Schauder Bases and Norm Ideals of Compact Operators II

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§1. In an earlier paper [6] we showed how problems concerning norm ideals of compact operators on a Hilbert space can often be solved quite easily if one first translates the problem to an equivalent one concerning the properties of a Schauder basis in a suitable sequence space. The purpose of this paper is to develop further the ideas and techniques introduced in [6]. In particular we solve completely the problem of “intermediate” ideals first studied by Mitiagin [8].

The origin of the problem is as follows: Let $\mathcal{H}$ denote a separable infinite dimensional Hilbert space (we assume $\mathcal{H}$ is a real Hilbert space but the results hold, as usual, in the complex case). Let $B(\mathcal{H})$ denote the ring of all bounded linear operators on $\mathcal{H}$ and $K(\mathcal{H})$ the ideal in $B(\mathcal{H})$ consisting of all compact operators. A two-sided ideal J in $B(\mathcal{H})$ is called a *norm ideal* if there is a norm α defined on J for which

- (i) J is complete under α .
- (ii) $\alpha(T) = \|T\|$ if T is a one dimensional operator.
- (iii) $\alpha(U \circ T \circ V) \leq \|U\| \|V\| \alpha(T)$ for all $T \in J$ and all $U, V \in B(\mathcal{H})$.

The norm ideal J is called *minimal* if it contains no proper closed subspace which is a norm ideal in $B(\mathcal{H})$, and *maximal* if it is not itself a closed proper subspace of another norm ideal. If J is a maximal norm ideal, then it contains a unique minimal norm ideal $\dot{J}$, namely the closure of the finite dimensional operators on $\mathcal{H}$.

The simplest example of such a pair of norm ideals is the maximal ideal $B(\mathcal{H})$ and its minimal ideal $K(\mathcal{H})$. Now Calkin has shown [2] that there is no norm ideal (in fact, no two-sided ideal) properly contained between $K(\mathcal{H})$ and $B(\mathcal{H})$. At one time it was unknown whether this situation is characteristic of all pairs of minimal—maximal norm ideals. That this is not the case was shown by Mitiagin [8] who constructed a norm ideal properly contained between the minimal ideal $\dot{S}_\pi$ and the maximal ideal S_π of Gohberg and Krein [5]. The general problem of characterizing those pairs of minimal—maximal norm ideals which admit no intermediate norm ideals has remained open.

The purpose of this paper is to settle this problem by showing that for every pair of minimal—maximal norm ideals *other than the pair* $K(\mathcal{H}) - B(\mathcal{H})$ there exist intermediate norm ideals, hence norm ideals which are neither minimal nor maximal. In fact, our construction gives the first general method for the construction of such norm ideals.

The proof consists in reducing the problem to the construction of certain closed subspaces of symmetric solid Banach sequence spaces. As such, the paper is another contribution to the growing body of information concerning the relationship between the ideal structure of $B(\mathcal{H})$ and the structure of symmetric sequence spaces (see, e.g., [1], [4], [6], [9], [11], [12]). More particularly, it is a continuation of the author's earlier work [6] in this area.

§2. Notation and preliminary results. We will denote by α a *uniform cross-norm* [13] on $\mathcal{H} \otimes \mathcal{H}$, the finite dimensional operators on $\mathcal{H}$, and denote by $\mathcal{H} \otimes_{\alpha} \mathcal{H}$ the completion of $\mathcal{H} \otimes \mathcal{H}$ with respect to α . If $\alpha \geq \lambda$ (the operator norm on $B(\mathcal{H})$), then α' , the *associate norm* of α , [13], is also a uniform crossnorm for which $\alpha'' = \alpha$ [13]. Throughout the paper we assume this to be the case.

Schatten [13] and Gohberg and Krein [5] have shown that every minimal norm ideal J is of the form $J = \mathcal{H} \otimes_{\alpha} \mathcal{H}$ for some α , while every maximal ideal is of the form $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ (see [13] for the identification of $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ with a space of operators on $\mathcal{H}$). Thus every pair of minimal—maximal norm ideals is of the form $\mathcal{H} \otimes_{\alpha} \mathcal{H} - (\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$, where α denotes the norm on the ideals.

Let us also recall the relationship between two sided ideals in $B(\mathcal{H})$ and symmetric sequence spaces.

