Existence of Hyperinvariant Subspaces

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Introduction. Let $\mathfrak{X}$ be a complex Banach space, and let S be a bounded linear operator on $\mathfrak{X}$. A subspace (*i.e.* a closed linear manifold) $\mathfrak{M}$ of $\mathfrak{X}$ is *invariant* under S if $S\mathfrak{M} \subset \mathfrak{M}$. $\mathfrak{M}$ is a hyperinvariant subspace of S if it is invariant under any operator commuting with S . A well known unsolved problem is the invariant subspace problem: whether or not a bounded linear operator on a complex Banach space has a non-trivial invariant subspace. Another related unsolved problem is: whether or not a bounded linear operator which is not a multiple of the identity on a complex Banach space has a non-trivial hyperinvariant subspace. It is clear that a positive answer to the second problem will give a positive answer to the first one. While both problems are still unsolved, in quite a few cases (scalar operators, operators commuting with a compact operator, operators with some sort of growth condition on their resolvent, *etc.*) the answer is known to be in the affirmative.

Let T be a bounded linear operator on a complex Hilbert space $\mathfrak{H}$. Macaev and Gohberg–Krein (*cf.* Macaev [10]), and also Schwartz [15], generalizing earlier results of Livsic [9] and Sahnovic [14] proved: if $T = A + iB$, where $A = (T + T^*)/2$, $B = (T - T^*)/2i$, and $B \in \mathfrak{C}_p$ (a Von Neumann–Schatten class), then T has a non-trivial invariant subspace. In 1967, K. Kitano [8] improved this result to the case $T = A + B$ where $B \in \mathfrak{C}_p$ and A is a normal operator with spectrum on a C^2 -Jordan arc. H. Radjavi and P. Rosenthal [13] simplified Kitano's proof and showed that it is enough that $\sigma(A)$ or $\sigma(T)$ contains an exposed C^2 -Jordan arc. In this paper we generalize these results. We prove that if $T = A + B$ where $B \in \mathfrak{C}_p$, $\sigma(A)$ or $\sigma(T)$ contains an exposed C^2 -Jordan arc J , and $\|(\lambda - A)^{-1}\| \leq K [\text{dist}(\lambda, J)]^{-m}$, then T has a non-trivial hyperinvariant subspace. Several corollaries are also derived. Some of our results are closely related to earlier results of C. Apostol. We only became aware of Apostol's work after ours had been completed. Our proofs are different from those of [1] and are more similar to those of [8], [13], [15].

§1. Preliminaries. In this section we quote some definitions, theorems and lemmas which will be necessary for our work in section 2. Throughout the

letters $\mathbf{C}$, $\mathfrak{X}$, and $\mathfrak{H}$ stand for the complex plane, a complex Banach space, and a complex Hilbert space, respectively. The algebra of all bounded linear operators on $\mathfrak{X}$ (respectively, $\mathfrak{H}$) will be denoted by $\mathfrak{B}(\mathfrak{X})$ (respectively, $\mathfrak{B}(\mathfrak{H})$). For an operator $T \in \mathfrak{B}(\mathfrak{X})$, the symbols $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $R(\lambda; T)$ and $\mathfrak{N}(\lambda - T)$ will be used for the resolvent of T , the spectrum of T , the point spectrum of T , $(\lambda - T)^{-1}$ with $\lambda \in \rho(T)$, and the null space of $\lambda - T$, respectively. By an *open* (respectively, *closed*) *Jordan arc* we mean the set of all $z(t) = x(t) + iy(t)$, $t \in (a, b)$, (respectively, $t \in [a, b]$) where $z(t)$ is an injective bicontinuous function. A *Jordan curve* is a subset of $\mathbf{C}$ homeomorphic to the unit circle. If m is a positive integer, then a C^m -*Jordan arc (curve)* is a Jordan arc (curve) for which $z(t)$ has an m^{th} derivative. If $d^m z/dt^m$ is continuous, then the arc (curve) will be called a C^{m+} -*Jordan arc (curve)*.

The following theorem is due to Weyl. For a proof see [7].

1.1. Theorem. Let $T = A + B$ where $A, B, T \in \mathfrak{B}(\mathfrak{H})$ and B is a compact operator. Then $\sigma(T) \subset \sigma(A) \cup \sigma_p(T)$.

Let T be a compact operator on $\mathfrak{H}$. Then TT^* (and hence $(TT^*)^{1/2}$) is a non-negative compact operator. Let $\{\mu_1(T), \dots, \mu_n(T), \dots\}$ be the set of eigenvalues of $(TT^*)^{1/2}$, repeated according to multiplicity, arranged in decreasing order.

