Projections on Spaces of Holomorphic Functions in Balls

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I. Introduction. Throughout this paper, n will be a fixed positive integer, and $\mathbf{C}^n$ will be the vector space of all ordered n -tuples $z = (z_1, \dots, z_n)$ of complex numbers, with inner product

$$(1) \quad \langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$$

and norm

$$(2) \quad |z| = \langle z, z \rangle^{1/2}.$$

The unit ball B of $\mathbf{C}^n$ is then the set of all $z \in \mathbf{C}^n$ with $|z| < 1$. The class of all holomorphic functions with domain B will be denoted by $H(B)$. The letter ν will stand for Lebesgue measure on $\mathbf{C}^n = \mathbf{R}^{2n}$, normalized so that $\nu(B) = 1$, and $L^p(B)$ will stand for the usual Lebesgue spaces of functions with domain B , relative to the measure ν .

We associate to each complex number

$$(3) \quad s = \sigma + it \quad (\sigma > -1, -\infty < t < \infty)$$

a kernel K_s , by

$$(4) \quad K_s(z, w) = \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{n+1+s}} \quad (z \in B, w \in B).$$

Our object is to study the operators T_s that these kernels induce, where (for suitable f) $T_s f$ is defined by

$$(5) \quad (T_s f)(z) = \binom{n+s}{n} \int_B K_s(z, w) f(w) d\nu(w) \quad (z \in B).$$

The complex powers that occur in (4) are of course understood to be the usual principal branches: both numerator and denominator of (4) are 1 when $w = 0$. The binomial coefficient in (5) is $\Gamma(n+s+1)/\Gamma(n+1)\Gamma(s+1)$.

We now state our principal results.

Theorem. (i) For $1 \leq p < \infty$, T_s is a bounded linear operator on $L^p(B)$ if and only if

$$(6) \quad (1 + \sigma)p > 1.$$

If (6) holds, then T_s projects $L^p(B)$ onto $L^p(B) \cap H(B)$. In fact,

$$(7) \quad T_s f = f \quad \text{and} \quad T_s \bar{f} = \overline{f(0)}$$

for every $f \in L^p(B) \cap H(B)$.

(ii) When $\sigma > 0$, then the norm of T_s , regarded as an operator on $L^1(B)$, is

$$(8) \quad \left| \frac{\Gamma(n+1+s)\Gamma(\sigma)}{\Gamma^2\left(\frac{n+1+s}{2}\right)\Gamma(1+s)} \right|.$$

(iii) When $\sigma > -1/2$, then the norm of T_s , regarded as an operator on $L^2(B)$, is

$$(9) \quad \frac{\{\Gamma(1+2\sigma)\}^{1/2}}{|\Gamma(1+s)|}.$$

This will be proved at the end of the paper. However, we shall immediately deduce an interesting corollary from part (i). It furnishes an analogue of Marcel Riesz' theorem about H^p -spaces in the unit disc; note, however, that our result is also valid when $p = 1$. (Korányi and Vági have extended the M. Riesz theorem to H^p -spaces in B ; see Theorem 7.1 of [4]; their theorem seems to be more difficult than ours.)

Corollary. To every p , $1 \leq p < \infty$, corresponds a constant $M_p = M_p(n) < \infty$ with the following property:

If $f \in H(B)$, $f(0) = 0$, $u = \operatorname{Re} f$, and $u \in L^p(B)$, then

$$(10) \quad \int_B |f|^p \, dv \leq M_p \int_B |u|^p \, dv.$$

Proof. Fix p and choose s so that (6) holds. Then there exists $A_p < \infty$ such that $\|T_s g\|_p \leq A_p \|g\|_p$ for every $g \in L^p(B)$. If f is as in the statement of our corollary, define $f_r(z) = f(rz)$, for $0 < r < 1$. Then there is no problem about the existence of $T_s(f_r)$ or of $T_s(\bar{f}_r)$. By (7)

$$(11) \quad f_r = T_s(f_r) = T_s(f_r + \bar{f}_r) = 2T_s(u_r)$$

so that

$$(12) \quad \int_B |f_r|^p \, dv = 2^p \int_B |T_s(u_r)|^p \, dv \leq (2A_p)^p \int_B |u_r|^p \, dv.$$

Now (10) follows if we let $r \rightarrow 1$ in (12).