Calkin [2] has shown that there is a one-to-one correspondence between the so-called *ideal sets* of non-negative termed sequences in c_0 (the space of real sequences convergent to zero) and the proper two sided ideals in $B(\mathcal{H})$. This correspondence is obtained as follows: If I is a proper ideal in $B(\mathcal{H})$, then $I \subset K(\mathcal{H})$ [2]. Hence if $T \in I$, then $T = \sum_i a_i \phi_i \otimes \psi_i$ for some (a_i) in c_0 with $a_i \geq 0$ for all i and (ϕ_i) and (ψ_i) some orthonormal sets in $\mathcal{H}$ [13]. In this way there is associated with every $T \in I$ a sequence (a_i) in c_0 . The collection of all such sequences is an ideal set and, conversely, given an ideal set A the collection of all operators $T \in K(\mathcal{H})$ having the representation $T = \sum_i a_i \phi_i \otimes \psi_i$ for some $(a_i) \in A$ and some orthonormal sets (ϕ_i) and (ψ_i) is a proper two sided ideal in $B(\mathcal{H})$ [2].

It is useful to extend this correspondence to one between proper two sided ideals in $B(\mathcal{H})$ and sequence spaces. To do this, note that if A is any ideal set, then the set of all sequences $(a_i) \in c_0$ for which $(|a_i|) \in A$ is a sequence space J which is *solid* (i.e. $(a_i) \in J$ and $|b_i| \leq |a_i|$ for all $i \Rightarrow (b_i) \in J$) and *symmetric* (i.e. $(a_i) \in J \Rightarrow (a_{\pi(i)}) \in J$ for every permutation π of the positive integers) and which determines the same ideal in $B(\mathcal{H})$ which A does, namely the set of all $T \in K(\mathcal{H})$ which have the representation $T = \sum_i a_i \phi_i \otimes \psi_i$ for some $(a_i) \in J$ and some orthonormal sets (ϕ_i) and (ψ_i) .

Now let α be a uniform crossnorm and Φ_α the associated symmetric norming function [13], [5]. Gohberg and Krein [5] have shown that the sequence space associated with the proper maximal ideal $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ is the Banach space $J = \{(a_i) \in c_0 \mid \sup_n \Phi_\alpha(a_i)_{i=1}^n < +\infty\}$, while the sequence space associated with the minimal norm ideal $\mathcal{H} \otimes_\alpha \mathcal{H} \subset (\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ is

$$\dot{J} = \{(a_i) \in c_0 \mid \Phi_\alpha(a_i)_{i=1}^n \xrightarrow{m,n} 0\}.$$

We can describe these sequence spaces in another way as follows: Let μ be the sequence space which is the completion of the finitely non-zero sequences with respect to $\Phi_{\alpha'}$. Then $J = \mu^\times = \{(a_i) \mid \sum_i |a_i| |b_i| < +\infty \text{ for all } (b_i) \in \mu\}$, and $\dot{J}$ is the closure in $\mu^\times$ of the finitely non-zero sequences. If e_n is the sequence with 1 in the n^{th} place and zeros elsewhere, then the sequence (e_i) in μ is a symmetric basis for μ [14] whose sequence of coefficient functionals (e_i) in $\mu^\times$ forms a symmetric basis for $\dot{J}$. Thus we write $[e_i] = \dot{J} \subset \mu^\times$.

A consideration of this relationship between the pair of minimal—maximal norm ideals $\mathcal{H} \otimes_{\alpha'} \mathcal{H} = (\mathcal{H} \otimes_\alpha \mathcal{H})^*$ and the associated pair of Banach sequence spaces $[e_i] = \mu^\times$ suggests a construction of intermediate norm ideals through the construction of intermediate subspaces of sequence spaces (in fact, Mitiagin's proof [8] is accomplished in this fashion). In this paper we show how this can be done in the general case.

§3. Let α be a uniform crossnorm and suppose the minimal norm ideal $\mathcal{H} \otimes_\alpha \mathcal{H}$ is properly contained in the maximal norm ideal $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$. Moreover, suppose $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ is a proper norm ideal in $B(\mathcal{H})$ —i.e. $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^* \subset K(\mathcal{H})$. Then α is not equivalent to λ , the least crossnorm. As we have mentioned above, there exists a symmetric solid Banach sequence space μ having (e_i) as a basis and normed by the symmetric norming function $\Phi_{\alpha'}$, for which $\mu^\times$ is the sequence space associated with $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ and $[e_i] \subset \mu^\times$ the sequence space associated with $\mathcal{H} \otimes_\alpha \mathcal{H}$. Since, as we have noted, α is not equivalent to λ , then α' is not equivalent to γ (the greatest crossnorm [13]). It follows that the basis (e_i) for μ is not similar to (e_i) in ℓ^1 and hence $\mu^\times$ is a linear subspace of c_0 [6]. Moreover, since $[e_i] \neq \mu^\times$ the basis (e_i) for μ is not shrinking, by definition [14]. In particular (e_i) in μ is not similar to the basis (e_i) for c_0 . We mention these facts here since we will need them later and since they justify the restrictions we impose on the basis (e_i) for μ in the statements of our results.

