M_3 Functions

RICHARD J. O'MALLEY

Communicated by the Editors

The class M_1 of Baire 1 functions having the Darboux property is the subject of much research. (See for example [1].) This is partially due to the fact that M_1 contains several important types of functions, such as derivatives, approximate derivatives, Peano derivatives, and approximately continuous functions.

In his study [2] of derivatives, Zahorski defines a finite nested sequence of classes of functions of which M_1 is the largest and M_5 , the approximately continuous functions, the smallest. Intermediate is M_3 which he proves contains the derivatives. Further, C. Weil in [3] shows that approximate derivatives and Peano derivatives are in M_3 .

C. J. Neugebauer [4], meanwhile, presents two interesting characterizations of M_1 functions and one characterization of derivatives. These results are especially interesting in that they provide a good indication of how an M_1 function fails to be a derivative. Motivated by this work, in this paper we give necessary and sufficient conditions for a function to be in M_3 . Not only do these conditions provide a view of how M_3 functions differ from M_1 functions but, also, in a precise sense, how M_3 functions fail to be continuous. Finally, we use the results to get simple proofs that derivatives and approximate derivatives are M_3 functions. We will need several definitions. For completeness and comparison we include those of Neugebauer's paper.

Let I_0 be the interval $[0, 1]$ and $\mathcal{J}$ the collection of all non-degenerate closed subintervals of I_0 . We take $\mathcal{A}$ to be the collection of all sets A of the form $J \setminus Z$ where J belongs to $\mathcal{J}$ and Z is of measure zero. Whenever we use the notation J_n or A_n we will require tacitly that J_n is in $\mathcal{J}$ for every n , and A_n is in $\mathcal{A}$ for every n .

Definition 1. A Baire 1 function $f: I_0 \rightarrow \mathbf{R}$ is an M_3 function if for every x_0 and for every $\epsilon > 0$ we have $m(A_n)/d(x_0, A_n) \rightarrow 0$ for every sequence A_n such that i) $m(A_n) \rightarrow 0$ and ii) $A_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset$ for every n . Here m denotes Lebesgue measure and d the distance function.

Definition 2. A sequence J_n is said to converge to x_0 , denoted by $J_n \rightarrow x_0$, if i) $m(J_n) \rightarrow 0$ and ii) x_0 is in J_n for every n .

Definition 3. A sequence of sets L_n is said to w -converge to x_0 , denoted by $L_n w \rightarrow x_0$ if i) $m(L_n) \rightarrow 0$ and ii) there exists $\alpha > 0$ such that $m(L_n) > \alpha d(x_0, L_n)$ for every n .

Definition 4. A function $f: I_0 \rightarrow \mathbf{R}$ is said to have property C_1 if for every J in $\mathcal{J}$ there is a fixed point x_J in J^0 (the interior of J) such that for every x_0 in I_0 , $J_n \rightarrow x_0$ implies that $f(x_{J_n}) \rightarrow f(x_0)$.

Definition 5. A function $f: I_0 \rightarrow \mathbf{R}$ is said to have property C_3 relative to a collection of sets $\mathcal{L}$ if for every L there is a fixed point x_L in L such that for every x_0 in I_0 , $L_n w \rightarrow x_0$ implies that $f(x_{L_n}) \rightarrow f(x_0)$.

Neugebauer has proved that f is an M_1 function if and only if f has property C_1 . He has also established that f is an M_1 function if and only if for every a the sets $\{x: f(x) \geq a\}$ and $\{x: f(x) \leq a\}$ are G_δ 's with compact components. We will need the second characterization later. Now we prove:

Theorem 1. A function $f: I_0 \rightarrow \mathbf{R}$ is an M_3 function if and only if f has the C_3 property relative to $\mathcal{Q}$.

Proof. Suppose first that f has property C_3 relative to $\mathcal{Q}$. Let x_0 and $\epsilon > 0$ be fixed. If A_n is any sequence such that $A_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset$ for all n , we cannot have $f(x_{A_n}) \rightarrow f(x_0)$. Hence A_n cannot w -converge to x_0 , implying that $m(A_n)/d(x_0, A_n) \rightarrow 0$ if $m(A_n) \rightarrow 0$. Also f is a Baire 1 function since if f has the C_3 property relative to $\mathcal{Q}$ it has the C_1 property. Thus f is an M_3 function.