Remarks. (a) The kernel K_0 is the well-known Bergman kernel for B . [See (e) below for further details.] Our L^2 -result is therefore classical for $s = 0$. Note that the norm (9) is larger than 1 whenever $s \neq 0$ (so that these projec-

tions T_s are not orthogonal ones) and that this norm does not depend on the dimension n . Note also that the Bergman operator T_0 is bounded on every $L^p(B)$ with $1 < p < \infty$, but not on $L^1(B)$.

(b) The above corollary is obviously false for $p = \infty$. Our theorem can therefore not be extended to $L^\infty(B)$.

(c) For $n = 1$, Ahlfors [1; pp. 176, 183] observed that T_2 is a bounded projection on $L^1(B)$, and Shields and Williams [5; p. 295] showed this for all T_s with $s > 0$.

(d) For each n , some s minimizes the L^1 -norm (8). We have not determined the exact minimum.

(e) The kernel K_{n+1} has an interesting geometric interpretation which leads to the reproducing property of the Bergman kernel K_0 .

Fix $w \in B$, choose coordinates in $\mathbb{C}^n$ (by a unitary transformation) so that $w = (r, 0, \dots, 0)$, $r \geq 0$, put $\rho = (1 - r^2)^{1/2}$, and define

$$(13) \quad \Phi(z) = \left(\frac{z_1 - r}{1 - rz_1}, \frac{\rho z_2}{1 - rz_1}, \dots, \frac{\rho z_n}{1 - rz_1} \right).$$

Then Φ is a biholomorphic mapping of B onto B whose Jacobian is easily computed to be $|K_{n+1}(z, w)|$.

Now pick $f \in L^1(B) \cap H(B)$, put $F(z) = f(z)K_0(w, w)/K_0(z, w)$, and put $G = F \circ \Phi^{-1}$. Then

$$(14) \quad f(w) = F(w) = G(0) = \int_B G \, d\nu = \int_B F(z) |K_{n+1}(z, w)| \, d\nu(z).$$

Since

$$(15) \quad |K_{n+1}(z, w)| = \frac{K_0(z, w)K_0(w, z)}{K_0(w, w)},$$

it follows that

$$(16) \quad f(w) = \int_B f(z)K_0(w, z) \, d\nu(z) = (T_0 f)(w).$$

(A more extensive discussion of this and of related topics may be found in sections 7 and 8 of [6].)

(f) Our work on the operators T_s began with a study of the action of T_{n+1} on $L^1(B)$. In that case, the Jacobian property of $|K_{n+1}|$ that was mentioned in (e) shows that

$$(17) \quad \int_B |K_{n+1}(z, w)| \, d\nu(z) = 1$$

for every $w \in B$. It follows that T_{n+1} maps $L^1(B)$ to $L^1(B)$, with norm $\binom{2n+1}{n}$. (This agrees with (8).)

To prove that T_{n+1} has the reproducing property (7) we let P be the orthogonal projection of $\mathbb{C}^{2n+2}$ onto $\mathbb{C}^n$. The integral (5) is then equal to the Cauchy

integral of $f \circ P$ over the unit sphere in $\mathbf{C}^{2n+2}$. (See Proposition 7.2 in [3] for the relevant formula.) From this, (7) can be read off.

Another (even easier) possibility is to show that (5) is equal to the Bergman integral of $f \circ P$ over the unit ball of $\mathbf{C}^{2n+1}$.

These geometric considerations do not seem to apply to the larger class of operators that we are considering now, and we are therefore forced to make our proofs more computational.

II. Preparatory computations.

2.1. Notation. The letter h will be used for the function

$$(18) \quad h(z) = 1 - |z|^2 \quad (z \in \mathbf{C}^n).$$

For $k = 0, 1, 2, \dots$, H_k will denote the class of all members of the polynomial ring $\mathbf{C}[z_1, \dots, z_n]$ that are homogeneous of degree k .