1.2. Definition. Let T and $\mu_n(T)$, $n = 1, 2, \dots$, be as above, and $0 < p < \infty$. The *Von Neumann-Schatten ideal* $\mathfrak{C}_p$ of compact operators on $\mathfrak{H}$ is defined as follows: $\mathfrak{C}_p = \{T \in \mathfrak{B}(\mathfrak{H}) : T \text{ is compact and } |T|_p = [\sum_{n=1}^{\infty} (\mu_n(T))^p]^{1/p} < \infty\}$.

It is shown in section XI.9 of [5] that for $p \geq 1$, $|T|_p$ defines a norm on $\mathfrak{C}_p$ which makes it a Banach algebra. It is also a 2-sided ideal in $\mathfrak{B}(\mathfrak{H})$. It is clear that $p_1 < p_2$ implies $\mathfrak{C}_{p_1} \subset \mathfrak{C}_{p_2}$.

1.3. Definition. (XI.9.21, [5]). Let $T \in \mathfrak{C}_k$, $k > 1$ an integer, and $\{\lambda_i\}_{i=1}^{\infty}$ be an enumeration of the eigenvalues of T , repeated according to multiplicity. The *generalized determinant* $\delta_k(T)$ is defined by

$$\delta_k(T) = \prod_{i=1}^{\infty} \left[(1 + \lambda_i) \exp \left(-\lambda_i + \frac{\lambda_i^2}{2} + \dots + (-1)^{k-1} \frac{\lambda_i^{k-1}}{k-1} \right) \right].$$

1.4. Lemma, (XI.9.22, [5]). Let $k > 1$ be an integer and $T \in \mathfrak{C}_k$. Then

- (i) $\delta_k(T)$ is absolutely convergent;
- (ii) $\delta_k(T)$ depends continuously on T on $\mathfrak{C}_k$;
- (iii) there is a constant M_k , depending only on k , such that

$$|\delta_k(T)| \leq \exp (M_k |T|_k^k);$$

- (iv) there is a constant N_k , depending only on k , such that

$$||\delta_k(T)(I + T)^{-1}|| \leq \exp (N_k |T|_k^k)$$

if $-1 \notin \sigma(T)$.

1.5. Lemma, (VII.6.13, [4]). *If $\lambda \rightarrow T(\lambda)$ is an analytic operator-valued function defined on a domain D , then the function $\lambda \rightarrow T^{-1}(\lambda)$ is defined on an open subset of D and is analytic there. If $T(\lambda)$ is compact for every $\lambda \in D$ and if D is connected, then either $I - T(\lambda)$ has a bounded inverse for no point in D or else it has a bounded inverse except at a countable number of isolated points.*

1.6. Lemma, (Lemma 6.7 of [13]). *If $B \in \mathcal{C}_k$ and A is any bounded operator, then $\delta_k [BR(\lambda; A)]$ is an analytic function of λ on $\rho(A)$.*

1.7. Lemma, (Lemma 6.10 of [13]). *Let D be a bounded simply connected region in the complex plane with boundary C . Suppose that C is a C^2 -Jordan curve. Let $z_0 \in C$ and L be any closed line segment with z_0 as an endpoint which lies entirely in D except for the point z_0 and which is not tangent to C . Then, if f is any function analytic and satisfying $\operatorname{Re} f(z) \leq K[\operatorname{dist}(z, C)]^{-p}$ in D , where $p \geq 1$ is an integer and $K > 0$, it follows that there exists a constant $M > 0$ such that, for every $z \in D \cap L$,*

$$|f(z)| \leq M |z - z_0|^{-p-1}.$$

1.8. Definition. Let T be a bounded linear operator on $\mathfrak{X}$. We say $\sigma(T)$ has an *exposed part along an open arc* Γ , if there exists a simply connected open set D such that $\emptyset \neq \sigma(T) \cap D \subset \Gamma$. If $\sigma(T) \cap D = \Gamma$, we say $\sigma(T)$ contains an *exposed open arc*; [13].

1.9. Theorem, (Theorem 6.3 of [13]). *Let $T \in \mathcal{B}(\mathcal{H})$, and suppose that $\sigma(T)$ contains an exposed open C^2 -Jordan arc C . Let k be a positive integer. Suppose that for each point $z_0 \in C$ and each closed line segment L which meets $\sigma(T)$ only in $\{z_0\}$ and which is not tangent to C , there is a constant $K > 0$ such that for every $z \in L \setminus \{z_0\}$ we have*

$$||R(z; T)|| \leq \exp [K |z - z_0|^{-k-1}].$$

Then T has a non-trivial hyperinvariant subspace.

