolution operator with the kernel $\omega_\alpha |x|^{\alpha-n}$ where the constant ω_α is such that

$$I^\alpha I^\beta = I^{\alpha+\beta}, \quad \text{provided } \alpha + \beta < n$$

(see [12]). $M(g)$ denotes the Hardy-Littlewood maximal function, and $M_r(g) = (M(|g|^r))^{1/r}$.

Results presented in this paper are part of the author's Ph.D. thesis from Washington University ([9]). The author would like to express his gratitude to his advisors, Professors Mitch Taibleson and Guido Weiss for their assistance. In particular, the idea of studying the Triebel-Lizorkin spaces in the context of the boundedness properties of the commutator operator is due to Professor Weiss.

1. Boundedness properties of the commutator operator. Let $K(x) = \Omega(x)/|x|^n$ be a Calderón-Zygmund singular integral kernel, and let $Tg = K * g$ be the corresponding Calderón-Zygmund singular integral operator. Let $f \in L^1_{\text{loc}} \cap \mathcal{S}'$, and let

$$\begin{aligned} C_f g(x) &= T(f \cdot g)(x) - f(x)Tg(x), \\ C_f^\alpha g(x) &= I^\alpha(f \cdot g)(x) - f(x) \cdot I^\alpha g(x). \end{aligned}$$

We have the following two theorems:

Theorem 1.1 ([2], [6], [14]). *Let $1 < p < \infty$. Then, $f \in \text{BMO}$ if and only if C_f is a bounded operator on $L^p(\mathbf{R}^n)$.*

Theorem 1.2 ([1]). *Let $1 < p < q < \infty$, $(1/p) - (1/q) = \alpha/n$. Then, $f \in \text{BMO}$ if and only if C_f^α is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$.*

The proof of the “if” part of this theorem given in [1] is only valid for α an even integer. Later we will indicate how we can prove it in full generality.

In this section we will prove the following two results.

Theorem 1.3. *Let $1 < p < \infty$, $0 < \beta < 1$. Then, the following conditions are equivalent:*

- (a) $f \in \dot{\Lambda}_\beta$,
- (b) C_f is a bounded operator from $L^p(\mathbf{R}^n)$ to $\dot{F}_p^{\beta, \infty}$,
- (c) C_f is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, $1/p - 1/q = \beta/n$, if $1/p > \beta/n$.

Remark. The equivalence of (a) and (c) was proved in [6].

Theorem 1.4. *Let $1 < p < q < \infty$, $0 < \beta < 1$, $1/p - 1/q = \alpha/n$. Then, the following conditions are equivalent:*

- (a) $f \in \dot{\Lambda}_\beta$,
- (b) C_f^α is a bounded operator from $L^p(\mathbf{R}^n)$ to $\dot{F}_q^{\beta, \infty}$,
- (c) C_f^α is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$, $1/p - 1/r = (\alpha + \beta)/n$, if $1/p > (\alpha + \beta)/n$.

In the four theorems above, the norm of f , in the appropriate function space, is comparable to the operator norm of C_f and C_f^α respectively.

Remark. It is easy to see that in Theorems 1.1 and 1.2 the condition $p > 1$ is essential, even if we replace L^1 by H^1 on the domain side. In other words, it is not true that $f \in \text{BMO}$ implies that $C_f : H^1 \rightarrow L^1$ is bounded or that $C_f^\alpha : H^1 \rightarrow L^q$ is bounded, with $1 - 1/q = \alpha/n$.

We now proceed with proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. (a) $\Rightarrow$ (b): Assume $f \in \dot{\Lambda}_\beta$. This proof follows the ideas of J.-O. Strömberg, as published in [6]. We will use the following characterizations of relevant function spaces:

Lemma 1.5.

(a) For $0 < \beta < 1$, $1 \leq q \leq \infty$, we have:

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f - f_Q|^q \right)^{1/q},$$

for $q = \infty$ the formula should be interpreted appropriately.

(b) For $0 < \beta < 1$, $1 < p < \infty$, we have:

$$\|h\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |h - h_Q| \right\|_p.$$

The proof of the first part of (a) may be found in [3], pages 14 and 38, the proof of the second part of (a) in [8], and the proof of (b) in [11].

Fix a cube $Q = Q(x_Q, s)$, and $x \in Q$. Fix $g \in L^p$ and let $g_1 = g\chi_{2Q}$, $g_2 = g - g_1$. Observe that $C_f g = C_{(f-f_Q)} g$, and we have:

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q |C_f g - (C_f g)_Q| \\ &= \frac{1}{|Q|^{1+\beta/n}} \int_Q |C_{(f-f_Q)} g - (C_{(f-f_Q)} g)_Q| \\ &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |C_{(f-f_Q)} g - T((f-f_Q)g_2)(x_Q)| \\ &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |(f-f_Q)Tg| + \frac{2}{|Q|^{1+\beta/n}} \int_Q |T((f-f_Q)g_1)| \\ &\quad + \frac{2}{|Q|^{\beta/n}} \sup_{y \in Q} |T((f-f_Q)g_2)(y) - T((f-f_Q)g_2)(x_Q)| \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

We estimate these terms individually:

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |(f-f_Q)Tg| &\leq \frac{1}{|Q|^{\beta/n}} \sup_{y \in Q} |f(y) - f_Q| \frac{1}{|Q|} \int_Q |Tg| \\ &\leq C \|f\|_{\dot{\Lambda}_\beta} M(Tg)(x), \end{aligned}$$

where we used Lemma 1.5(a). Thus, $I \leq C \|f\|_{\dot{\Lambda}_\beta} M(Tg)(x)$.