2.2. Proposition. For $F \in L^1(B)$,

$$(19) \quad \int_B F \, d\nu = 2n \int_0^1 r^{2n-1} \, dr \int_S F(rw) \, d\tau(w)$$

where $S = S^{2n-1}$ is the unit sphere in $\mathbf{C}^n$ and τ is the rotation-invariant measure on S which is normalized so that $\tau(S) = 1$. Also,

$$(20) \quad \int_B F \, d\nu = \int_B d\nu(z) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}z) \, d\theta.$$

Proof. (19) is just integration in polar coordinates. To prove (20), apply Fubini's theorem to the right side and note that the transformation $z \rightarrow e^{i\theta}z$ does not affect the measure ν .

2.3. We recall the binomial expansion

$$(21) \quad (1 - \lambda)^{-\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \gamma)}{k! \Gamma(\gamma)} \lambda^k \quad (|\lambda| < 1),$$

valid for $\gamma \neq 0, -1, -2, \dots$. With $\gamma = n + 1 + s$, it follows from (4) and (5) that

$$(22) \quad (T_s f)(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n + k + 1 + s)}{n! k! \Gamma(1 + s)} \int_B h^s(w) \langle z, w \rangle^k f(w) \, d\nu(w).$$

2.4. Proposition. If $f_k \in H_k$, $g_i \in H_i$, $k \neq j$, $\operatorname{Re} \alpha > -1$, and $z \in B$, then

$$(23) \quad \int_B h^\alpha f_k \bar{g}_i \, d\nu = 0,$$

$$(24) \quad f_k(z) = \binom{n+k-1}{n-1} \int_S \langle z, w \rangle^k f_k(w) \, d\tau(w),$$

$$(25) \quad \int_B |f_k|^2 d\nu = \binom{n+k+\alpha}{\alpha} \int_B h^\alpha |f_k|^2 d\nu,$$

$$(26) \quad \int_B h^\alpha(w) \langle z, w \rangle^k f_k(w) d\nu(w) = \frac{n! k! \Gamma(1+\alpha)}{\Gamma(n+k+1+\alpha)} f_k(z),$$

$$(27) \quad \int_B h^\alpha(w) |\langle z, w \rangle|^{2k} d\nu(w) = \frac{n! k! \Gamma(1+\alpha)}{\Gamma(n+k+1+\alpha)} |z|^{2k}.$$

Proof. Since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (h^\alpha f_k \bar{g}_i)(e^{i\theta} z) d\theta = (h^\alpha f_k \bar{g}_i)(z) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-i)\theta} d\theta,$$

(23) follows from (20).

Now apply (22) to f_k , with $s = 0$. Since $T_0 f_k = f_k$, the orthogonality relations (23) show that

$$(28) \quad f_k(z) = \binom{n+k}{n} \int_B \langle z, w \rangle^k f_k(w) d\nu(w).$$

Now (19), applied to (28), gives (24).

Next,

$$\int_B h^\alpha |f_k|^2 d\nu = 2n \int_0^1 (1-r^2)^\alpha r^{2n+2k-1} dr \int_S |f_k|^2 d\tau$$

so that

$$(29) \quad \int_B h^\alpha |f_k|^2 d\nu = \frac{n\Gamma(1+\alpha)\Gamma(n+k)}{\Gamma(n+k+1+\alpha)} \int_S |f_k|^2 d\tau.$$

Let (29') be (29) with 0 in place of α . Division of (29') by (29) gives (25).

Similarly, the left side of (26) is

$$2n \int_0^1 (1-r^2)^\alpha r^{2n+2k-1} dr \int_S \langle z, w \rangle^k f(w) d\tau(w);$$

hence (24) gives (26).

Finally, (27) is a special case of (26), obtained by putting $f_k(w) = \langle w, z \rangle^k$.

2.5. Proposition. *If $\sigma > -1$ and $f_k \in H_k$, then*

$$(30) \quad T_s f_k = f_k,$$

$$(31) \quad T_s \bar{f}_k = \overline{f_k(0)}.$$

Proof. By (22) and (23),

$$(32) \quad (T_s f_k)(z) = \frac{\Gamma(n+k+1+s)}{n! k! \Gamma(1+s)} \int_B h^s(w) \langle z, w \rangle^k f_k(w) d\nu(w).$$

Hence (26) gives (30).