Now, suppose that f is an M_3 function. We will show how to choose x_A . This process will be more detailed than the above argument. More precisely, since f is also an M_1 function, it must be possible to select x_J in J^0 , for each J in $\mathcal{J}$, in such a way that f has property C_1 . The method of selecting x_J , which was developed in [4], will be modified here.

Let f_n be a sequence of continuous functions which converges pointwise to f on I_0 . For each n , it is possible to choose a $1/n > \delta(n) > 0$ for which $m(f_n(I)) < 1/n$ for any interval I with $m(I) < \delta_n$. Further, we may pick closed intervals $J_{n1}, J_{n2}, \dots, J_{nj(n)}, m(J_{ni}) < \delta(n), j = 1, 2, \dots, j(n)$, so that $I = \bigcup_{1 \leq i \leq j(n)} J_{ni}$ and every $0 < x < 1$ is in the interior of some $J_{ni}, j = 1, \dots, j(n)$.

From the set $\mathcal{J}$ we remove all intervals J which are not contained in a J_{ni} for some $n = 1, 2, \dots$ and $j = 1, \dots, j(n)$. We label the remaining set $\mathcal{J}^*$. Similarly, let $\mathcal{Q}^* = \{A: J \text{ in } \mathcal{J}^* \text{ with } m(J \setminus A) = 0\}$. It follows from the conditions on J_{ni} that if $A_n w \rightarrow x_0$ there is an M such that for all $m > M$, A_m is in $\mathcal{Q}^*$.

With each fixed A in $\mathcal{Q}^*$ we associate an integer n , an interval, and two points. For n we select the largest integer n for which there is a J_{ni} containing A . For this n , we pick one such J_{ni} containing A to be the interval associated with A . Then consider $d(f_n(J_{ni}), \overline{f(A)})$ where $\overline{f(A)}$ is the closure of $f(A)$. Since f_n is continuous and J_{ni} is a closed interval there are points y_n in J_{ni} and Z_A

in $\overline{f(A)}$ such that $d(f_n(J_{n_i}), \overline{f(A)}) = d(f_n(y_n), Z_A)$. Let these be the points associated with A . Finally, we choose x_A to be any point of A satisfying $d(f(x_A), Z_A) < 1/n$.

For each fixed A in $\mathcal{Q} \setminus \mathcal{Q}^*$ we select x_A to be any point of A .

Now let x_0 be fixed and A_k be any sequence w -converging to x_0 . There is an $\alpha > 0$ for which $m(A_k) > \alpha d(x_0, A_k)$ for all k . As mentioned previously, for all but finitely many k we have that A_k is in $\mathcal{Q}^*$. Hence we will assume that A_k is in $\mathcal{Q}^*$ for all k . Also, if n_k is the sequence of integers associated with A_k , then $n_k \rightarrow +\infty$ as $k \rightarrow +\infty$.

Denoting by J_{n_k} , y_{n_k} and Z_k the sequences associated with A_k we take I_k to be the smallest interval containing both x_0 and J_{n_k} . We have

$$m(I_k) = m(J_{n_k}) + d(x_0, J_{n_k}) < (1 + \alpha^{-1})m(J_{n_k})$$

since $A_k \subset J_{n_k}$. Hence, since $m(J_{n_k}) < \delta(n_k)$ and $m(f_{n_k}(I)) < 1/n_k$ for any $m(I) < \delta(n_k)$, we have

$$m(f_{n_k}(I_k)) < M/n_k,$$

where M is any fixed integer greater than $1 + \alpha^{-1}$. Therefore, for any x in J_{n_k} , $d(f_{n_k}(x), f_{n_k}(x_0)) < M/n_k$ because x_0 is in I_k and $J_{n_k} \subset I_k$. But $f_{n_k}(x_0) \rightarrow f(x_0)$; thus, we have

$$d(f_{n_k}(J_{n_k}), f(x_0)) \rightarrow 0 \quad \text{and} \quad d(f_{n_k}(y_{n_k}), f(x_0)) \rightarrow 0.$$

Meanwhile, since f is an M_3 function and $A_k w \rightarrow x_0$, $d(\overline{f(A_k)}, f(x_0)) \rightarrow 0$ giving $d(f_{n_k}(y_{n_k}), Z_k) = d(f_{n_k}(J_{n_k}), \overline{f(A_k)}) \rightarrow 0$. Therefore, $d(f(x_0), f(x_{A_k})) \rightarrow 0$ because

$$d(f(x_0), f(x_{A_k})) \leq d(f(x_0), f_{n_k}(y_{n_k})) + d(f_{n_k}(y_{n_k}), Z_k) + d(Z_k, f(x_{A_k})).$$

This completes the proof of Theorem 1.

