Small Random Perturbations of Dynamical Systems and Applications to Parabolic Equations

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Introduction. This work is concerned with a stochastic differential system of the form

$$(0.1) \quad d\xi(t) = b(\xi(t)) dt + \epsilon \sigma(\xi(t)) dw(t), \quad \epsilon > 0$$

as $\epsilon \rightarrow 0$, and with applications to parabolic equations of the form

$$(0.2) \quad \frac{\partial u}{\partial t} = L_\epsilon u,$$

where

$$(0.3) \quad L_\epsilon u \equiv \frac{\epsilon^2}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}$$

and

$$b = (b_1, \dots, b_n), \quad \sigma = (\sigma_{ij})_{i,j=1}^n, \quad a_{ij} = \sum_{k=1}^n \sigma_{ik} \sigma_{jk}.$$

In 1967 Varadhan [7] studied the fundamental solution of (0.2) as $\epsilon \rightarrow 0$ in the special case $b \equiv 0$ by deep analytical techniques. He then derived, in [8], probabilistic results of the form

$$(0.4) \quad \begin{aligned} \overline{\lim}_{\epsilon \rightarrow 0} [2\epsilon^2 \log P_x(Q)] &\leq - \inf_{\omega \in Q_x} I(\omega) && \text{if } Q \text{ is closed,} \\ \underline{\lim}_{\epsilon \rightarrow 0} [2\epsilon^2 \log P_x(Q)] &\geq - \inf_{\omega \in Q_x} I(\omega) && \text{if } Q \text{ is open,} \end{aligned}$$

where Q is a subset of the space of continuous functions, and $I(\omega)$ is a certain functional depending on σ . Finally (in [8]) he used (0.4) in order to derive the asymptotic behavior, as $\epsilon \rightarrow 0$, of Green's function of (0.2) (when $b \equiv 0$) in any cylinder $\{(x, t); x \in \Omega, t > 0\}$.

In 1970 Ventsel and Freidlin [11] established, by a relatively simple probabilistic method, two probabilistic estimates of the form

$$(0.5) \quad \begin{aligned} \lim_{\epsilon \rightarrow 0} \{2\epsilon^2 \log P_x[\rho_t(\xi^\epsilon, \phi) < \delta]\} &\geq -I_t(\phi), \\ \overline{\lim}_{\epsilon \rightarrow 0} \{2\epsilon^2 \log P_x[d_t(\xi^\epsilon, \Phi_0) > \delta]\} &\leq -I_0 \quad (I_0 = \sup_{\phi \in \Phi_0} I_t(\phi)). \end{aligned}$$

(More details will be given in Section 1.) They used these estimates to (i) derive the asymptotic behavior, as $\epsilon \rightarrow 0$, of the solution u_ϵ of

$$\begin{aligned} \frac{\partial u}{\partial t} &= L_\epsilon u \quad \text{if } x \in \Omega, t > 0, \\ u(x, 0) &= 0 \quad \text{if } x \in \Omega, \\ u(x, t) &= 1 \quad \text{if } x \in \partial\Omega, t > 0; \end{aligned}$$

and (ii) study the “exit problem.” The estimates (0.5) have been subsequently used also to solve other problems (see [4], [9], [10]).

The “exit problem” is concerned with the following situation: Suppose all the solutions of $dx/dt = b(x)$ which begin in Ω never leave Ω . Denote by Σ_x^ϵ the set of points on $\partial\Omega$ where the solution ξ^ϵ of (0.1) with $\xi(0) = x \in \Omega$ leaves Ω . Find the asymptotic behavior of Σ_x^ϵ as $\epsilon \rightarrow 0$.

In Section 1 we state the Ventsel–Freidlin probabilistic estimates (the second one in a more general form as needed for subsequent applications).

In Sections 2 and 3 we extend the results of Varadhan in [7], [8] regarding the asymptotic behavior, as $\epsilon \rightarrow 0$, of the fundamental solution and Green’s function, to the case of a general vector b . The proofs employ the estimates of Section 1. The inequalities in (0.4) can also be generalized to the case of any b , and (0.5) can be viewed as a special case of this generalization.