When $k = 0$, (31) follows from (30). When $k > 0$, then $f_k(0) = 0$, and (31) follows from (22) and (20).

The following fact about the gamma function will be needed in Proposition 2.7 and in Section 4.3.

2.6. Proposition. *If $x > 0$, $u > 0$, $-\infty < y < \infty$, then*

$$(33) \quad \frac{|\Gamma(x + iy + u)|^2}{\Gamma(u)\Gamma(2x + u)} \leq 1.$$

For fixed x and y , the left side of (33) converges to 1 as $u \rightarrow +\infty$.

Proof. Euler's integral $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ shows that $|\Gamma(x + iy)| \leq \Gamma(x)$ if $x > 0$. Hence it is enough to prove (33) with $y = 0$. In that case, (33) just states that $\log \Gamma$ is convex on the positive real axis, a well-known fact that also follows directly from Euler's integral. The assertion about the limit is a consequence of Stirling's formula [7; p. 150].

2.7. Proposition. (a) *If $0 < c < 1 + \sigma$, then*

$$(34) \quad \sup_{z \in B} h^c(z) \int_B |K_s(z, w)| h^{-c}(w) dv(w) = A(c, s, n)$$

where $A(c, s, n) = n! \Gamma(1 + \sigma - c) \Gamma(c) / |\Gamma((n + 1 + s)/2)|^2$.

(b) *If $0 < c + \sigma < 1 + \sigma$, then*

$$(35) \quad \sup_{w \in B} h^c(w) \int_B |K_s(z, w)| h^{-c}(z) dv(z) = A(c + \sigma, s, n).$$

(c) *If $\sigma > 0$, then*

$$(36) \quad \sup_{w \in B} \int_B |K_s(z, w)| dv(z) = \frac{n! \Gamma(\sigma)}{\left| \Gamma\left(\frac{n + 1 + s}{2}\right) \right|^2}.$$

(d) *If $-\infty < t < \infty$, then*

$$(37) \quad \sup_{z \in B} \frac{1}{1 + |\log h(z)|} \int_B |K_{it}(z, w)| dv(w) = A_0$$

where $A_0 = n! / |\Gamma((n + 1 + it)/2)|^2$.

Proof. Assume that $c < 1 + \sigma$. Put $n + 1 + s = 2\gamma$. The integrand in (34) is then the square of the absolute value of

$$\frac{h(w)^{(\sigma-c)/2}}{(1 - \langle z, w \rangle)^\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \gamma)}{k! \Gamma(\gamma)} h^{(\sigma-c)/2} \langle z, w \rangle^k.$$

Here (21) was used. Parseval's theorem, combined with (23) and (27), shows therefore that the integral in (34) is equal to

$$(38) \quad \frac{n! \Gamma(1 + \sigma - c)}{|\Gamma(\gamma)|^2} \sum_{k=0}^{\infty} \frac{|\Gamma(k + \gamma)|^2}{\Gamma(n + k + 1 + \sigma - c)} \cdot \frac{|z|^{2k}}{k!}.$$

If $c > 0$, (21) gives

$$(39) \quad h^{-c}(z) = \frac{1}{\Gamma(c)} \sum_{k=0}^{\infty} \Gamma(k + c) \cdot \frac{|z|^{2k}}{k!}.$$

Since $2 \operatorname{Re}(k + \gamma) = (n + k + 1 + \sigma - c) + (k + c)$, Proposition 2.6 shows that no term of the series (38) is larger than the corresponding term in (39), and that their ratio tends to 1 as $k \rightarrow \infty$. Since $h^{-c}(z) \rightarrow \infty$ as $|z| \rightarrow 1$, it follows that the ratio of these two series tends to 1 as $|z| \rightarrow 1$. This proves (a).

The identity

$$(40) \quad |K_s(z, w)| h^{-c}(z) = h^c(w) |K_s(w, z)| h^{-\sigma-c}(z)$$

shows that (b) is a consequence of (a).