Note. The proof presented in [13] for this theorem holds in a more general situation than the theorem states: it suffices T be a bounded linear operator on a Banach space, the arc C a C^{1+} -Jordan arc, k a function of $z_0 \in C$, and the growth condition on $R(z, T)$ holding on a dense subset of C .

1.10. Definition. Let T be an operator on $\mathfrak{X}$, and J be an open arc along which $\sigma(T)$ has an exposed part. We say the resolvent of T satisfies *growth condition* $G_{m,K}$ ($m > 0$ integer, $K > 0$) near J , if

$$||R(\lambda; T)|| \leq K[\operatorname{dist}(\lambda, J)]^{-m}$$

for all $\lambda \in U \cap \rho(T)$, where $U \supset J$ is an open set (not necessarily containing the endpoints of J).

§2. Existence of hyperinvariant subspaces. We begin with the following theorem.

2.1. Theorem. Assume $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, $\sigma_p(T) = \emptyset$, $\sigma(A)$ contains an exposed open C^2 -Jordan arc J and A has growth condition $G_{m,K}$ near J . Let $z_0 \in J$ and L be any closed line segment which is not tangent to J with z_0 as an endpoint and $L \cap \sigma(T) = \{z_0\}$. Then there is a constant $K' > 0$ such that

$$||R(z; T)|| \leq \exp (K' |z - z_0|^{-mk-1}), \text{ for all } z \in L \setminus \{z_0\}.$$

Proof. We can assume without loss of generality that $k > 1$, for $\mathcal{C}_p \subset \mathcal{C}_q$ if $p \leq q$. Since $\sigma_p(T) = \emptyset$, by Weyl's Theorem, $\sigma(T) \subset \sigma(A)$. Let z_0 and L be as above. Let D be an open disk tangent to J at z_0 , which lies in the open set $U \cap \rho(A)$ and meets L , where U is as in Definition 1.10. By shortening L if necessary we can assume that $L \setminus \{z_0\} \subset D$. Let C denote the boundary of D . Let $\delta(z) = \delta_k(-BR(z; A))$, $z \in \rho(A)$. Then, by Lemma 1.6, δ is analytic on $\rho(A)$. For every $z \in \rho(A)$ we have

$$(1) \quad R(z; T) = R(z; A + B) = R(z; A)(1 - BR(z; A))^{-1}.$$

(Note that $\rho(T) \supset \rho(A)$. The identity can be checked by multiplying each side of (1) by $(z - T)$). This shows that $-1 \notin \sigma(-BR(z; A))$, and then the absolute convergence of δ_k (Lemma 1.4 (i)) implies that $\delta(z) \neq 0$ on $\rho(A)$. Therefore, for every $z \in \rho(A)$,

$$(2) \quad R(z; T) = (\delta(z))^{-1}R(z; A)\delta(z)(1 - BR(z; A))^{-1}.$$

We use the growth condition $G_{m,K}$ to estimate $||R(z; T)||$ by estimating each term in the factorization (2). Using Lemma 1.4 (iii) there is a constant $K_1 > 0$ such that

$$(3) \quad |\delta(z)| \leq \exp (K_1 |BR(z; A)|_k^k), \quad z \in \rho(A).$$

Since $|MN|_k \leq |M|_k|N|$ for $M \in \mathcal{C}_k$ and $N \in \mathcal{B}(\mathcal{H})$, using (3) on D we have

$$(4) \quad |\delta(z)| \leq \exp (K_1 |B|_k^k ||R(z; A)||^k) \leq \exp (K_1 |B|_k^k K^k [\text{dist}(z, J)]^{-mk}) \\ = \exp (K_2 [\text{dist}(z, J)]^{-mk}), \quad K_2 = K_1 |B|_k^k K^k.$$

Since for $z \in D$, $\text{dist}(z, J) \geq \text{dist}(z, C)$, it follows from (4) that

$$(5) \quad |\delta(z)| \leq \exp (K_2 [\text{dist}(z, C)]^{-mk}), \quad z \in D.$$

Now there is an analytic function $\alpha(z)$ on D such that $\exp(\alpha(z)) = \delta(z)$. (This is true simply because $\delta(z)$ is a non-vanishing analytic function in the simply connected region D). Then $|\delta(z)| = \exp(\text{Re } \alpha(z))$, and inequality (5) implies

$$(6) \quad \text{Re } \alpha(z) \leq K_2 [\text{dist}(z, C)]^{-mk}, \quad z \in D.$$