In order to simplify the statement of the results which follow, let us agree upon the following notation: *throughout the paper μ will denote a separable solid symmetric Banach sequence space. Then the sequence of unit vectors (e_i) forms a symmetric basis for μ . Assume also that the basis (e_i) for μ is not similar to the basis (e_i) in ℓ^1 , so that both μ and $\mu^\times$ are vector subspaces of c_0 . If (a_i) is an element of some normal sequence space ν , then we denote the norm of (a_i) in ν by $\|(a_i)\|_\nu$.*

We begin with several results which will be used frequently throughout the paper. The first two are well known and follow easily from Abel's partial summation formula.

Lemma 1. Let (r_i) and (s_i) be non-increasing sequences of non-negative numbers. If π is any permutation of the positive integers and (ϵ_i) is any ± 1 sequence, then

$$\left| \sum_{i=1}^n \epsilon_i r_i s_{\pi(i)} \right| \leq \sum_{i=1}^n r_i s_i \quad \text{for all } n.$$

Lemma 2. Let (r_i) , (s_i) , and (t_i) be three sequences of positive numbers. If (t_i) is non-increasing and

$$\sum_{i=1}^m r_i \leq \sum_{i=1}^m s_i \quad \text{for all } m, \text{ then } \sum_{i=1}^m r_i t_i \leq \sum_{i=1}^m s_i t_i \quad \text{for all } m.$$

The following result is due to K. Fan [3]. Recall that if T is a compact operator on $\mathcal{H}$, then by the s -numbers of T we mean the sequence $(s_i(T))$ of eigenvalues (repeated according to multiplicity) of $(T^*T)^{1/2}$ [5].

Lemma 3. Let $T \in K(\mathcal{H})$ and let (ϕ_i) and (ψ_i) be any orthonormal sets in $\mathcal{H}$. Then

$$\sum_{i=1}^m \langle T\phi_i, \psi_i \rangle \leq \sum_{i=1}^m s_i(T) \quad \text{for all } m.$$

Finally we will also need the following simple observation.

Lemma 4. If (e_i) in μ is not similar to (e_i) in c_0 , then given any $\delta > 0$ there exists $N(\delta)$ such that whenever $(a_i) \in \mu$ with $\|(a_i)\|_\mu = 1$, there exist at most $N(\delta)$ terms a_n of the sequence (a_i) for which $|a_n| \geq \delta$.

Proof. Suppose for some $\delta > 0$ no such $N(\delta)$ exists. Then there is a sequence $\{(a_i^{(n)})\}_{n=1}^\infty$ in μ for which $\|(a_i^{(n)})\|_\mu = 1$ for all n , but for which $|a_i^{(n)}| \geq \delta$ for $i = 1, 2, \dots, q(n)$, where $\lim_n q(n) = +\infty$. Then $1 = \|(a_i^{(n)})\|_\mu \geq \|\sum_{i=1}^{q(n)} a_i^{(n)} e_i\|_\mu \geq \delta \|\sum_{i=1}^{q(n)} e_i\|_\mu \geq \delta \|\sum_{i=1}^n e_i\|_\mu$ for all $n = 1, 2, \dots$. But this says the basis (e_i) for μ is of type P [14], and being an unconditional basis must therefore be similar to the basis (e_i) for c_0 [14], a contradiction.

In view of our remarks at the beginning of this section and in §2 on the pair of minimal—maximal ideals $\mathcal{H} \otimes_\alpha \mathcal{H} \subset (\mathcal{H} \otimes_\alpha \mathcal{H})^*$ and their corresponding sequence spaces $[e_i] \subset \mu^\times$, it is clear that we are interested only in those sequence spaces μ for which $[e_i] \neq \mu^\times$; i.e. those for which (e_i) in μ is not shrinking.

It is a well known result in Schauder basis theory that if (x_i, f_i) is a basis for a Banach space E (f_i denotes the coefficient functionals for (x_i) in E^*), then (x_i) is shrinking (i.e. $[f_i] = E^*$) if and only if the following is true: If (z_k) is a bounded sequence in E for which $\langle f_i, z_k \rangle \xrightarrow{k} 0$ for every i , then (z_k) converges weakly to zero [10]. Our next result shows that in symmetric solid Banach sequence spaces a stronger result holds.

Proposition 1. The basis (e_i) for μ is not shrinking if and only if there exists

a sequence (z_k) in μ for which $\|z_k\|_{c_0} \xrightarrow{k} 0$ but for which (z_k) is not weakly convergent to zero.