To get II, we need the following modification of a lemma from [6]:

Lemma 1.6. *With the above notation, we have:*

$$\|(f - f_Q)g_1\|_r \leq C|Q|^{1/r+\beta/n}\|f\|_{\dot{\Lambda}_\beta} M_r(g)(x).$$

Proof.

$$\begin{aligned} \frac{1}{|Q|} \int_{\mathbf{R}^n} |f - f_Q|^r |g_1|^r &= C \frac{1}{|2Q|} \int_{2Q} |f - f_Q|^r |g|^r \\ &\leq C \left(\sup_{y \in 2Q} |f(y) - f_Q| \right)^r M_r^r(g)(x) \\ &\leq C|Q|^{r\beta/n} \|f\|_{\dot{\Lambda}_\beta}^r M_r^r(g)(x). \end{aligned} \quad \square$$

Using the boundedness of T , together with the Hölder's inequality, we obtain

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T((f - f_Q)g_1)| &\leq \frac{1}{|Q|^{1+\beta/n}} \|T((f - f_Q)g_1)\|_r \cdot |Q|^{1-1/r} \\ &\leq C|Q|^{-\beta/n-1/r} \|(f - f_Q)g_1\|_r \\ &\leq C\|f\|_{\dot{\Lambda}_\beta} M_r(g)(x). \end{aligned}$$

Thus, $\text{II} \leq C\|f\|_{\dot{\Lambda}_\beta} M_r(g)(x)$.

III: We need the following well known fact (see, e.g., [3], p. 14). Let $Q^* \subset Q$. Then

$$|f_{Q^*} - f_Q| \leq C\|f\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n}.$$

We now estimate III.

$$\begin{aligned} &|T((f - f_Q)g_2)(y) - T((f - f_Q)g_2)(x_Q)| \\ &= \left| \int_{\mathbf{R}^n} (K(y - z) - K(x_Q - z)) (f(z) - f_Q) g_2(z) dz \right| \\ &\leq C \int_{(2Q)^c} \frac{|y - x_Q|}{|x_Q - z|^{n+1}} |f(z) - f_Q| |g(z)| dz \\ &\leq C \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} 2^{-k} |2^k Q|^{-1} (|f(z) - f_{2^k Q}| + |f_{2^k Q} - f_Q|) |g(z)| dz \\ &\leq C \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{\beta/n} \|f\|_{\dot{\Lambda}_\beta} M(g)(x) + C\|f\|_{\dot{\Lambda}_\beta} \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{\beta/n} M(g)(x) \\ &\leq C\|f\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=2}^{\infty} 2^{-k+\beta k} M(g)(x) \\ &= C\|f\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M(g)(x). \end{aligned}$$

Thus, $\text{III} \leq C \|f\|_{\dot{\Lambda}_\beta} M(g)(x)$.

Putting these estimates together, we obtain

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |C_f g - (C_f g)_Q| \leq C \|f\|_{\dot{\Lambda}_\beta} (M(Tg)(x) + M_r(g)(x) + M(g)(x)).$$

We now take the supremum over all Q such that $x \in Q$, and L^p the norm of both sides, and using Lemma 1.5(b) we conclude that

$$\|C_f g\|_{\dot{F}_p^{\beta,\infty}} \leq C \|f\|_{\dot{\Lambda}_\beta} \|g\|_p.$$

Thus, (a) $\Rightarrow$ (b) is proved.

(b) $\Rightarrow$ (a): We are going to show, that there exist s, t , $1 < s, t < \infty$, such that $1/s - 1/t = \beta/n$, and C_f is bounded from $L^s(\mathbf{R}^n)$ to $L^t(\mathbf{R}^n)$, and $\|C_f\|_{L^s \rightarrow L^t} \leq C \|C_f\|_{L^p \rightarrow \dot{F}_q^{\beta,\infty}}$.

Claim 1.7. *There exists β_0 , $0 \leq \beta_0 \leq \beta$ such that $1/p - (\beta - \beta_0)/n > 0$ and $\beta_0/n + 1/p < 1$.*

Proof. For $0 \leq z \leq \beta$, $z \mapsto 1/p - (\beta - z)/n$ is increasing and, if $z = \beta$, it is positive. Thus, there are two possibilities:

- (a) for $z = 0$, $1/p - (\beta - z)/n = 1/p - \beta/n > 0$, in which case we let $\beta_0 = 0$,
- (b) there exists z_0 , $0 \leq z_0 < \beta$ such that $1/p - (\beta - z_0)/n = 0$. Then, $z_0/n + 1/p = \beta/n < 1$. Then, there exists $\beta_0 > z_0$ such that $\beta_0/n + 1/p < 1$, and $1/p - (\beta - \beta_0)/n > 1/p - (\beta - z_0)/n = 0$.