When $\sigma > 0$, then $c = 0$ is admissible in (b). This proves (c).

Finally, the computation that led to (38) is valid when $\sigma = c = 0$. The integral in (37) is therefore equal to

$$(41) \quad 1 + \frac{n!}{|\Gamma(\gamma)|^2} \sum_{k=1}^{\infty} \frac{|\Gamma(k + \gamma)|^2}{\Gamma(n + k + 1) \Gamma(k)} \cdot \frac{|z|^{2k}}{k}$$

where now $2\gamma = n + 1 + it$. On the other hand,

$$(42) \quad 1 + |\log h(z)| = 1 + \sum_{k=1}^{\infty} k^{-1} |z|^{2k}.$$

Comparison of these two series shows, as in the proof of (a), that (d) holds.

III. An L^p -inequality.

3.1. Lemma. *Let μ be a positive measure on a measure space X . Suppose $Q : X \times X \rightarrow [0, \infty)$ is measurable, $1 < p < \infty$, $1/p + 1/q = 1$, and suppose that there exists a measurable function $g : X \rightarrow (0, \infty)$ and constants a and b such that the pointwise estimates*

$$(43) \quad \int_X Q(x, y) g(y)^q d\mu(y) \leq [ag(x)]^q \quad (x \in X)$$

and

$$(44) \quad \int_X Q(x, y) g(x)^p d\mu(x) \leq [bg(y)]^p \quad (y \in X)$$

hold. Then the equation

$$(45) \quad (Tf)(x) = \int_X Q(x, y) f(y) d\mu(y)$$

defines a bounded operator T on $L^p(\mu)$, with $\|T\| \leq ab$. (The case $p = 2$ occurs in [2].)

Proof. It suffices to establish the inequality

$$(46) \quad \int_X (Tf)^p d\mu \leq (ab)^p \int_X f^p d\mu$$

for all measurable $f: X \rightarrow [0, \infty)$. Fix such an f , write $Q \cdot f = Q^{1/q} g \cdot Q^{1/p} f/g$, and apply Hölder's inequality and (43) to obtain

$$(Tf)(x) \leq ag(x) \left\{ \int_X Q(x, y) (f/g)^p(y) d\mu(y) \right\}^{1/p}.$$

Hence

$$\int_X (Tf)^p d\mu \leq a^p \int_X (f/g)^p(y) d\mu(y) \int_X Q(x, y) g(x)^p d\mu(x),$$

and (46) follows from (44).

IV. Proof of the theorem.

4.1. The case $p = 1$. If $\sigma < 0$ and $z \in B$, then $K_s(z, w)$ is not a bounded function in B , and therefore $T_s f$ fails to exist for some $f \in L^1(B)$.

So assume $\sigma \geq 0$. The duality between L^1 and L^∞ shows that T_s is a bounded operator on $L^1(B)$ if and only if

$$(47) \quad \sup_{w \in B} \left| \binom{n+s}{n} \right| \int_B |K_s(z, w)| d\nu(z) < \infty$$

and that the norm of T_s on $L^1(B)$ equals the left side of (47) if (47) holds. We see from parts (c) and (d) of Proposition 2.7 that (47) holds if and only if $\sigma > 0$. Moreover, (36) shows that the left side of (47) is equal to (8). Hence part (ii) of the Theorem is proved.

Now assume $\sigma > 0$. Proposition 2.5 shows that $T_s f = f$ and $T_s \bar{f} = \overline{f(0)}$ for all $f \in H(B)$ that are polynomials. Since the polynomials are dense in $L^1(B) \cap H(B)$ and since T_s is continuous on $L^1(B)$, we have proved part (i) of the Theorem for $p = 1$.

4.2. The case $1 < p < \infty$. If $(1 + \sigma)p \leq 1$, then $\sigma q \leq -1$, where $1/p + 1/q = 1$. Hence

$$(48) \quad \int_B |K_s(z, w)|^q d\nu(w) = \infty \quad (z \in B)$$

and $T_s f$ fails to exist, for some $f \in L^p(B)$.