We now consider some ideas which arise naturally from this theorem and Neugebauer's results.

The initial idea concerns the difference between properties C_1 and C_3 . A study of this difference leads to the following question:

What type of function f would we obtain if we demanded that in every J in $\mathcal{J}$ there was a fixed point x_J in J^0 such that for every x_0 in I_0 , $f(x_{J_n}) \rightarrow f(x_0)$ if $m(J_n) \rightarrow 0$ and $d(x_0, J_n) \rightarrow 0$?

The answer to this question is immediate: the function would have to be continuous. This shows that not only does property C_3 point out the difference between M_1 functions and M_3 functions but also the difference between M_3 functions, such as derivatives, and continuous functions.

We next discuss how we may somewhat weaken the C_3 property relative to $\mathcal{Q}$. For this purpose we must recall a well-known property of derivatives, approximate derivatives and Peano derivatives—the Denjoy–Clarkson property. This property requires that for every $I = (a, b)$ and every J in $\mathcal{J}$:

$$m(J \cap f^{-1}(I)) > 0 \quad \text{or} \quad J \cap f^{-1}(I) = \emptyset.$$

We prove

Theorem 2. *Let $f: I_0 \rightarrow \mathbf{R}$ be a function having the Denjoy–Clarkson property. Then f is an M_3 function if and only if f has the C_3 property relative to $\mathcal{J}$.*

Proof. $(\Rightarrow)$ This is obvious from Theorem 1.

$(\Leftarrow)$ Let x_0 and $\epsilon > 0$ be fixed. Consider any sequence of sets A_n such that i) $A_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset$ for all n and ii) $m(A_n) \rightarrow 0$. With each A_n there is a unique J_n in $\mathcal{J}$ with $A_n \subset J_n$ and $m(J_n \setminus A_n) = 0$. From the Denjoy–Clarkson property we have that

$$J_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset.$$

Hence $f(x_{J_n})$ cannot converge to $f(x_0)$, which implies that J_n does not w -converge to x_0 . But $m(J_n) \rightarrow 0$, so we must have $m(J_n)/d(x_0, J_n) \rightarrow 0$. Since $m(J_n)/d(x_0, J_n) = m(A_n)/d(x_0, A_n)$ we have that f is an M_3 function.

For the next step we must introduce the concept of metric density.

Definition 6. For a measurable set E and a fixed point x_0 we define the upper and lower densities of E at x_0 , $D^-(E, x_0)$, $D_-(E, x_0)$, as follows:

$$D^-(E, x_0) = \limsup_{J \rightarrow x_0} m(E \cap J)/m(J) \quad \text{and}$$

$$D_-(E, x_0) = \liminf_{J \rightarrow x_0} m(E \cap J)/m(J).$$

When $D^-(E, x_0) = D_-(E, x_0) = 1$ we say E has density 1 at x_0 .

Definition 7. A function $f: I_0 \rightarrow k$ is said to have an approximate C_1 property if i) for every x_0 there is a measurable set $E(x_0)$, containing x_0 , having density 1 at x_0 and ii) for every J there is an x_J in J^0 such that $f(x_{J_n}) \rightarrow f(x_0)$ if $J_n \rightarrow x_0$ and J_n has endpoints in $E(x_0)$ for all n .

Definition 8. Similar to Definition 7 we define an approximate C_3 property relative to $\mathcal{A}$. Noting that if A is in $\mathcal{A}$, then $\bar{A}$ is a closed interval, we demand that $f(x_{A_n}) \rightarrow f(x_0)$ if $A_n \rightarrow x_0$ and $\bar{A}_n$ has endpoints in $E(x_0)$.

We are able to show:

Theorem 3. *A function $f: I_0 \rightarrow \mathbf{R}$ is an M_i function $i = 1, 3$ if and only if f has an approximate C_i property $i = 1, 3$.*

Proof. $(\Rightarrow)$ For $i = 1, 3$ the necessity follows from Neugebauer's theorem and Theorem 1.