The claim is proved. $\square$

Claim 1.7 gives us a β_0 . If this $\beta_0 = 0$, then $1/p - \beta/n > 0$. Choose $1 < q < \infty$ such that $1/p - 1/q = \beta/n$. Then, by Sobolev-Besov embedding,

$$\|C_f g\|_q = \|C_f g\|_{\dot{F}_q^{0,2}} \leq C \|C_f g\|_{\dot{F}_p^{\beta,\infty}} \leq C \|C_f\|_{L^p \rightarrow \dot{F}_q^{\beta,\infty}} \|g\|_p.$$

Thus, we obtained the pair s, t (p, q) indicated above Claim 1.7. If $\beta_0 > 0$, choose $1 < r < \infty$ such that $1/p - 1/r = (\beta - \beta_0)/n$, and $1 < q < \infty$ such that $1/q - 1/p = \beta_0/n$. Both of these choices are possible by the properties of β_0 . We use the Sobolev-Besov embedding twice, and obtain:

$$\|C_f g\|_{\dot{F}_r^{\beta_0,2}} \leq C \|C_f g\|_{\dot{F}_p^{\beta,\infty}} \leq C \|C_f\|_{L^p \rightarrow \dot{F}_p^{\beta,\infty}} \|g\|_p \leq C \|C_f\|_{L^p \rightarrow \dot{F}_p^{\beta,\infty}} \|g\|_{\dot{F}_q^{\beta_0,2}},$$

so, $C_f : \dot{F}_q^{\beta_0,2} \rightarrow \dot{F}_r^{\beta_0,2}$ is bounded. By duality (see [4]) we get that $C_f^* : \dot{F}_{r'}^{-\beta_0,2} \rightarrow \dot{F}_{q'}^{-\beta_0,2}$ is bounded. Observe that $C_f^* = -C_{\bar{f}}$, so, we have that $C_f :$

$\dot{F}_{r'}^{-\beta_0,2} \rightarrow \dot{F}_q^{-\beta_0,2}$ is bounded. By interpolation (see, again [4]), we get that $C_f : \dot{F}_s^{0,2} \rightarrow \dot{F}_t^{0,2}$ is bounded, with $1/s = (1/q + 1/r')/2 = (1 + \beta/n)/2$, $1/t = (1/q' + 1/r)/2 = (1 - \beta/n)/2$. Thus, we get $C_f : L^s \rightarrow L^t$ is bounded, with $1/s - 1/t = \beta/n$.

Now, it follows that $f \in \dot{\Lambda}_\beta$ (see [6], Example 2), and $\|f\|_{\dot{\Lambda}_\beta} \leq C \|C_f\|_{L^p \rightarrow \dot{F}_p^{\beta,\infty}}$.

Finally, the equivalence of (a) and (c) was proved in [6] (Example 2).

We now turn to the proof of Theorem 1.4:

Proof of Theorem 1.4. (a) $\Rightarrow$ (b): Fix $g \in L^p(\mathbf{R}^n)$ and recall that x_Q denotes the center of a cube Q , and $g_1 = g\chi_{2Q}$, $g_2 = g - g_1$.

$$\begin{aligned} \|C_f^\alpha g\|_{\dot{F}_q^{\beta,\infty}} &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |C_f^\alpha g - (C_f^\alpha g)_Q| \right\|_q \\ &= C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |C_{(f-f_Q)}^\alpha g - (C_{(f-f_Q)}^\alpha g)_Q| \right\|_q \\ &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |C_{(f-f_Q)}^\alpha g - I^\alpha((f-f_Q)g_2)(x_Q)| \right\|_q \\ &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |(f-f_Q)I^\alpha g| \right\|_q \\ &\quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+(\alpha+\beta)/n}} \int_Q |I^\alpha((f-f_Q)g) - I^\alpha((f-f_Q)g_2)(x_Q)| \right\|_p. \end{aligned}$$

The last inequality follows from the following lemma, with β in the place of γ .

Lemma 1.8 ([3], pp. 71–72). *Let $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$. Suppose for each cube Q we have a function h^Q , defined on this cube. Then, for $0 \leq \gamma$:*

$$\left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\gamma/n}} \int_Q |h^Q| \right\|_q \leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\gamma/n+\alpha/n}} \int_Q |h^Q| \right\|_p,$$

where the constant C depends only on p, q, α and n . □

This lemma, as proved in [3], specifies that the functions h^Q have the form $h - h_Q$, but the proof carries over immediately to the more general case above.

Thus,

$$\begin{aligned}
& \|C_f^\alpha g\|_{\dot{F}_q^{\beta,\infty}} \\
& \leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |(f - f_Q)I^\alpha g| \right\|_q \\
& \quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+(\alpha+\beta)/n}} \int_Q |I^\alpha((f - f_Q)g_1)| \right\|_p \\
& \quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(\alpha+\beta)/n}} \sup_{y \in Q} |I^\alpha((f - f_Q)g_2)(y) - I^\alpha((f - f_Q)g_2)(x_Q)| \right\|_p \\
& = \text{I} + \text{II} + \text{III}.
\end{aligned}$$

Now, we make a temporary assumption that $\alpha + \beta < 1$, and estimate each part separately, as in the previous proof.