Proof. Suppose (e_i) is not shrinking. Then there exists an element (b_i) in $\mu^\times$ but not in $[e_i] \subset \mu^\times$. As we mentioned earlier, our assumptions on μ imply that $\mu^\times$ is contained in c_0 and $\mu^\times$ is a symmetric solid sequence space. Hence we may assume that the sequence (b_i) decreases monotonically to zero. Since $(b_i) \notin [e_i]$, there exists an $\epsilon > 0$ such that $\|(b_i)_{i=n}^\infty\|_{\mu^\times} \geq 2\epsilon$ for all n . Thus there exists an increasing sequence $(p(n))$ of integers for which $\|\sum_{i=p(n)+1}^{p(n+1)} b_i e_i\|_{\mu^\times} \geq 3\epsilon/2$ for all n , and, correspondingly, a sequence (w_n) in μ such that $w_n = \sum_{i=p(n)+1}^{p(n+1)} a_i e_i$ with $\|w_n\|_\mu = 1$ and $\langle (b_i), w_n \rangle = \sum_{i=p(n)+1}^{p(n+1)} a_i b_i \geq 3\epsilon/2$ for all n . Clearly, by using the fact that (e_i) is a symmetric basis for μ and applying Lemma 1, we may assume that $a_{p(n)+1} \geq a_{p(n)+2} \geq \dots \geq a_{p(n+1)} \geq 0$ for all n .

Now by Lemma 4, at most N_1 terms of each sequence w_n are $\geq \frac{1}{2}$. However $\sum_{i=p(n)+1}^{p(n+1)} a_i b_i \geq 3\epsilon/2$ for all n , where $b_i \downarrow 0$. Thus if n_1 is chosen large enough we can get

$$\begin{aligned} \frac{3}{2}\epsilon &\leq \sum_{i=p(n_1)+1}^{p(n_1+1)} a_i b_i = \sum_{i=p(n_1)+1}^{p(n_1)+N_1} a_i b_i + \sum_{i=p(n_1)+N_1+1}^{p(n_1+1)} a_i b_i \\ &\leq N_1 b_{p(n_1)+N_1+1} + 1 + \sum_{i=p(n_1)+N_1+1}^{p(n_1+1)} a_i b_i, \end{aligned}$$

where $0 \leq a_i \leq \frac{1}{2}$ for $i = p(n_1) + N_1 + 1, \dots, p(n_1 + 1)$. If $b_{p(n_1)+N_1+1} \leq \epsilon/2N_1$, then $\sum_{i=p(n_1)+N_1+1}^{p(n_1+1)} a_i b_i \geq \epsilon$, and $0 \leq a_i < \frac{1}{2}$ for all terms a_i in this sum. Setting $z_1 = \sum_{i=p(n_1)+N_1+1}^{p(n_1+1)} a_i e_i$, we have $\|z_1\|_{c_0} < \frac{1}{2}$, while $\langle (b_i), z_1 \rangle \geq \epsilon$.

In this fashion, applying Lemma 4 as above with $\delta = 1/(k+1)$ for $k = 1, 2, \dots$, we then get a sequence (z_k) in μ for which $z_k = \sum_{i=q(k)+1}^{q(k+1)} a_i e_i$ for some increasing sequence $(q(k))$, $a_{q(k)+1} \geq \dots \geq a_{q(k+1)}$, $\langle (b_i), z_k \rangle \geq \epsilon > 0$, and $\|z_k\|_{c_0} < 1/k$ for all k . Thus $\|z_k\|_{c_0} < 1/(k+1)$ while (z_k) is not weakly convergent to zero.

The converse is clear from the more general result quoted previous to Proposition 1.

Remark. Proposition 1 is clearly true also in the case where (e_i) in μ is similar to (e_i) in ℓ^1 .

For easy reference, let us summarize the results in Proposition 1 and its proof: If (e_i) is non-shrinking in μ , then there exists a bounded sequence (z_k) in μ for which

- (i) $z_k = \sum_{i=p(k)+1}^{p(k+1)} a_i e_i$ for an increasing sequence $(p(k))$,
- (ii) $a_{p(k)+1} \geq a_{p(k)+2} \geq \dots \geq a_{p(k+1)} \geq 0$ for all k ,
- (iii) $\lim_k \|z_k\|_{c_0} = 0$, and
- (iv) there exists $(b_i) \in \mu^\times$, $(b_i) \notin [e_i]$, such that $(b_i) \downarrow 0$ and $\langle (b_i), z_k \rangle \geq \epsilon > 0$ for all k .

We will now use the sequence (z_k) to construct another sequence (w_k) , having essentially the same properties as (z_k) , but which has $\langle (w_k), e_i \rangle \downarrow 0$ for all k .