$(\Leftarrow)$ For $i = 1$ suppose f has an approximate C_1 property. To show that f is an M_1 function, we need only show that f is a Baire 1 function. This is because then f is clearly M_1 according to the second characterization of M_1 functions mentioned after Definition 5.

That f is a Baire 1 function is also immediate from a result by Snyder [6] via the following observation: For $J = [a, b]$ in $\mathcal{J}^*$ let x_J be the point associated with J such that f has the approximate C_1 property. Define $\Phi: \{(u, v), v > u\} \rightarrow \mathbf{R}$ as follows:

$$\Phi(a, b) = f(x_J) \quad \text{where} \quad J = [a, b].$$

This Φ satisfies the hypothesis of Snyder's Theorem 1, and his conclusion gives that f is Baire 1.

Suppose that f has an approximate C_3 property. Then we have that f is an M_1 function.

Let x_0 and $\epsilon > 0$ be fixed and A_n a sequence of sets from $\mathcal{A}$ such that i) $m(A_n) \rightarrow 0$ and ii) $A_n \cap \{x: d(f(x), f(x_0)) < \epsilon\} = \emptyset$ for all n . Assume there is an $\alpha > 0$ such that $m(A_n) > \alpha d(x_0, A_n)$ for all n . Then $U = \bigcup_{1 \leq n < \infty} A_n$ is a measurable set and $D^-(U, x_0) \geq \alpha/(1 + \alpha)$. For, as in Theorem 1, let I_n be the smallest interval containing x_0 and A_n . Then

$$m(I_n) = m(A_n) + d(x_0, A_n) < (1 + \alpha^{-1})m(A_n).$$

Since $E(x_0)$ has density 1 at x_0 , there is an N such that for $n > N$,

$$\begin{aligned} m(E(x_0) \cap I_n) &> \left(\frac{\alpha + 2}{2\alpha + 2}\right)m(I_n) \\ &= \left(\frac{\alpha + 2}{2\alpha + 2}\right)m(A_n) + \left(\frac{\alpha + 2}{2\alpha + 2}\right)d(x_0, A_n). \end{aligned}$$

Then

$$\begin{aligned} m(E(x_0) \cap A_n) &> \left(\frac{\alpha + 2}{2\alpha + 2}\right)m(A_n) + \left(\frac{\alpha + 2}{2\alpha + 2} - 1\right)d(x_0, A_n) \\ &= \left(\frac{\alpha + 2}{2\alpha + 2}\right)m(A_n) - \left(\frac{\alpha}{2\alpha + 2}\right)d(x_0, A_n) \\ &> \left(\frac{\alpha + 2}{2\alpha + 2}\right)m(A) - \left(\frac{1}{2\alpha + 2}\right)m(A_n) \\ &= \frac{1}{2}m(A_n). \end{aligned}$$

Therefore, for $n > N$ we may choose sets $A_n' \subset A_n$ such that A_n' is in $\mathcal{A}$ and $\overline{A_n'}$ has endpoints in $E(x_0)$ and $m(A_n') > 1/2 m(A_n)$. Clearly, $f(x_{A_n'})$ does not converge to $f(x_0)$, but $A_{n_k}'w \rightarrow x_0$ if $A_n w \rightarrow x_0$. This contradiction shows that f is an M_3 function.

Next, the same reasoning as in Theorem 2 allows us to state:

Theorem 4. *Let $f: I_0 \rightarrow \mathbf{R}$ be a function having the Denjoy-Clarkson property. Then f is an M_3 function if and only if f has an approximate C_3 property when $\mathcal{A}$ is restricted to $\mathcal{J}$.*

However, even if a function g has an approximate C_3 property relative to a certain choice of x_A in A for each A in $\mathcal{A}$, we are not assured that g has the C_3 property relative to $\mathcal{A}$ for this choice of x_A , as the following points out.

In each of the intervals $I_n = [1/2^{n+1}, 1/2^n]$, $n = 1, \dots$, we center an interval J_n with $m(J_n) = (1/n)(1/2^{n+1})$. Then $\bigcup_{1 \leq n < \infty} J_n$ has density zero at 0 and $m(J_n)/d(x_0, J_n) \rightarrow 0$. For each n we define f to be 1 on the middle third of J_n , zero at the endpoints of J_n , and linear on the rest of J_n . For all other x we define $f(x)$ to be zero. Then clearly f is an M_3 function and is continuous except at zero; in fact, f is approximately continuous. For I_n we pick x_{I_n} to be any point of I_n for which $f(x_{I_n}) = 1$. For all other A in $\mathcal{A}$ we use the following criteria:

If A contains any point x where $f(x) = 0$, we pick x_A to be such a point.