I: For each $x \in Q$ we get by Lemma 1.5(a),

$$\begin{aligned}
\frac{1}{|Q|^{\beta/n}} \frac{1}{|Q|} \int_Q |(f - f_Q)I^\alpha g| & \leq \frac{1}{|Q|^{\beta/n}} \left(\sup_{y \in Q} |f(y) - f_Q| \right) \left(\frac{1}{|Q|} \int_Q |I^\alpha g| \right) \\
& \leq C \|f\|_{\dot{\Lambda}_\beta} M(I^\alpha g)(x),
\end{aligned}$$

so $I \leq C \|f\|_{\dot{\Lambda}_\beta} \|I^\alpha g\|_q \leq C \|f\|_{\dot{\Lambda}_\beta} \|g\|_p$.

II: Choose r , $1 < r < p$, and $\bar{r}$ such that $(1/r) - (1/\bar{r}) = \alpha/n$. Such $\bar{r}$ exists, since $r < p < n/\alpha$. We proceed as in the previous proof,

$$\begin{aligned}
\frac{1}{|Q|^{1+(\alpha+\beta)/n}} \int_Q |I^\alpha((f - f_Q)g_1)| & \leq \frac{1}{|Q|^{1+(\alpha+\beta)/n}} \|I^\alpha((f - f_Q)g_1)\|_{\bar{r}} |Q|^{1/\bar{r}'} \\
& \leq C |Q|^{-1-\alpha/n-\beta/n+1-1/\bar{r}} \|(f - f_Q)g_1\|_r \\
& = C |Q|^{-1/r-\beta/n} \|(f - f_Q)g_1\|_r \\
& \leq C \|f\|_{\dot{\Lambda}_\beta} M_r(g)(x),
\end{aligned}$$

by Lemma 1.6. Thus, $\text{II} \leq C \|f\|_{\dot{\Lambda}_\beta} \|M_r(g)\|_p \leq C \|f\|_{\dot{\Lambda}_\beta} \|g\|_p$.

III: We proceed as in the previous proof. Let $y \in Q$,

$$\begin{aligned}
& \frac{1}{|Q|^{(\alpha+\beta)/n}} |I^\alpha((f-f_Q)g_2)(y) - I^\alpha((f-f_Q)g_2)(x_Q)| \\
& \leq \frac{C}{|Q|^{(\alpha+\beta)/n}} \int_{(2Q)^c} \frac{|y-x_Q|}{|x_Q-z|^{n+1-\alpha}} |f(z) - f_Q| \cdot |g(z)| dz \\
& \leq \frac{C}{|Q|^{(\alpha+\beta)/n}} \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} 2^{-k} |2^k Q|^{-1+\alpha/n} |f(z) - f_Q| \cdot |g(z)| dz \\
& \leq C \sum_{k=2}^{\infty} 2^{-k+\kappa\alpha+k\beta} \frac{1}{|2^k Q|^{\beta/n}} \cdot \frac{1}{|2^k Q|} \int_{2^k Q} |f(z) - f_{2^k Q}| \cdot |g(z)| dz \\
& \quad + C \sum_{k=2}^{\infty} 2^{-k+\kappa\alpha} \frac{1}{|Q|^{\beta/n}} |2^k Q|^{\beta/n} \|f\|_{\dot{\Lambda}_\beta} \frac{1}{|2^k Q|} \int_{2^k Q} |g(z)| dz \\
& \leq C \|f\|_{\dot{\Lambda}_\beta} M(g)(x).
\end{aligned}$$

So, $\text{III} \leq C \|f\|_{\dot{\Lambda}_\beta} \|g\|_p$. Thus,

$$\|C_f^\alpha g\|_{\dot{F}_q^{\beta,\infty}} \leq C \|f\|_{\dot{\Lambda}_\beta} \|g\|_p.$$

Now, we get rid of the assumption $\alpha + \beta < 1$. For this, we need the following easy observation:

Fact 1.9. *There exist $m, k_0 \geq 0$ integers, and $\delta > 0$ such that $\alpha = \delta + m\beta/2^{k_0}$, with $\beta + \delta < 1$.*

Now, observe that since $\beta + \beta/2^{k_0} < 1$, the part of the theorem already proved says that if $1/u - 1/w = \beta_0/n$, then $C_f^{\beta_0} : L^u \rightarrow \dot{F}_w^{\beta,\infty}$ is bounded, with $\beta_0 = \beta/2^{k_0}$, $1 < u, w < \infty$.

Fact 1.10. *Suppose $C_f^\gamma : L^u \rightarrow \dot{F}_w^{\beta,\infty}$ is bounded for every u, w , $1/u - 1/w = \gamma/n$, $1 < u, w < \infty$, $0 < \gamma < n - \beta_0$. Then $C_f^{\gamma+\beta_0} : L^u \rightarrow \dot{F}_v^{\beta,\infty}$ is bounded for every u, v , $1/u - 1/v = (\gamma + \beta_0)/n$, $1 < u, v < \infty$.*

Proof. Choose $u < n/(\gamma + \beta_0)$, $g \in L^u$.

$$\begin{aligned}
C_f^{\gamma+\beta_0} g &= I^\gamma I^{\beta_0}(fg) - f I^\gamma I^{\beta_0} g \\
&= I^\gamma(I^{\beta_0} fg - f I^{\beta_0} g) + I^\gamma f I^{\beta_0} g - f I^\gamma I^{\beta_0} g \\
&= I^\gamma(C_f^{\beta_0} g) + C_f^\gamma(I^{\beta_0} g) \\
&= Ag + Bg.
\end{aligned}$$