In Section 4 we deal with the exit problem, already mentioned above.

1. The estimates of Ventsel–Freidlin. Consider a stochastic differential system of n equations

$$(1.1) \quad d\xi^\epsilon(t) = b(\xi^\epsilon(t)) dt + \epsilon \sigma(\xi^\epsilon(t)) dw(t)$$

where $w(t)$ is n -dimensional Brownian motion, $b = (b_1, \dots, b_n)$ and σ is an $n \times n$ matrix (σ_{ij}) . Let $a \equiv (a_{ij}) = \sigma\sigma^*$, where σ^* is the transpose of σ . We shall assume throughout this work that:

(A) $a(x)$, $b(x)$ are uniformly Lipschitz continuous in compact subsets of $\mathbf{R}^n$, and

$$\begin{aligned} |a(x)| &\leq M, & |a(x) - a(y)| &\leq M |x - y|^\alpha, \\ |b(x)| &\leq M, & |b(x) - b(y)| &\leq M |x - y|^\alpha \end{aligned}$$

for all x, y in $\mathbf{R}^n$, where M, α are positive constants and $\alpha \leq 1$, and

$$(1.2) \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq m |\xi|^2 \quad (m \text{ positive constant})$$

for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$.

Notice that, since (1.2) holds, $a(x)$ is Hölder (or Lipschitz) continuous if and only if $\sigma(x)$ is Hölder (or Lipschitz) continuous. We also note that the local Lipschitz continuity of $a(x)$ is needed only to ensure the existence of a unique solution of (1.1), given $\xi^*(0)$. Since (1.2) holds, there exists a unique solution of (1.1) in the sense of Stroock and Varadhan [6] when the condition of Lipschitz continuity of $a(x)$ is removed from (A). Since all the results of this paper can be adapted to this notion of solution, they remain true when we omit from (A) the restriction of Lipschitz continuity of $a(x)$.

For x, y in $\mathbf{R}^n$, set $\rho(x, y) = |x - y|$.

Denote by C_{T_1, T_2} the space of all continuous functions $\phi(t)$, $T_1 \leq t \leq T_2$, with range in $\mathbf{R}^n$, and set

$$\rho_{T_1, T_2}(\phi, \psi) = \min_{T_1 \leq t \leq T_2} \rho_t(\phi(t), \psi(t))$$

if ϕ, ψ belong to C_{T_1, T_2} . If Φ is a subset of C_{T_1, T_2} , let

$$d_{T_1, T_2}(\psi, \Phi) = \inf_{\phi \in \Phi} \rho_{T_1, T_2}(\psi, \phi).$$

If $\phi \in C_{T_1, T_2}$, we set $I_{T_1, T_2}(\phi) = \infty$ in case ϕ is not absolutely continuous, and

$$I_{T_1, T_2}(\phi) = \int_{T_1}^{T_2} \left\| \frac{d\phi}{dt} - b(\phi(t)) \right\|^2 dt$$

in case ϕ is absolutely continuous; here

$$\begin{aligned} \left\| \frac{d\phi}{dt} - b(\phi(t)) \right\|^2 &= \left\| \sigma^{-1}(\phi(t)) \left[\frac{d\phi}{dt} - b(\phi(t)) \right] \right\|^2 \\ &= \left[\frac{d\phi}{dt} - b(\phi(t)) \right]^* \sigma^{-1}(\phi(t)) \left[\frac{d\phi}{dt} - b(\phi(t)) \right]. \end{aligned}$$

We write

$$C_T = C_{0, T}, \rho_T(\phi, \psi) = \rho_{0, T}, d_T(\psi, \Phi) = d_{0, T}(\psi, \Phi), I_T(\phi) = I_{0, T}(\phi).$$