If A contains no point where $f(x) = 0$, we pick any x of A to be x_A .

Then f has an approximate C_3 property relative to this choice of x_A if we choose $E(0)$ to be any closed set having density 1 at 0 but missing the points $1/2^n$, $n = 1, \dots$. However, since $m(I_n) = d(0, I_n)$, f does not have the C_3 property relative to $\mathcal{A}$.

Finally, we end the paper by showing:

Theorem 5. *If $f: I_0 \rightarrow \mathbf{R}$ is a derivative, then f is an M_3 function.*

Proof. By the remark before Theorem 2 we need only show that f has the C_3 property relative to $\mathcal{J}$. Let F be a continuous function such that $F'(x) = f(x)$ for all x in I_0 . For $J = [a, b] \subset I_0$ we select x_J to be that point of J^0 such that $f(x_J)(b - a) = F(b) - F(a)$. That is, we use the mean value property of derivatives. To show that f has the C_3 property relative to this choice of x_J we check this at $x = 0$ assuming that $F(0) = 0$ and $f(0) = 0$. This will be sufficiently general. Let $I_n = [a_n, b_n]$, $w \rightarrow 0$. Consider

$$\begin{aligned} f(x_{I_n}) &= \frac{F(b_n) - F(a_n)}{b_n - a_n} \\ &= \frac{F(b_n)}{b_n} \cdot \frac{b_n}{b_n - a_n} + \frac{F(a_n)}{a_n} \cdot \frac{(-a_n)}{b_n - a_n}. \end{aligned}$$

Since for all n , $b_n - a_n > (\alpha)(a_n) > 0$ for some $\alpha > 0$, we have

$$0 < \frac{b_n}{b_n - a_n} < 1 + \frac{1}{\alpha}, \quad \frac{-1}{\alpha} < \frac{-a_n}{b_n - a_n} < 0.$$

Also, $F(b_n)/b_n \rightarrow f(0) = 0$, and $F(a_n)/a_n \rightarrow f(0) = 0$. Hence, $f(x_{I_n}) \rightarrow f(0) = 0$.

Similarly we have: If $f: I_0 \rightarrow \mathbf{R}$ is an approximate derivative, then f is an M_3 function.

Proof. Let $F: I_0 \rightarrow \mathbf{R}$ be a function such that for each x_0 in I_0 there is a set $E(x_0)$ having density 1 at x_0 for which

$$\lim_{\substack{x \rightarrow x_0 \\ x \text{ in } E(x_0) \setminus \{x_0\}}} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Now f has the mean value property [5] and the Denjoy-Clarkson property [3]. By Theorem 4 we need only show that f has an approximate C_3 property when $\mathcal{Q}$ is restricted to $\mathcal{J}$.

As in Theorem 5, for $J = [a, b] \subset I_0$ we select x_J in J^0 so that

$$f(x_J) = \frac{F(b) - F(a)}{b - a}.$$

For $E(x)$ we select the same $E(x)$ mentioned above. From this point the proof is exactly the same as in Theorem 5.

REFERENCES

1. A. M. BRUCKNER & J. G. CEDER, *Darboux continuity*, Jber. Deutsch. Math-Verein., **67** (1965), 93-117.
2. Z. ZAHORSKI, *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1-54.
3. C. WEIL, *On properties of derivatives*, Trans. Amer. Math. Soc. **114** (1965), 363-376.
4. C. J. NEUGEBAUER, *Darboux functions of Baire class 1 and derivatives*, Proc. Amer. Math. Soc. **13** (1962), 838-843.
5. C. GOFFMAN & C. J. NEUGEBAUER, *On approximate derivatives*, Proc. Amer. Math. Soc. **11** (1960), 962-966.
6. L. E. SNYDER, *Approximate Stolz angle limits*, Proc. Amer. Math. Soc. **17** (1966), 416-422.

University of Wisconsin, Milwaukee

Date communicated: JANUARY 18, 1974