- (A) $C_f^{\beta_0} : L^u \rightarrow \dot{F}_r^{\beta, \infty}$; $1/u - 1/r = \beta_0/n$, so $A : L^u \rightarrow \dot{F}_r^{\beta+\gamma, \infty} \hookrightarrow \dot{F}_v^{\beta, \infty}$, where $1/r - 1/v = \gamma/n$ and thus $1/u - 1/v = (\gamma + \beta_0)/n$.
- (B) $I^{\beta_0} : L^u \rightarrow \dot{F}_u^{\beta_0, 2} \hookrightarrow L^s$; $1/u - 1/s = \beta_0/n$, so $B : L^u \rightarrow \dot{F}_v^{\beta, \infty}$, where $1/s - 1/v = \gamma/n$ and thus $1/u - 1/v = (\beta_0 + \gamma)/n$. The embeddings used are the Sobolev-Besov embeddings ([5]). $\square$

Now, the hypothesis of Fact 1.10 is satisfied for $\gamma = \delta$ (since $\beta + \delta < 1$). We then apply Fact 1.10 m times to increase this γ by β_0 , m times. Thus, by induction, (a) $\Rightarrow$ (b) is proved for $\delta + m\beta_0 = \alpha$. Observe that we also get that $\|C_f^\alpha\|_{L^p \rightarrow \dot{F}_q^{\beta, \infty}} \leq C\|f\|_{\dot{A}_\beta}$.

(b) $\Rightarrow$ (a): Choose a C^∞ function a with the following properties:

$$\text{supp } a \subset Q(0, 3), \quad \int_{\mathbf{R}^n} a = 0,$$

$$|a(x)| \leq 1, \quad \left| \frac{\partial}{\partial x_i} a(x) \right| \leq 1, \quad i = 1, \dots, n,$$

a is radial, and there exists $x \notin B(0, 1)$ with $a(x) \neq 0$.

Here, $B(0, 1)$ is the ball centered at 0 with radius 1. It follows that a is an atom for the cube $Q(0, 1)$ for the space $\dot{F}_{q'}^{-\beta, 1}$, and $\|a\|_{\dot{F}_{q'}^{-\beta, 1}} = 1$ (see [13], pp. 17–18).

We know that $(\dot{F}_{q'}^{-\beta, 1})^* = \dot{F}_q^{\beta, \infty}$ (see [4], p. 60). Define:

$$\nu(x) = \int_{\mathbf{R}^n} |y - x|^{\alpha-n} (\chi_{B(0,1)}(x)a(y) - a(x)\chi_{B(0,1)}(y)) dy.$$

Let $\nu_t(x) = 1/t^n \nu(x/t)$. By change of variables, we get

$$f * \nu_t(x_0) = \omega_\alpha^{-1} t^{-n-\alpha} \int_{\mathbf{R}^n} C_f^\alpha(\chi_{B(x_0, t)})(x) a'(x) dx,$$

where $a'(x) = a((x - x_0)/t)$. So, for $x_0 \in \mathbf{R}^n$,

$$\begin{aligned} (1.1) \quad |f * \nu_t(x_0)| &\leq C t^{-n-\alpha} \|a'\|_{\dot{F}_{q'}^{-\beta, 1}} \|C_f^\alpha(\chi_{B(x_0, t)})\|_{\dot{F}_q^{\beta, \infty}} \\ &\leq C t^{-n-\alpha} t^{\beta+n/q'} \|C_f^\alpha\|_{L^p \rightarrow \dot{F}_q^{\beta, \infty}} \cdot t^{n/p} \\ &= C \|C_f^\alpha\|_{L^p \rightarrow \dot{F}_q^{\beta, \infty}} \cdot t^\beta. \end{aligned}$$

We used the homogeneity properties of the spaces L^p and $\dot{F}_{q'}^{-\beta, 1}$:

$$\|a'\|_{\dot{F}_{q'}^{-\beta, 1}} = t^{\beta+n/q'} \|a\|_{\dot{F}_{q'}^{-\beta, 1}},$$

and

$$\|\chi_{B(x_0, t)}\|_p = t^{n/p} \|\chi_{B(0, 1)}\|_p.$$

Let $\hat{\nu}$ be the Fourier transform of ν .

Lemma 1.11. $\hat{\nu}$ has the Tauberian property; that is, for every $\xi \in \mathbf{R}^n$, $\xi \neq 0$, there exists $t > 0$ such that $\hat{\nu}(t\xi) \neq 0$.