Theorem 1.1. Let (A) hold. For any $\delta > 0$, $x \in \mathbf{R}^n$ and for any $\phi \in C_T$, with $\phi(0) = x$,

$$(1.3) \quad \lim_{\epsilon \rightarrow 0} \{2\epsilon^2 \log [P_x(\rho_T(\xi^\epsilon, \phi) < \delta)]\} \geq -I_T(\phi);$$

more precisely, if $\int_0^T (1 + |d\phi/ds|^2) ds \leq K < \infty$ then

$$(1.4) \quad P_x(\rho_T(\xi^\epsilon, \phi) < \delta) > \frac{1}{2} \exp \left\{ -\frac{I_T(\phi)}{2\epsilon^2} - \frac{C\delta^\alpha K}{2\epsilon^2} - \frac{(4CK)^{1/2}}{\epsilon} \right\}$$

provided $C_0 \epsilon^2 T \leq (1/4)\delta^2$, where C, C_0 are positive constants depending only on M, m .

The notation P_x indicates the probability of an event, given that $\xi^*(0) = x$.

In [11] the inequality (1.4) is stated (for $\alpha = 1$) without the factor $1/2$. The authors, however, neglect to estimate from below the probability of the event

$$A = \{ \sup_{0 \leq s \leq T} |z_s| \geq \delta \}$$

(i.e., they tacitly take $P(A) = 1$). To do things correctly, we use the martingale inequality to deduce that

$$P(A) \leq \frac{C_0 \epsilon^2 T}{\delta^2} \leq 1/4 \quad \text{if} \quad C_0 \epsilon^2 T \leq (1/4) \delta^2.$$

Then, together with the other estimates in [11], we obtain (1.4).

Let Γ and Δ be disjoint sets in $\mathbf{R}^n$; Γ is compact and Δ is a finite union of closed domains with C^2 boundary $\partial\Delta$.

For any set A , write $\rho(x, A) = \inf_{y \in A} \rho(x, y)$.

Denote by $\Phi_{x,T}(\Gamma, \Delta)$ the set of all $\phi \in C_T$ satisfying:

1) $\phi(0) = x$;

2) ϕ intersects Γ ;

3) if $\bar{t} = \min \{t; \phi(t) \in \Gamma\}$ then $\phi(s) \notin \Delta$ for all $0 \leq s \leq \bar{t}$, i.e., ϕ intersects Γ before it can possibly intersect Δ .

Let I_0 be a positive number and denote by Φ_0 the subset of $\Phi_{x,T}(\Gamma, \Delta)$ consisting of all ϕ with $I_T(\phi) \leq I_0$.

Theorem 1.2. *Let (A) hold. For any $\delta > 0$, $x \in \mathbf{R}^n \setminus \Delta$,*

$$(1.5) \quad \overline{\lim}_{\epsilon \rightarrow 0} \{2\epsilon^2 \log [P_x(d_T(\xi^\epsilon, \Phi_0) > \delta; \xi^\epsilon \in \Phi_{x,T}(\Gamma, \Delta))]\} \leq -I_0.$$

If the set Φ_0 is empty, then the condition $d_T(\xi^\epsilon, \Phi_0) > \delta$ is trivially true, i.e., (1.5) becomes:

$$\overline{\lim}_{\epsilon \rightarrow 0} \{2\epsilon^2 \log [P_x(\xi^\epsilon \in \Phi_{x,T}(\Gamma, \Delta))]\} \leq -I_0.$$

In [11] the theorem is stated only in case $\Phi_{x,T}(\Gamma, \Delta)$ is the set of all $\phi \in C_T$ with $\phi(0) = x$. Both theorems 1.1 and 1.2 are stated in [11] under the stronger assumptions that the second derivatives of a_{ij} and the first derivatives of b_i are continuous and bounded. However, a close look at the proof shows that it suffices to assume the smoothness conditions of (A).

Proof. We shall give the proof only in case the set Φ_0 is non-empty; the proof in case Φ_0 is empty can be obtained by minor modifications.