Let n be an integer and $\delta > 0$. Since $\|z_k\|_{c_0} \xrightarrow{k} 0$ there is a k_0 for which $a_{p(k_0)+1} < \delta/n$ and $p(k_0) + 1 \geq n$. Consider the sequence

$$w_\delta = (\underbrace{a_{p(k_0)+1}, a_{p(k_0)+1}, \dots, a_{p(k_0)+1}}_{n \text{ places}}, a_{p(k_0)+1}, a_{p(k_0)+1}, a_{p(k_0)+2}, \dots, a_{p(k_0)+1}, 0, 0, \dots).$$

Clearly $\langle (b_i), w_\delta \rangle \geq \langle (b_i), z_{k_0} \rangle \geq \epsilon > 0$, while

$$\|w_\delta - (\underbrace{0, 0, \dots, 0}_{n \text{ places}}, a_{p(k_0)+1}, \dots, a_{p(k_0)+1}, 0, 0, \dots)\|_\mu < \delta.$$

Using this observation and Proposition 1 we have

Corollary 1. *Let (e_i) be a non-shrinking basis in μ . Then there exists a bounded sequence (w_k) in μ with the following properties:*

- (i) $w_k = \sum_{i=1}^{q(k+1)} a_i^{(k)} e_i$, where $a_1^{(k)} \geq a_2^{(k)} \geq \dots \geq a_{q(k+1)}^{(k)} \geq 0$ and $(q(k+1))$ is an increasing sequence,
- (ii) Both $\langle (b_i), w_k \rangle \geq \epsilon$ and $\langle (b_i), \sum_{i=q(k)+1}^{q(k+1)} a_i^{(k)} e_i \rangle > \epsilon$. For all k ,
- (iii) $\|w_k - \sum_{i=q(k)+1}^{q(k+1)} a_i^{(k)} e_i\|_\mu < 1/2^k$, and
- (iv) $\|w_{k+1}\|_{c_0} \leq \|w_k\|_{c_0} < 1/2^k$ for all k .

We can now define the subspaces of $\mu^\times$ which are crucial to our construction of intermediate norm ideals.

Recall that if $(c_i) \in c_0$, then the sequence $(\hat{c}_i)$ is defined to be the rearrangement of the non-zero terms of $(|c_i|)$ into a monotonically decreasing sequence.

Definition. Let (w_k) be the sequence in μ defined in Corollary 1. For any subsequence (w_{k_n}) of (w_k) , let

$$A(w_{k_n}) = \{(c_i) \in \mu^\times \mid \langle (\hat{c}_i), w_{k_n} \rangle \xrightarrow{n} 0\}.$$

Theorem 1. *Each set $A(w_{k_n})$ forms a closed solid symmetric subspace of $\mu^\times$ for which $[e_i] \subset A(w_{k_n}) \subset \mu^\times$.*

Proof. Clearly we need only prove the result for the set $A(w_k)$ itself. For the sake of notational simplicity we write $A = A(w_k)$.

The fact that A is a vector subspace of $\mu^\times$ is clear from Lemma 1 and the definition of A , in view of the fact that $w_k = \sum_{i=1}^{q(k+1)} a_i^{(k)} e_i$, where $a_1^{(k)} \geq \dots \geq a_{q(k+1)}^{(k)} \geq 0$.

Clearly A is a symmetric sequence space, by definition. Suppose $(c_i) \in A$ and $|d_i| \leq |c_i|$ for all i . One easily checks that $\hat{d}_i \leq \hat{c}_i$ for all i ; hence $0 \leq \langle \hat{d}_i, w_k \rangle \leq \langle \hat{c}_i, w_k \rangle$ for all k , implying $(d_i) \in A$. That is, A is also a solid subspace.

To show that A is closed, let $(c_i) \in \mu^\times$ and suppose $\{(c_i^{(n)})\}_{n=1}^\infty$ is a sequence in A which converges to (c_i) in $\mu^\times$. Then $\|(c_i^{(n)}) - (c_i)\|_{\mu^\times} \xrightarrow{n} 0$.

Using the fact that A is a symmetric solid subspace of $\mu^\times$ we may assume $c_i > 0$. Thus there is a permutation π for which $(c_{\pi(i)}) = (\hat{c}_i)$. Using the fact that $\mu^\times$ is a symmetric space, we then have that $\|(c_{\pi(i)}^{(n)}) - (c_i)\|_{\mu^\times} \xrightarrow{n} 0$, where

$(c_{\pi(i)})^{(n)} \in A$ for all n since A is symmetric. Since (w_k) is a bounded sequence in μ , we then have that

$$\langle (c_{\pi(i)})^{(n)} - c_i, w_k \rangle \xrightarrow{n} 0, \text{ uniformly over } k = 1, 2, \dots.$$

That is, $|\Sigma_{i=1}^{\infty} c_{\pi(i)}^{(n)} \langle w_k, e_i \rangle - \Sigma_{i=1}^{\infty} \hat{c}_i \langle w_k, e_i \rangle| \xrightarrow{n} 0$ uniformly over $k = 1, 2, \dots$. Hence by Lemma 1 and the fact that $(c_{\pi(i)})^{(n)} \in A$ for all n , we conclude that (c_i) is also in A , proving that A is closed in $\mu^\times$.