Proof. Observe that ν is radial, and thus $\hat{\nu}$ is radial. So, if there exists ξ such that for every $t > 0$, $\hat{\nu}(t\xi) = 0$, we would have $\hat{\nu} \equiv 0$, and thus $\nu \equiv 0$. By assumption, there exists an open set U with $B(0,1) \cap U = \emptyset$, and $a(x) \neq 0$ for $x \in U$. On the other hand, if $x \notin B(0,1)$,

$$\nu(x) = \int_{\mathbf{R}^n} |y-x|^{\alpha-n} (-a(x)\chi_{B(0,1)}(y)) dy = -a(x) \int_{B(0,1)} |y-x|^{\alpha-n} dy.$$

Thus, $\nu(x) \neq 0$ for $x \in U$. □

Thus, $\hat{\nu}$ has the Tauberian property, and by (1.1),

$$\sup_{t>0} t^{-\beta} \|f * \nu_t\|_{\infty} < C \|C_f^{\alpha}\|_{L^p \rightarrow \dot{F}_q^{\beta,\infty}}.$$

Using Calderón's reproducing formula due to Janson and Taibleson ([7]), we get

$$f = f_0 + P$$

with $f_0 \in \dot{\Lambda}_{\beta}$, $\|f_0\|_{\dot{\Lambda}_{\beta}} \leq C \|C_f^{\alpha}\|_{L^p \rightarrow \dot{F}_q^{\beta,\infty}}$ and P a polynomial. From part (a) $\Rightarrow$ (b) in this proof, we get that $C_{f_0}^{\alpha} : L^p \rightarrow \dot{F}_q^{\beta,\infty}$ is bounded, and thus $C_P^{\alpha} : L^p \rightarrow \dot{F}_q^{\beta,\infty}$ is bounded. Let $P(x) = \sum_{|\gamma| \leq M} a_{\gamma} x^{\gamma}$. We will show that P is actually a constant. Fix a , a smooth atom for the space $\dot{F}_{q'}^{-\beta,1}$, $x_0 \in \mathbf{R}^n$, $s > 0$.

$$\begin{aligned} & \langle C_P^{\alpha} \chi_{Q(tx_0,ts)}, a(\cdot/t) \rangle \\ &= C \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (P(y) - P(x)) |y-x|^{\alpha-n} \chi_{Q(x_0,s)}\left(\frac{y}{t}\right) a\left(\frac{x}{t}\right) dy dx \\ &= C \sum_{i=0}^M \sum_{|\gamma|=i} a_{\gamma} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (y^{\gamma} - x^{\gamma}) |y-x|^{\alpha-n} \chi_{Q(x_0,s)}\left(\frac{y}{t}\right) a\left(\frac{x}{t}\right) dy dx \\ &= C t^{n+\alpha} \sum_{i=0}^M t^i \sum_{|\gamma|=i} a_{\gamma} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (y^{\gamma} - x^{\gamma}) |y-x|^{\alpha-n} \chi_{Q(x_0,s)}(y) a(x) dy dx \\ &= C t^{n+\alpha} \sum_{i=0}^M t^i d_i. \end{aligned}$$

On the other hand,

$$\left| \left\langle C_P^{\alpha} \chi_{Q(tx_0,ts)}, a\left(\frac{\cdot}{t}\right) \right\rangle \right| \leq \|C_P^{\alpha}\|_{L^p \rightarrow \dot{F}_q^{\beta,\infty}} \|\chi_{Q(tx_0,ts)}\|_p \left\| a\left(\frac{\cdot}{t}\right) \right\|_{\dot{F}_{q'}^{-\beta,1}} \leq C t^{\alpha+\beta+n}$$

(s, x_0 fixed). Thus, $|\sum_{i=0}^M t^i d_i| \leq C t^\beta$ with $0 < \beta < 1$, so $d_i \equiv 0$ for every i . In particular,

$$\langle C_{P_i}^\alpha \chi_{Q(x_0, s)}, a \rangle = 0$$

for every i , where P_i is the homogeneous part of P of degree i . Thus,

$$\langle C_{P_i}^\alpha g, h \rangle = 0 \quad \forall g \in L^p, h \in \dot{F}_{q'}^{-\beta, 1}.$$

This means that $\|C_{P_i}^\alpha g\|_{\dot{F}_q^{\beta, \infty}} = 0$, so $C_{P_i}^\alpha g$ is a constant a.e. for every $g \in L^p(\mathbf{R}^n)$.

Let $i > 0$, and suppose $A \subset \{x : 1 < |x| < 2\}$. It is then easy to observe that $C_{P_i}^\alpha \chi_A$ is continuous on the complement of $\{x : (\frac{1}{2}) < |x| < 3\}$. Thus, $C_{P_i}^\alpha \chi_A$ is constant on this set. Let $A = \{x : P_i(x) > 0\} \cap \{x : 1 < |x| < 2\}$, and $B = \{x : P_i(x) < 0\} \cap \{x : 1 < |x| < 2\}$. Since $i > 0$, we have $P_i(0) = 0$.

$$(1.2) \quad C_{P_i}^\alpha \chi_A(0) = C \int_A P_i(y) |y|^{\alpha-n} dy \geq 0,$$

$$(1.3) \quad C_{P_i}^\alpha \chi_B(0) = C \int_B P_i(y) |y|^{\alpha-n} dy \leq 0.$$

Suppose $x_1 \in A, x_2 \in B$.

$$(1.4) \quad C_{P_i}^\alpha \chi_A(tx_1) = C \int_A P_i(y) |y - tx_1|^{\alpha-n} dy - C P_i(tx_1) \int_A |y - tx_1|^{\alpha-n} dy,$$

$$(1.5) \quad C_{P_i}^\alpha \chi_B(tx_2) = C \int_B P_i(y) |y - tx_2|^{\alpha-n} dy - C P_i(tx_2) \int_B |y - tx_2|^{\alpha-n} dy.$$