At this point we use Lemma 1.7 in order to find a constant K_3 such that $|\alpha(z)| \leq K_3 |z - z_0|^{-mk-1}$ for $z \in L \setminus \{z_0\}$. Thus $\text{Re } \alpha(z) \geq -K_3 |z - z_0|^{-mk-1}$, and hence

$$(7) \quad |(\delta(z))^{-1}| \leq \exp (K_3 |z - z_0|^{-mk-1}), \quad z \in L \setminus \{z_0\}.$$

Now using (2) and (7), for every $z \in L \setminus \{z_0\}$ we have

$$\begin{aligned}
 \|R(z; T)\| &= \|(\delta(z))^{-1}R(z; A)\delta(z)(1 - BR(z; A))^{-1}\| \\
 &\leq [\exp(K_3 |z - z_0|^{-mk-1})] \|R(z; A)\| \|\delta(z)(1 - BR(z; A))^{-1}\| \\
 (8) \quad &\leq K[\exp(K_3 |z - z_0|^{-mk-1})][\text{dist}(z, J)]^{-m} \exp(K_4 | -BR(z; A)|_k^k) \\
 &\text{(by Lemma 1.4 (iv) and the growth condition on } R(z; A)) \\
 &\leq K[\exp(K_3 |z - z_0|^{-mk-1})][\text{dist}(z, J)]^{-m} \exp(K_4 |B|_k^k \|R(z; A)\|^k) \\
 &\leq K[\exp(K_3 |z - z_0|^{-mk-1})][\text{dist}(z, J)]^{-m} \exp(K_5 [\text{dist}(z, J)]^{-mk}).
 \end{aligned}$$

Now use the fact that L is not tangent to J and hence $|z - z_0|/\text{dist}(z, J)$ is bounded for $z \in L \setminus \{z_0\}$. Using this, and the fact that $x < e^x$ for $x > 0$, and $mk \geq k$, it follows from (8) that there exists a constant $K' > 0$ such that

$$\|R(z; T)\| \leq \exp(K' |z - z_0|^{-mk-1}), \quad \text{for } z \in L \setminus \{z_0\}.$$

2.2. Theorem. Suppose that $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, and that $R(z; A)$ has growth condition $G_{m, K}$ near an open C^2 -Jordan curve J , along which $\sigma(T)$ has an exposed part. If $\sigma(T)$ contains more than one point, then T has a non-trivial hyperinvariant subspace.

Proof. We can assume $k \geq 1$. If $\sigma_p(T) \neq \emptyset$, then, for $\lambda \in \sigma_p(T)$, the eigenspace of T corresponding to λ is a non-trivial hyperinvariant subspace; (note that $T \neq \lambda I$ because $\sigma(T)$ contains at least two points). So suppose $\sigma_p(T) = \emptyset$. Also assume that $\sigma(T)$ is connected, (for otherwise the Riesz decomposition theorem gives a non-trivial hyperinvariant subspace). Since $\sigma(T)$ has an exposed part along J , the connectedness of $\sigma(T)$ implies that $\sigma(T) \cap J \supset J'$, where J' is an open subarc of J . Thus $\sigma(T)$ contains an exposed open arc J' . By Weyl's Theorem, $\sigma(T) \subset \sigma(A) \cup \sigma_p(T) = \sigma(A)$, which shows that $\sigma(A)$ also contains J' as an exposed arc. This together with Theorem 2.1 implies that for each $z_0 \in J'$ and closed line segment L with z_0 as an endpoint which is not tangent to J' , there exists a constant $K' > 0$ such that

$$\|R(z; T)\| \leq \exp(K' |z - z_0|^{-mk-1}), \quad \text{for every } z \in L \setminus \{z_0\}.$$

Now applying Theorem 1.9 we obtain the theorem.

2.3. Theorem. Assume that $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, $\sigma(A)$ has an exposed part along an open C^2 -Jordan arc J , $\sigma(A)$ has at least two accumulation points, one of them on J , and that the resolvent of A has growth condition $G_{m, K}$ near J . Then T has a non-trivial hyperinvariant subspace.

Proof. In view of Theorem 2.2 it is enough to show that $\sigma(T)$ satisfies the required conditions. First $\sigma(T)$ contains at least two points. To see this suppose that $\sigma(T) = \{\lambda_0\}$ and let $D = \mathbb{C} \setminus \{\lambda_0\}$. Then for every $\lambda \in D$ we have

$$(1) \quad \lambda - A = (\lambda - T) + B = (\lambda - T)[I + R(\lambda; T)B].$$

Let $T(\lambda) = -R(\lambda; T)B$, $\lambda \in D$; then, by the second part of Lemma 1.5, $(\lambda - A)$ has a bounded inverse in D except for a countable set of isolated points, which is impossible because $\sigma(A)$ has at least two accumulation points.