Let $0 < \mu < (1/2) \text{dist}(\Delta, \Gamma)$, $J > 0$, and define

$$\Gamma_\mu = \{x \in \mathbf{R}^n; \rho(x, \Gamma) < \mu\},$$

$$\Phi_{\mu,J} = \{\phi \in \Phi_{x,T}(\Gamma_\mu, \Delta); I_T(\phi) \leq J\}.$$

We claim that for every $\phi \in \Phi_{\mu, J}$ there exists a function ψ in $\Phi_{x, T}(\Gamma, \Delta)$ such that

$$(1.6) \quad |I_T(\phi) - I_T(\psi)| \leq \tilde{C}(J + 1)\mu,$$

$$(1.7) \quad \rho_T(\phi, \psi) \leq \tilde{C}(J + 1)\mu,$$

where $\tilde{C}$ is a constant independent of J, μ .

Indeed, let s_0 denote the first time ϕ intersects Γ_μ . If

$$\begin{aligned} \tilde{\psi}(s) &= \begin{cases} \phi(s) & \text{if } 0 \leq s < s_0, \\ \text{straight segment of length } \mu \text{ connecting } \phi(s_0) \text{ to } \Gamma \text{ and} \\ \text{then traced back to } \phi(s_0), & \text{for } s_0 < s < s_0 + 2\mu, \\ \phi(s - 2\mu) & \text{if } s_0 + 2\mu < s < T + 2\mu, \end{cases} \\ \psi(s) &= \tilde{\psi}\left(\frac{s(T + 2\mu)}{T}\right), \end{aligned}$$

then (1.6), (1.7) hold.

We now proceed as in [11] and construct a random polygonal curve $\ell^e(t)$ (depending on h_0, r_0). Suppose

$$(1.8) \quad d_T(\xi^e, \Phi_0) > \delta, \xi^e \in \Phi_{x, T}(\Gamma, \Delta).$$

Then, for any small $\nu > 0, \mu > 0$,

$$\ell^e \in \Phi_{x, T}(\Gamma_\mu, \Delta'), \quad \rho(\ell^e, \xi^e) < \delta/2$$

provided h_0, r_0 are sufficiently small; here Δ' is the subset of Δ consisting of all points x with $\rho(x, \mathbf{R}^n \setminus \Delta) > \nu$.

Suppose $I_T(\ell^e) \leq J$. Then, by (1.6), (1.7) (with Δ replaced by Δ'), there exists a curve ψ in $\Phi_{x, T}(\Gamma, \Delta')$ such that

$$\begin{aligned} I_T(\psi) &\leq J + \tilde{C}(J + 1)\mu, \\ \rho_T(\psi, \ell^e) &\leq \tilde{C}(J + 1)\mu. \end{aligned}$$

Further, since $\partial\Delta$ is in C^2 , we can modify ψ into a curve ψ^* in C_T such that

$$\begin{aligned} \psi^* &\in \Phi_{x, T}(\Gamma, \Delta), \\ I_T(\psi^*) &\leq I_T(\psi) + C\nu, \\ \rho_T(\psi^*, \psi) &\leq C\nu, \end{aligned}$$

where $C = C_0(J + 1)$ and C_0 is a constant independent of J, ν . Indeed, let Ω_0 be a δ_0 -neighborhood of $\partial\Delta$, δ_0 small. Let Λ be a diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$ such that Λ maps $\mathbf{R}^n \setminus \Delta'$ onto $\mathbf{R}^n \setminus \Delta$ and its Jacobian is equal to identity $+ O(\nu)$. Λ can be constructed by “pushing” the points of Ω_0 along the normals to $\partial\Delta$ in an appropriate smooth manner, while leaving the points of $\mathbf{R}^n \setminus \Omega_0$ unchanged. Now take $\psi^*(t) = \Lambda\psi(t)$.

If $J + \tilde{C}(J + 1)\mu + C\nu \leq I_0$ and $\tilde{C}(J + 1)\mu + C\nu < \delta/2$, then $I_T(\psi^*) \leq I_0$ (so that $\psi^* \in \Phi_0$) and $\rho_T(\xi^\epsilon, \psi^*) < \delta$; i.e., $d_T(\xi^\epsilon, \Phi_0) < \delta$. Since this contradicts (1.8), we must have

$$(1.9) \quad I_T(\ell^\epsilon) \geq J, \quad J = I_0 - h,$$

where h can be taken arbitrarily small (if μ, ν are arbitrarily small).