If $t > 3$, both (1.4) and (1.5) become independent of t , and thus, both (1.4) and (1.5) have limits as $t \rightarrow \infty$, and these limits are equal to (1.2) and (1.3) respectively. In (1.4) and (1.5), the first terms tend to 0 as $t \rightarrow \infty$, and the second terms are negative and positive respectively. Thus, the limit of (1.4) is ≤ 0 and the limit of (1.5) is ≥ 0 . Comparing this with (1.2) and (1.3), we get that (1.2) = (1.3) = 0. Thus, both A and B have measure 0, and being open sets, are empty. So, $P_i(x) \equiv 0$ on $\{x : 1 < |x| < 2\}$, and being a polynomial, $P_i \equiv 0$. Thus, polynomial P is a constant, and therefore $P \in \dot{\Lambda}_\beta$, and thus finally $f \in \dot{\Lambda}_\beta$.

(a) $\Rightarrow$ (c): By definition, if $f \in \dot{\Lambda}_\beta$, then $|f(y) - f(x)| \leq \|f\|_{\dot{\Lambda}_\beta} |y - x|^\beta$. So,

$$|C_f^\alpha g(x)| \leq C \|f\|_{\dot{\Lambda}_\beta} \int_{\mathbf{R}^n} |y - x|^{\alpha+\beta-n} |g(y)| dy = C \|f\|_{\dot{\Lambda}_\beta} I^{\alpha+\beta}(|g|)(x),$$

and $I^{\alpha+\beta} : L^p \rightarrow L^r$ is bounded, provided $1/p - 1/r = (\alpha + \beta)/n$.

(c) $\Rightarrow$ (a): We follow the method in [6]. Choose $z_0 \in \mathbf{R}^n$ such that in $Q(z_0, 2)$, $|x|^{n-\alpha}$ can be written as the absolutely convergent Fourier series, $|x|^{n-\alpha} = \sum_{m=0}^{\infty} a_m e^{i\langle v_m, x \rangle}$, with $\sum |a_m| < \infty$. Choose $x_0 \in \mathbf{R}^n$, $t > 0$, and let $Q = Q(x_0, t)$, $Q^{z_0} = Q(x_0 + z_0 t, t)$.

$$\begin{aligned} & \frac{1}{|Q|} \frac{1}{|Q^{z_0}|} \int_Q \left| \int_{Q^{z_0}} (f(y) - f(x)) dy \right| dx \\ &= \frac{1}{t^{2n}} \int_Q s(x) \left(\int_{Q^{z_0}} (f(y) - f(x)) |y - x|^{\alpha-n} |y - x|^{n-\alpha} dy \right) dx \end{aligned}$$

where $s(x) = \overline{\text{sgn}(\int_{Q^{z_0}} (f(y) - f(x)) dy)}$. For $x \in Q$, $y \in Q^{z_0}$ we have $(y - x)/t \in Q(z_0, 2)$, so this expression is equal to

$$\begin{aligned} & \frac{t^{n-\alpha}}{t^{2n}} \int_Q s(x) \left(\int_{Q^{z_0}} (f(y) - f(x)) |y - x|^{\alpha-n} \left| \frac{y - x}{t} \right|^{n-\alpha} dy \right) dx \\ &= t^{-n-\alpha} \sum_{m=0}^{\infty} a_m \int_Q s(x) \left(\int_{Q^{z_0}} (f(y) - f(x)) |y - x|^{\alpha-n} e^{i\langle v_m, y/t \rangle} dy \right) e^{-i\langle v_m, x/t \rangle} dx \\ &= Ct^{-n-\alpha} \left| \sum_{m=0}^{\infty} a_m \int_{\mathbf{R}^n} C_f^\alpha(\chi_{Q^{z_0}} e^{i\langle v_m, \cdot/t \rangle})(x) \chi_Q(x) e^{-i\langle v_m, x/t \rangle} s(x) dx \right| \\ &\leq Ct^{-n-\alpha} \sum_{m=0}^{\infty} |a_m| \|C_f^\alpha(\chi_{Q^{z_0}} e^{i\langle v_m, \cdot/t \rangle})\|_r \cdot \|\chi_Q\|_{r'} \\ &\leq Ct^{-n-\alpha} \|C_f^\alpha\|_{L^p \rightarrow L^r} t^{n/p} t^{n-n/r} \sum_{m=0}^{\infty} |a_m| \\ &= Ct^\beta \|C_f^\alpha\|_{L^p \rightarrow L^r}. \end{aligned}$$

We conclude that

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |f - f_{Q^{z_0}}| \leq C \|C_f^\alpha\|_{L^p \rightarrow L^r}.$$

From Lemma 1.5(a) we conclude that $f \in \dot{\Lambda}_\beta$, and $\|f\|_{\dot{\Lambda}_\beta} \leq C \|C_f^\alpha\|$. Theorem 1.4 is proved. $\square$

2. Generalized commutator operator. Suppose $0 < \beta < k \leq n$, k an integer (in particular, $k = [\beta] + 1$), and n the dimension of the ambient space. Let us define

$$\begin{aligned} C_{f,k} g(x) &= \int_{\mathbf{R}^n} \Delta_{h/k}^k f(x) K(h) g(x+h) dh, \\ C_{f,k}^\alpha g(x) &= \int_{\mathbf{R}^n} \Delta_{h/k}^k f(x) K^\alpha(h) g(x+h) dh. \end{aligned}$$

We have the following two theorems, which generalize the theorems from the previous section, where we considered the particular case $k = 1$.