Now we can assume that $\sigma_p(T) = \emptyset$, for otherwise the result is clear. This together with Weyl's Theorem implies that $\sigma(T) \subset \sigma(A)$. Now suppose $\sigma(T) \cap J = \emptyset$. Then, since $\sigma(A)$ has an exposed part along J , we have an open simply connected set $G \supset J$ such that

$$(2) \quad \emptyset \neq G \cap \sigma(A) = J \cap \sigma(A).$$

Now $\sigma(T) \cap J = \emptyset$ and $\sigma(T) \subset \sigma(A)$ imply that $\sigma(T) \cap G = \emptyset$. Using relation (1) for $\lambda \in G$, together with Lemma 1.5, we find that $\sigma(A) \cap G = \sigma(A) \cap J$ is a countable set with no accumulation point, which is a contradiction to the assumption that $\sigma(A)$ has an accumulation point on J . So $\sigma(T) \cap J \neq \emptyset$. Since $\sigma(T) \subset \sigma(A)$, it follows that $\sigma(T)$ has an exposed part along J . Now the conditions of Theorem 2.2 are fulfilled.

2.4. Corollary. *Let $T = A + B$ where $B \in \mathcal{C}_k$, $k > 0$, A is a spectral operator of finite type whose spectrum contains an exposed open C^2 -Jordan arc J . Then T has a non-trivial hyperinvariant subspace.*

Proof. By Theorem XV.6.7 of [6], the resolvent of A has growth condition $G_{m,k}$ near the spectrum of A , and hence near the arc J . Now the corollary follows from the above theorem.

Now let us consider the case in which $T = A + B$, $A \in \mathcal{B}(\mathcal{H})$ is any operator, $B \in \mathcal{B}(\mathcal{H})$ is compact, and $\sigma(T) = \{\lambda_0\}$ is a single point. From the relation

$$\lambda - A = (\lambda - T)[I + R(\lambda; T)B], \quad \lambda \neq \lambda_0$$

and Lemma 1.5 we conclude that $\sigma(A)$ is a countable set of isolated points with λ_0 as a possible accumulation point. Moreover, if $\lambda \in \sigma(A)$ and $\lambda \neq \lambda_0$, then we can show that the dimension of the null space of $\lambda - A$ is finite. For otherwise there exists an infinite sequence $\{e_n\}_{n=1}^\infty$ of orthonormal vectors in $\mathcal{H}$ such that $Ae_n = \lambda e_n$, and since $\{e_n\}_{n=1}^\infty$ converges weakly to zero and B is compact, it follows that $Be_n \rightarrow 0$ strongly. But we have

$$\|(\lambda - T)e_n\| = \|(\lambda - A)e_n - Be_n\| = \|Be_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which shows that $\lambda \in \sigma(T)$. But $\lambda \neq \lambda_0$ and $\sigma(T) = \{\lambda_0\}$, which is a contradiction. So $\dim \mathfrak{N}(\lambda - A) < \infty$ for all $\lambda \in \sigma(A)$ other than λ_0 . This leads to the following corollary.

2.5. Corollary. *Let $T = A + B$ where $B \in \mathcal{C}_k$, $k > 0$, A is a spectral operator of finite type whose spectrum lies on $\bigcup_{i=1}^n J_i$ where the J_i are C^2 -Jordan curves. If T is not a multiple of the identity, then it has a non-trivial hyperinvariant subspace.*

Proof. If $\sigma(T)$ is disconnected or $\sigma_p(T) \neq \emptyset$, the result is clear, so suppose $\sigma(T)$ is connected and $\sigma_p(T) = \emptyset$. Then, by Weyl's Theorem $\sigma(T) \subset \sigma(A) \subset$

$\bigcup_{i=1}^n J_i$. Now if $\sigma(T)$ contains more than one point, then connectedness of $\sigma(T)$ implies that $\sigma(T)$, and hence $\sigma(A)$, contains an exposed open C^2 -Jordan arc, and therefore T has a non-trivial hyperinvariant subspace, by Corollary 2.4. So suppose $\sigma(T) = \{\lambda_0\}$. Then, by replacing T by $T - \lambda_0$ and A by $A - \lambda_0$, without loss of generality we can assume $\sigma(T) = \{0\}$. Now $\sigma(A)$ is a countable set of isolated points with zero as a possible accumulation point. We have $A = S^{-1}(M + N)S$, where M is a normal operator, N a nilpotent of some finite order m and $MN = NM$. Thus $\sigma(A) = \sigma(M + N) = \sigma(M)$. It follows from the above discussion that M is compact. So

$$STS^{-1} = M + N + SBS^{-1} = N + C,$$

where C is a compact operator. Now $(STS^{-1})^m = N^m + C_1 = C_1$, where C_1 is also compact. So STS^{-1} commutes with a compact operator and hence by Lomonosov's result (cf. Corollary 8.24 of [13]) it has a non-trivial hyperinvariant subspace, which implies that T has a non-trivial hyperinvariant subspace.