All that remains is to show that $[e_i] \subset A \subset \mu^\times$.

To prove that $[e_i] \subset A$, it suffices to show that $e_i \in A$ for every i . To see this, note that $|\langle w_k, \hat{e}_i \rangle| \leq \|w_k\|_{e_0} < 1/2^k$, $k = 1, 2, \dots$, $i = 1, 2, \dots$. Therefore $\lim_k \langle w_k, \hat{e}_i \rangle = 0$ for $i = 1, 2, \dots$, and $e_i \in A$ for all i .

We show next that the inclusions $[e_i] \subset A(w_{k_n}) \subset \mu^\times$ proved in Theorem 1 are actually proper for some subsequence (w_{k_n}) .

Theorem 2. *Let (w_k) be the sequence in μ defined in Corollary 1. Then there exists a subsequence (w_{k_n}) of (w_k) for which $[e_i] \neq A(w_{k_n}) \neq \mu^\times$.*

Proof. Since there exists $(b_i) \in \mu^\times$ for which $\langle \hat{b}_i, w_k \rangle \geq \epsilon > 0$ for all k , it is clear that $A(w_{k_n}) \neq \mu^\times$ for all subsequences (w_{k_n}) .

To prove that $[e_i] \neq A(w_{k_n})$ for some (w_{k_n}) , let us first note that if $z_k = \Sigma_{i=q(k)+1}^{\infty} a_i^{(k)} e_i$ then $\|w_k - z_k\|_\mu < 1/2^k$ for all k . Hence if $(c_i) \in \mu^\times$ has the property that $\langle (\hat{c}_i), z_{k_n} \rangle \xrightarrow{n} 0$ for some subsequence (z_{k_n}) of (z_k) , then $\langle (c_i), w_{k_n} \rangle \xrightarrow{n} 0$ also. Thus we will show that there exists $(c_i) \in \mu^\times$, $(c_i) \notin [e_i]$, for which $\langle (\hat{c}_i), z_{k_n} \rangle \xrightarrow{n} 0$ for some subsequence (z_{k_n}) , thereby proving the theorem.

For simplicity of notation, let $z_k = \Sigma_{i=q(k)+1}^{\infty} a_i^{(k)} e_i$ and recall that $a_{q(k)+1} \geq \dots \geq a_{q(k+1)} \geq 0$ for all k . Again let $(b_i) \in \mu^\times$ be as in Corollary 1. Since $\|\Sigma_{i=n}^{\infty} b_i e_i\|_{\mu^\times} \geq \epsilon > 0$ for all n , there exists an s_1 for which $\|(b_1, \dots, b_{s_1}, 0, 0, \dots)\|_{\mu^\times} \geq \epsilon/2$. Let $r_1 = 0$ and define $z_{k_1} = z_1$.

Now choose z_{k_2} so that $\langle z_{k_2}, e_i \rangle = 0$ for $1 \leq i \leq s_1 (= (s_1 - r_1))$. Since $(b_i) \downarrow 0$ and $\|\Sigma_{i=n}^{\infty} b_i e_i\|_{\mu^\times} \geq \epsilon$ for all n , we can find integers r_2 and s_2 such that

- (i) $s_1 \leq r_2 < s_2$,
- (ii) $s_2 - r_2 \geq q(k_2 + 1) - q(k_2)$,
- (iii) $\|(b_{r_2+1}, b_{r_2+2}, \dots, b_{s_2}, 0, 0, \dots)\|_{\mu^\times} \geq \epsilon/2$, and
- (iv) $\Sigma_{i=1}^{q(k_2+1)-q(k_2)} a_{q(k_2)+i} \circ b_{r_2+i} < \frac{1}{2}$.

Similarly, choose z_{k_3} so that $\langle z_{k_3}, e_i \rangle = 0$ for $1 \leq i \leq (s_1 - r_1) + (s_2 - r_2)$. Again, we can find integers r_3 and s_3 so that

- (i) $s_2 \leq r_3 < s_3$,
- (ii) $s_3 - r_3 \geq q(k_3 + 1) - q(k_3)$,
- (iii) $\|(b_{r_3+1}, b_{r_3+2}, \dots, b_{s_3}, 0, 0, \dots)\|_{\mu^\times} \geq \epsilon/2$, and
- (iv) $\Sigma_{i=1}^{q(k_3+1)-q(k_3)} a_{q(k_3)+i} b_{r_3+i} < \frac{1}{3}$.