Theorem 2.1. *Let $1 < p < q < \infty$; $(1/p) - (1/q) = \beta/n$. Under the assumptions above, the following are equivalent:*

- (a) $f = f_1 + P$, with $f_1 \in \dot{\Lambda}_\beta$ and P a polynomial of degree less than k .
- (b) $C_{f,k} : L^p \rightarrow L^q$ is a bounded operator.

Theorem 2.2. *Let $1 < p < q < \infty$; $(1/p) - (1/q) = (\alpha + \beta)/n$. Under the assumptions above, the following are equivalent:*

- (a) $f = f_1 + P$, with $f_1 \in \dot{\Lambda}_\beta$ and P a polynomial of degree less than k .
- (b) $C_{f,k}^\alpha : L^p \rightarrow L^q$ is a bounded operator.

If, in particular, $k = [\beta] + 1$, then (a) in both theorems says that $f \in \dot{\Lambda}_\beta$.

Proof of Theorem 2.1. (a) $\Rightarrow$ (b): If the degree of P is less than k , then $\Delta_h^k P = 0$. Thus, we need to show that if $f \in \dot{\Lambda}_\beta$, then $C_{f,k}$ is bounded from L^p to L^q . If $f \in \dot{\Lambda}_\beta$, then $|\Delta_h^k f(x)| \leq \|f\|_{\dot{\Lambda}_\beta} \cdot |h|^\beta$, so

$$|C_{f,k} g(x)| \leq C \|f\|_{\dot{\Lambda}_\beta} \int_{\mathbf{R}^n} \frac{|h|^\beta}{|h|^n} |g(x+h)| dh = C \|f\|_{\dot{\Lambda}_\beta} I^\beta(|g|)(x).$$

Since I^β is bounded from L^p to L^q , (b) follows.

(b) $\Rightarrow$ (a): We are going to use the characterization of the Besov spaces from [10].

Theorem 2.3 ([10]). *Let $0 < \beta < k$, with k an integer. Suppose $f \in \mathcal{S}' \cap L_{\text{loc}}^1$. The following are then equivalent:*

- (a) $f = f_1 + P$, where $f_1 \in \dot{B}_{\infty}^{\beta,\infty}$ and P is a polynomial of degree less than k .
- (b) There exists $z_0 \in \mathbf{R}^n$ such that

$$\sup_{t>0} t^{-\beta} \sup_{x_0 \in \mathbf{R}^n} \frac{1}{|Q|} \frac{1}{|Q^{z_0}|} \left(\int_Q \left| \int_{Q^{z_0}} (\Delta_{(y-x)/k}^k f(x)) dy \right| dx \right) \leq C < \infty,$$

where $Q = Q(x_0, t)$, and $Q^{z_0} = Q(x_0 + z_0 t, t)$.

If these conditions hold, then $\|f_1\|_{\dot{B}_{\infty}^{\beta,\infty}}$ is comparable with the best possible C in (b).

We are going to show that condition (b) of Theorem 2.3 holds. We follow the proof of the last part of Theorem 1.4. We choose z_0 such that $1/K(x) = \sum_{m=0}^{\infty} a_m e^{i\langle v_m, x \rangle}$ with $\sum |a_m| < \infty$, if $x \in Q(z_0, 2)$. We choose $x_0 \in \mathbf{R}^n$, $t > 0$, and we let $Q = Q(x_0, t)$, $Q^{z_0} = Q(x_0 + z_0 t, t)$. Then,

$$\begin{aligned}
& \left| \frac{1}{|Q|} \frac{1}{|Q^{z_0}|} \int_Q \int_{Q^{z_0}} (\Delta_{y-x}^k f(x)) dy dx \right| \\
&= \frac{1}{t^{2n}} \left| \int_Q \int_{Q^{z_0}} [\Delta_{y-x}^k f(x)] K(y-x) \frac{1}{K(y-x)} dy dx \right| \\
&= t^{-n} \left| \sum_{m=0}^{\infty} a_m \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} (\Delta_{y-x}^k f(x)) K(y-x) \chi_{Q^{z_0}}(y) e^{i\langle v_m, y \rangle/t} dy \right) \right. \\
&\quad \left. \times \chi_Q(x) e^{-i\langle v_m, x \rangle/t} dx \right| \\
&\leq t^{-n} \sum_{m=0}^{\infty} |a_m| \|C_{f,k}(\chi_{Q^{z_0}} \cdot e^{i\langle v_m, \cdot \rangle/t})\|_q \|\chi_Q\|_{q'} \\
&\leq t^{-n} \|C_{f,k}\|_{L^p \rightarrow L^q} |Q^{z_0}|^{1/p} |Q|^{1-1/q} \sum_{m=0}^{\infty} |a_m| \\
&= C \|C_{f,k}\|_{L^p \rightarrow L^q} t^\beta.
\end{aligned}$$

Thus, condition (b) of Theorem 2.3 holds. □

The proof of Theorem 2.2 is identical.

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Received: October 19, 1993.