2.6. Definition. Let $A \in \mathfrak{B}(\mathcal{H})$. A is called hyponormal if $A^*A \geq AA^*$.

It is known [17] that a hyponormal operator A satisfies the growth condition $\|R(\lambda; A)\| = [\text{dist}(\lambda, \sigma(A))]^{-1}$, for $\lambda \notin \sigma(A)$.

2.7. Corollary. Suppose that $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, where A is a hyponormal operator such that $\sigma(A)$ has an exposed part along an open C^2 -Jordan arc J . If $\sigma(T) \cap J \neq \emptyset$, (which is the case if $\sigma(A)$ has an accumulation point on J), and if T is not a multiple of the identity, then it has a non-trivial hyperinvariant subspace.

Proof. Suppose $\sigma(T)$ is connected and $\sigma_p(T) = \emptyset$; (otherwise the corollary is trivial). If $\sigma(T) = \{\lambda_0\}$, then, by the above, $\sigma(A)$ is a countable set and hence $\mu(\sigma(A)) = 0$, where μ is the two dimensional Lebesgue measure. Now it follows from Putnam's theorem [12] that A is normal, hence $\lambda_0 - T$ is compact and T has a non-trivial hyperinvariant subspace by Lomonosov's result. So assume that $\sigma(T)$ contains at least two points. Since $\sigma_p(T) = \emptyset$, by Weyl's Theorem $\sigma(T) \subset \sigma(A)$. Since $\sigma(T)$ is connected and $\sigma(T) \cap J \neq \emptyset$, it follows that $\sigma(T)$, and hence $\sigma(A)$, contains an exposed open C^2 -Jordan arc J' , $J' \subset J$. Now by Theorem 2.2 T has a non-trivial hyperinvariant subspace.

2.8. Corollary. Let $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, $\sigma(A) \subset J$, where J is a C^2 -Jordan curve, and suppose that the resolvent of A satisfies growth condition $G_{m,K}$ near J . If T is not a multiple of the identity, then it has a nontrivial hyperinvariant subspace.

Proof. We can suppose that $\sigma_p(T) = \emptyset$; then, by Weyl's Theorem, $\sigma(T) \subset \sigma(A) \subset J$. If $\sigma(T)$ contains more than one point, then the result follows from Theorem 2.2, so suppose that $\sigma(T)$ is a single point. By a translation we can

assume $\sigma(T) = \{0\}$. We can also assume that J is tangent to the x -axis at the origin. Let $\mathcal{C}_\infty$ denote the closed 2-sided ideal of compact operators on $\mathcal{H}$ (the underlying Hilbert space). Consider the Calkin algebra $\mathfrak{B}(\mathcal{H})/\mathcal{C}_\infty$, [3], and let $\dot{S}$ denote the element of $\mathfrak{B}(\mathcal{H})/\mathcal{C}_\infty$ corresponding to the $S \in \mathfrak{B}(\mathcal{H})$. Then $\dot{T} = \dot{A}$. Since $\emptyset \neq \sigma(\dot{T}) \subset \sigma(T) = \{0\}$, it follows that $\sigma(\dot{T}) = \{0\}$. Now we have

$$(1) \quad \|R(\lambda; \dot{T})\| = \|R(\lambda; \dot{A})\| \leq \|R(\lambda; A)\| \leq K [\text{dist}(\lambda, J)]^{-m}.$$

Now we claim that (1) implies that $\dot{T}$ is nilpotent. For the case $J = x$ -axis, this follows from a theorem of Naimark ([11], page 208. An equivalent of Naimark's theorem is: Let Q be a quasinilpotent element of a normed algebra $\mathfrak{A}$. Suppose that for some integer $n \geq 1$, there is a constant $C > 0$ such that $\|R(\lambda; Q)\| \leq C/(|\lambda| |\sin \theta|)^n$ for all sufficiently small $|\lambda|$ and $\theta = \arg \lambda \neq 0, \pi$. Then $Q^{n+1} = 0$). If J is not necessarily a line, let α and β be two points on J at distance s from 0. Construct a quadrilateral Γ_s (as in the figure) cutting J orthogonally at α and β with transverse sides of length $2s$. Then

$$(\dot{T} - \alpha)^m (\dot{T} - \beta)^m = \frac{1}{2\pi i} \int_{\Gamma_s} (\lambda - \alpha)^m (\lambda - \beta)^m R(\lambda; \dot{T}) d\lambda.$$