So assume $(1 + \sigma)p > 1$. Then

$$(49) \quad -\frac{\sigma}{p} < \frac{1 + \sigma}{q}.$$

Hence we can choose δ so that

$$(50) \quad 0 < \delta q < 1 + \sigma \quad \text{and} \quad -\sigma < \delta p < 1.$$

If $g(z) = h^{-\delta}(z)$, it follows from (50) and Proposition 2.7(a) that

$$(51) \quad \int_B |K_*(z, w)| g(w)^q d\nu(w) \leq A(\delta q, s, n)g(z)^q \quad (z \in B),$$

and it follows from Proposition 2.7(b) that

$$(52) \quad \int_B |K_*(z, w)| g(z)^p d\nu(z) \leq A(\delta q + \sigma, s, n)g(w)^p \quad (w \in B).$$

Lemma 3.1 can thus be applied, with $Q = |K_*|$. It shows that T_* is a bounded operator on $L^p(B)$. The proof of (7) is now the same as in the case $p = 1$.

4.3. Determination of the L^2 -norm. We now assume that $\sigma > -1/2$. By (23), the spaces $h^{\sigma-i't}H_k$ ($k = 0, 1, 2, \dots$) are pairwise orthogonal in $L^2(B)$. They span a closed subspace Y of $L^2(B)$. By (22), T_* annihilates $Y^\perp$.

Fix $f \in L^2(B)$, and let $Pf = \sum h^{\sigma-i't}f_k$ be its orthogonal projection into Y . Then $T_*f = T_*Pf$. Hence (22), (23), and (26) (with $\alpha = 2\sigma$) imply that

$$\begin{aligned} (T_*f)(z) &= \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1+s)}{n!k!\Gamma(1+s)} \int_B h^{2\sigma}(w) \langle z, w \rangle^k f_k(w) d\nu(w) \\ &= \frac{\Gamma(1+2\sigma)}{\Gamma(1+s)} \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1+s)}{\Gamma(n+k+1+2\sigma)} f_k(z). \end{aligned}$$

By Parseval's theorem and (25) (with $\alpha = 2\sigma$) it follows that

$$\begin{aligned} (53) \quad \int_B |T_*f|^2 d\nu &= \frac{\Gamma(1+2\sigma)}{|\Gamma(1+s)|^2} \sum_{k=0}^{\infty} \frac{|\Gamma(n+k+1+s)|^2}{\Gamma(n+k+1)\Gamma(n+k+1+2\sigma)} \int_B h^{2\sigma} |f_k|^2 d\nu. \end{aligned}$$

Hence (33) gives

$$(54) \quad \int_B |T_*f|^2 d\nu \leq \frac{\Gamma(1+2\sigma)}{|\Gamma(1+s)|^2} \sum_{k=0}^{\infty} \int_B h^{2\sigma} |f_k|^2 d\nu.$$

The sum in (54) is $\int_B |Pf|^2 d\nu \leq \int_B |f|^2 d\nu$. Hence

$$(55) \quad \int_B |T_*f|^2 d\nu \leq \frac{\Gamma(1+2\sigma)}{|\Gamma(1+s)|^2} \int_B |f|^2 d\nu.$$

On the other hand, if $g_k \in H_k$ and $f = h^{\sigma-i't}g_k$, (53) shows that

$$(56) \quad \int_B |T_*f|^2 d\nu = \frac{\Gamma(1+2\sigma)}{|\Gamma(1+s)|^2} \cdot \frac{|\Gamma(n+k+1+s)|^2}{\Gamma(n+k+1)\Gamma(n+k+1+2\sigma)} \int_B |f|^2 d\nu.$$

Now the proof of (iii) is complete, by (55), (56), and Proposition 2.6.

Added in Proof. After completion of the present paper, we became aware of a Research Announcement by E. M. Stein, *Singular integrals and estimates for the Cauchy–Riemann equations*, Bull. Amer. Math. Soc. **79** (1973), 440–445. Theorem 3 of that paper contains part of what we prove: If $1 < p < \infty$, then T_0 is a bounded linear operator on $L^p(B)$.

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