Continuing in this fashion we see that there exists a subsequence (z_{k_n}) of (z_k) such that

- (i) $\langle z_{k_n}, e_i \rangle = 0$ for $1 \leq i \leq (s_1 - r_1) + (s_2 - r_2) + \dots + (s_{n-1} - r_{n-1})$,
- (ii) $s_n \leq r_{n+1} < s_{n+1}$ for all n ,

- (iii) $s_k - r_k \geq q(n_k + 1) - q(n_k)$,
- (iv) $\|(b_{r_k+1}, b_{r_k+2}, \dots, b_{s_k}, 0, 0, \dots)\|_{\mu^\times} \geq \epsilon/2$,
- (v) $\sum_{j=1}^{q(n_k+1)-q(n_k)} a_{q(n_k)+j} b_{r_k+j} < 1/k$.

Then setting

$$(\hat{c}_i) = (b_1, \dots, b_{s_1}, b_{r_2}, \dots, b_{s_2}, b_{r_3}, \dots, b_{s_3}, \dots),$$

we see that $(c_i) \in \mu^\times$ but $(c_i) \notin [e_i]$ since for any N there exist $s_k \geq r_k \geq N$ for which $\|\sum_{i=r_k}^{s_k} b_i e_i\|_{\mu^\times} \geq \epsilon/2$. However, by construction it is clear that $\langle (\hat{c}_i), z_{k_n} \rangle < 1/n$ for $n \geq 2$, implying that $(c_i) \in A(w_{k_n})$, as we noted earlier.

Let $(v_n) = (w_{k_n})$, where (w_{k_n}) is the subsequence of (w_k) chosen above. Then, in summary, there exists an increasing sequence $(\rho(n))$ for which

- 1) $v_n = \sum_{i=1}^{\rho(n+1)} d_i^{(k)} e_i$, where $d_1^{(k)} \geq d_2^{(k)} \geq \dots \geq d_{\rho(n+1)}^{(k)}$
- 2) $A(v_n) = \{(c_i) \in \mu^\times \mid \langle (\hat{c}_i), v_n \rangle \xrightarrow{n} 0\}$ is a closed solid symmetric subspace of $\mu^\times$ for which $[e_i] \not\subset A(v_n) \not\subset \mu^\times$.

We can now apply this result to the construction of intermediate norm ideals.

Let α be a uniform crossnorm and $\mathcal{H} \otimes_\alpha \mathcal{H}$ and $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ the associated minimal and maximal norm ideals. If we assume $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ is a proper ideal in $B(\mathcal{H})$, then according to the ideas discussed in §2 there exists a separable solid symmetric Banach sequence space μ in which (e_i) is a symmetric non-shrinking basis (which is not similar to (e_i) in ℓ^1) and for which $\mu^\times$ is the sequence space associated with $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ and $[e_i] \subset \mu^\times$ is the sequence space associated with $\mathcal{H} \otimes_\alpha \mathcal{H}$.

Let (v_n) be the sequence in μ whose properties are listed above. We now associate with $A(v_n)$ an ideal in $B(\mathcal{H})$.

Definition. Let $I(A(v_n)) = \{T \in (\mathcal{H} \otimes_{\alpha'} \mathcal{H})^* \mid T = \sum_{i=1}^\infty a_i \phi_i \otimes \psi_i \text{ for } (\phi_i) \text{ and } (\psi_i) \text{ orthonormal sequences and } (a_i) \in A(v_n)\}$.

We claim that $I(A(v_n))$ is the desired intermediate ideal.

Theorem 3. $I(A(v_n))$ is a norm ideal in $B(\mathcal{H})$ which is closed in $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ and for which

$$\mathcal{H} \otimes_\alpha \mathcal{H} \not\subset I(A(v_n)) \not\subset (\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*.$$

Proof. Clearly $I(A(v_n))$ is an ideal in $B(\mathcal{H})$ in view of the Calkin characterization of ideals in $B(\mathcal{H})$ discussed in §2, since $A(v_n)$ is a solid symmetric subspace of $\mu^\times$.

That $I(A(v_n))$ is properly contained between $\mathcal{H} \otimes_\alpha \mathcal{H}$ and $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ is also clear since $A(v_n)$ is properly contained between $[e_i]$ (the sequence space associated with $\mathcal{H} \otimes_\alpha \mathcal{H}$) and $\mu^\times$ (the sequence space associated with $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$).

We need only show that $I(A(v_n))$ is closed in $(\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$. To do this, let $T \in (\mathcal{H} \otimes_{\alpha'} \mathcal{H})^*$ and $\|T - T_n\|_\alpha \xrightarrow{n} 0$ for some sequence $(T_n) \subset I(A(v_n))$. Since T is compact, $T = \sum_{i=1}^\infty s_i(T) e_i \otimes f_i$ for some orthonormal sequences (e_i) and (f_i) in $\mathcal{H}$.