It can be seen that there exists a constant M (independent of s) such that

$$(2) \quad \|(\dot{T} - \alpha)^m (\dot{T} - \beta)^m\| \leq M(s + \epsilon(s))$$

where $\epsilon(s) \rightarrow 0$ as $s \rightarrow 0$: (a similar argument is used by J. G. Stampfli [16] and R. G. Bartle [2]). If we let $s \rightarrow 0$ in (2) we obtain $\dot{T}^{2m} = 0$. Now $\dot{T}$ nilpotent

implies T is polynomially compact. Since T commutes with a compact operator, it follows from Lomonosov's result (cf. Corollary 8.24 of [13]) that T has a non-trivial hyperinvariant subspace.

For further results we need the following lemma.

2.9. Lemma. *Let $M = N + B$, where M is a normal operator, $N^m = 0$, and B is compact (respectively, $B \in \mathcal{C}_k$). Then M is compact (respectively, $M \in \mathcal{C}_{mk}$).*

Proof. Let $\tilde{M}$ denote the element of Calkin algebra $\mathfrak{B}(\mathcal{H})/\mathcal{C}_\infty$ corresponding to M . Then $\tilde{M}^m = 0$. Since $\tilde{M}$ is normal, it follows that $\tilde{M} = 0$ and hence M is compact. Assume $B \in \mathcal{C}_k$. Since M is a compact normal operator, there exists a decomposition of $\mathcal{H}$ into an orthogonal direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $M = M_1 \oplus 0$, where $\ker M_1 = 0$, and M_1 is a diagonal operator with respect to an orthonormal basis $\{e_n\}_{n=1}^P$, $1 \leq P \leq \infty$, consisting of eigenvectors corresponding to nonzero eigenvalues of M . Let $\alpha_n = (M_1 e_n, e_n)$, $1 \leq n \leq P$. Since $N^m = 0$, it follows that $M^m \in \mathcal{C}_k$. Now

$$|M|_{mk}^{mk} = \sum_{n=1}^P |\alpha_n|^{mk} = \sum_{n=1}^P (|\alpha_n|^m)^k = |M^m|_k^k < \infty,$$

which shows $M \in \mathcal{C}_{mk}$.

The first theorem using the above techniques is due to Schwartz [15]; the next corollary generalizes his result.

2.10. Corollary. *Let $T - T^* = N + B$, where $N^m = 0$, $B \in \mathcal{C}_k$, $k > 0$. If T is not a multiple of the identity, then it has a non-trivial hyperinvariant subspace.*

Proof. Since $T - T^*$ is normal, by the above lemma $T - T^* = B_1$, where $B_1 \in \mathcal{C}_{mk}$. Now $T = ((T + T^*)/2) + ((T - T^*)/2)$ and hence the required result follows from Corollary 2.5.

2.11. Corollary. *Let $1 - T^*T = N + B$, $N^m = 0$, $B \in \mathcal{C}_k$, $k > 0$: If T is not a multiple of the identity, then it has a non-trivial hyperinvariant subspace.*

Proof. Since $1 - T^*T$ is normal (in fact, Hermitian), $1 - T^*T \in \mathcal{C}_{mk}$, by Lemma 2.9. It is known (see after Corollary 8.25 of [13]) that if $1 - T^*T \in \mathcal{C}_p$, then it has a non-trivial hyperinvariant subspace.

Remarks. 1). Theorems 2.1, 2.2 and 2.3 can be improved by requiring that A satisfy a weaker growth condition than $G_{m,K}$ near the exposed part J . Namely, it suffices to assume that the set Δ of those points $z_0 \in J$ for which there are two open tangent disks $D_1(z_0)$ and $D_2(z_0)$ to the arc J at z_0 , (one on each side of the arc J , as in the figure), and there are two positive constants $m(z_0)$ and $K(z_0)$ such that

$$||R(\lambda; A)|| \leq K(z_0)[\text{dist}(\lambda, J)]^{-m(z_0)},$$

for $\lambda \in D_1(z_0) \cup D_2(z_0)$, is dense in J .

By raising $m(z_0)$ and considering smaller disks we can assume that $K(z_0) = 1$ for all $z_0 \in \Delta$. If Δ contains a subarc J_1 of J , then by an argument similar to the one given by N. Dunford in XVI.5.11 of [6], one can see that the union of those open subarcs of J_1 on which $m(z)$ is constant is an open dense subset of J_1 . Thus for such Δ there is an open subarc of J near which $R(\lambda; A)$ has the growth condition $G_{m, \kappa}$, and hence we don't get any improvement.

One can justify the above strengthened forms of the theorems by considering the note given after Theorem 1.9, taking an open tangent disk in place of D in Lemma 1.7, and verifying that the proofs of Theorems 2.1, 2.2 and 2.3 remain valid under the weaker hypothesis.

2). As we mentioned in the introduction of the paper, results similar to those of this paper (except Lemma 2.9 and Corollaries 2.7, 2.10 and 2.11) can be found with different proofs among the earlier results of C. Apostol [1]. It would be worthwhile to state some of the corresponding results of C. Apostol here. According to our notations and definitions these results are:

Theorem 1. Let $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, $\sigma_p(T) = \emptyset$, and let $\sigma(A)$ contain an exposed open C^{2+} -Jordan arc J near which $R(\lambda; A)$ has growth condition $G_{m, \kappa}$. Then there exists a neighborhood U of J and a positive constant M such that for $\lambda \in U \setminus J$ we have

$$||R(\lambda; T)|| \leq \exp \left\{ \frac{M}{[\text{dist}(\lambda, J)]^{mk+1}} \right\}$$

Note. The above theorem is implicitly contained in [1].

Theorem 2. Let $T = A + B$, $B \in \mathcal{C}_k$, $k > 0$, and let $\sigma(A) \subset J$ where J is a

finite union of C^{2+} -Jordan curves. Suppose that $R(\lambda; A)$ has growth condition $G_{m,K}$ near J . Then T has a non-trivial invariant subspace.

Corollary. Let A be a spectral operator of finite type whose spectrum lies in a finite union of C^{2+} -Jordan curves. Let $B \in \mathcal{C}_k$, $k > 0$, Then $T = A + B$ has a non-trivial invariant subspace.

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REFERENCES

1. C. APOSTOL, *On the growth of resolvent, perturbation and invariant subspaces*, Rev. Roum. Math. Pures Appl. **16** (1971), 161–172.
2. R. G. BARTLE, *Spectral localization of operators in Banach spaces*, Math. Annalen **153**, (1964), 261–269.
3. J. CALKIN, *Two sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. **42** (1941), 839–873.
4. N. DUNFORD & J. T. SCHWARTZ, *Linear operators, Part I*, Interscience Publishers, New York, 1958.
5. ———, *Linear operators, Part II*, Interscience Publishers, New York 1963.
6. ———, *Linear operators, Part III*, Interscience Publishers, New York 1971.
7. P. R. HALMOS, *A Hilbert space problem book*, D. Van Nostrand Co., Princeton 1967.
8. K. KITANO, *Invariant subspaces of some non-self-adjoint operators*, Tôhoku Math. J. **20** (1967), 313–322.
9. M. S. LIVSIC, *On the spectral resolution of linear non-self-adjoint operators*, Amer. Math. Soc. Translations, series 2, **5** (1957), 67–114.
10. V. I. MACAEV, *On a class of completely continuous operators*, Soviet Math. Dokl. **2** (1961), 972–975.
11. M. A. NAIMARK, *Normed rings*, Walters-Noordhoff Publishing Groningen, the Netherlands, 1970.
12. C. R. PUTNAM, *An inequality for the area of hyponormal spectra*, Math. Ziets. **116** (1970), 323–330.
13. H. RADJAVI & P. ROSENTHAL, *Invariant subspaces*, Springer-Verlag, Berlin, 1973.
14. L. A. SAHNOVIC, *A study of the "triangular form" of non-self-adjoint operators*, Izv. Vyss. Ucebn. Zaved. Matematika **4** (11) (1959), 141–149 (Russian).
15. J. T. SCHWARTZ, *Subdiagonalization of operators in Hilbert space with compact imaginary part*, Comm. Pure Appl. Math. **15** (1962), 159–172.
16. J. G. STAMPFLI, *A local spectral theory for operators III: Resolvents, spectral sets and similarity*, Trans. Amer. Math. Soc. **168** (1972), 133–151.
17. T. YOSHINO, *On the spectrum of a hyponormal operator*, Tôhoku Math. J. Ser. 2, **17** (1965), 305–309.

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