By Lemma 3, $\Sigma_{i=1}^m \langle (T - T_n)\phi_i, \psi_i \rangle \leq \Sigma_{i=1}^m s_i(T - T_n)$ for all m and for all n . For any fixed n look at $(\langle (T - T_n)\hat{e}_i, f_i \rangle) = (\langle (T - T_n)\phi_i, \psi_i \rangle)$ for some other orthonormal sequences (ϕ_i) and (ψ_i) (note that (ϕ_i) and (ψ_i) are actually just reorderings of (e_i) and (f_i) followed by suitable multiplication by ± 1).

Then, by the above, for all m

$$\Sigma_{i=1}^m \langle (T - T_n)\hat{e}_i, f_i \rangle = \Sigma_{i=1}^m \langle (T - T_n)\phi_i, \psi_i \rangle \leq \Sigma_{i=1}^m s_i(T - T_n).$$

By Lemma 2, given any $(a_i) \in \mu$ with $\|(a_i)\|_\mu = 1$ and (a_i) monotonically decreasing we have

$$\Sigma_{i=1}^m a_i \langle (T - T_n)e_i, f_i \rangle \leq \Sigma_{i=1}^m a_i s_i(T - T_n).$$

It follows that

$$\|(\langle (T - T_n)e_i, f_i \rangle)\|_{\mu^\times} \leq \|(s_i(T - T_n))\|_{\mu^\times} = \|T - T_n\|_\alpha.$$

In particular, then, since $T = \Sigma_i s_i(T)e_i \otimes f_i$ and $\|T - T_n\|_\alpha \xrightarrow{n} 0$,

$$\|(s_i(T)) - \langle T_n e_i, f_i \rangle\|_{\mu^\times} \xrightarrow{n} 0.$$

Now $T_n \in I(A(v_k))$ for all n , implying $(s_i(T_n)) \in A(v_k)$ for all n , by definition. That is,

$$\langle (s_i(T_n)), v_k \rangle = \Sigma_{i=1}^\infty s_i(T_n) \langle v_k, e_i \rangle \xrightarrow{k} 0 \quad \text{for all } n.$$

Applying to T_n the same argument applied previously to $(T - T_n)$, we have

$$\Sigma_{i=1}^m \langle T_n e_i, f_i \rangle \leq \Sigma_{i=1}^m s_i(T_n) \quad \text{for all } m.$$

By Lemma 2 (recalling that $v_k = \Sigma_{i=1}^{\rho(k+1)} d_i^{(k)}$, where $d_i^{(k)} \geq \dots \geq d_{\rho(k+1)}^{(k)} \geq 0$ for all k),

$$\Sigma_{i=1}^m \langle T_n \hat{e}_i, f_i \rangle \langle v_k, e_i \rangle \leq \Sigma_{i=1}^m s_i(T_n) \langle v_k, e_i \rangle$$

for all m and for all k . Thus for any n ,

$$\Sigma_{i=1}^\infty \langle T_n \hat{e}_i, f_i \rangle \langle v_k, e_i \rangle \leq \Sigma_{i=1}^\infty s_i(T_n) \langle v_k, e_i \rangle \xrightarrow{k} 0,$$

since $T_n \in I(A(v_k)) \Rightarrow (s_i(T_n)) \in A(v_k)$.

That is, $(\langle T_n e_i, f_i \rangle) \in A(v_k)$ for all n . Since $A(v_k)$ was shown in Theorem 1 to be closed in $\mu^\times$, it follows from the above that $(s_i(T))$, being the limit of the sequence $\{(T_n e_i, f_i)\}_{n=1}^\infty$, must also be in $A(v_k)$, implying $T \in I(A(v_k))$. Therefore $I(A(v_k))$ is a closed subspace of $(\mathfrak{K} \otimes_\alpha \mathfrak{K})^*$ and hence a norm ideal. The proof is complete.

§4. Remarks. In this paper we have given a general method of constructing norm ideals of compact operators on a Hilbert space which are neither minimal nor maximal. An inspection of the proof shows that in our approach the essential property which the constructed subspaces A of $\mu^\times$ must have is the following:

If $(a_i) \in A$ and (b_i) is any sequence for which there is a constant k with the property that $\sum_{i=1}^n \hat{b}_i \leq k \sum_{i=1}^n \hat{a}_i$ for all n , then $(b_i) \in A$.

Any solid symmetric closed subspace A of $\mu^\times$ which possesses the above property determines a norm ideal in $B(\mathcal{H})$. If $[e_i] \not\subseteq A \not\subseteq \mu^\times$ then, of course, the associated norm ideal is neither minimal nor maximal.

In a later paper we will show how the use of certain eigenvalue inequalities enables one to develop even more fully the relationship between the theory of norm ideals and the theory of Banach sequence spaces.

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