We have proved that (1.8) implies (1.9). Hence

$$P_x[\rho_T(\xi^\epsilon, \Phi_0) > \delta, \xi^\epsilon \in \Phi_{x,T}(\Gamma, \Delta)] \leq P_x[I_T(\ell^\epsilon) \geq I_0 - h].$$

We can now proceed, as in [11], to evaluate the right-hand side. This completes the proof.

Denote by $\Phi_{x,T}^*(\Gamma, \Delta)$ the subset of $\Phi_{x,T}(\Gamma, \Delta)$ consisting of all the curves ϕ that actually do intersect the set Δ (of course, only after intersecting Γ at some preceding time). Let Φ_0^* be the subset of $\Phi_{x,T}^*(\Gamma, \Delta)$ consisting of all ϕ with $I_T(\phi) \leq I_0$.

Later on we shall need the following variant of Theorem 1.2.

Theorem 1.2'. *Let (A) hold. For any $\delta > 0, x \in \mathbf{R}^n \setminus \Delta$,*

$$(1.10) \quad \overline{\lim}_{\epsilon \rightarrow 0} \{2\epsilon^2 \log P_x[d_T(\xi^\epsilon, \Phi_0^*) > \delta; \xi^\epsilon \in \Phi_{x,T}^*(\Gamma, \Delta)]\} \leq -I_0.$$

Proof. Arguing as before we conclude that if

$$(1.11) \quad d_T(\xi^\epsilon, \Phi_0^*) > \delta, \quad \xi^\epsilon \in \Phi_{x,T}^*(\Gamma, \Delta)$$

and if $I_T(\ell^\epsilon) \leq J$, then there is a curve ψ^* in $\Phi_{x,T}(\Gamma, \Delta)$ such that

$$\rho_T(\xi^\epsilon, \psi^*) < k, \quad I_T(\psi^*) \leq J + h$$

where k, h are any given positive and small numbers, provided h_0, r_0 are sufficiently small.

Modify ψ^* into $\psi^{**} \in \Phi_{x,T}^*(\Gamma, \Delta)$ as follows: If $\psi^* \in \Phi_{x,T}^*(\Gamma, \Delta)$, take $\psi^{**} = \psi^*$. If $\psi^* \notin \Phi_{x,T}^*(\Gamma, \Delta)$, then notice that, since $\xi^\epsilon \in \Phi_{x,T}^*(\Gamma, \Delta)$, $\xi^\epsilon(t_1) \in \Gamma$ and $\xi^\epsilon(t_2) \in \Delta$ for some $0 \leq t_1 < t_2 \leq T$. Therefore $\psi^*(t_1)$ and $\psi^*(t_2)$ belong to k -neighborhoods of Γ and Δ respectively. Modify ψ^* into ψ^{**} so that ψ^{**} intersects Γ at $t_1 + 0(k)$ and intersects Δ for the first time at $t_2 + 0(k)$, and

$$|I_T(\psi^*) - I_T(\psi^{**})| \leq C_1 k,$$

$$\rho_T(\psi^*, \psi^{**}) \leq C_1 k$$

(cf. the modification from ϕ into ψ in the proof of Theorem 1.2).

We conclude that, if k is sufficiently small,

$$\rho_T(\xi^\epsilon, \psi^{**}) < \delta, \quad I_T(\psi^{**}) < J + 2h.$$

Since $\psi^{**} \in \Phi_{x,T}^*(\Gamma, \Delta)$, we get a contradiction to the first part of (1.11) if $J + 2h \leq I_0$. Therefore, if $I_T(\ell^\epsilon) \leq J$ then $J \geq I_0 - 2h$, i.e., (1.11) implies that $I_T(\ell^\epsilon) \geq I_0 - 2h$.

We can now proceed precisely as in the proof of Theorem 1.2.

Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary $\partial\Omega$. Consider the parabolic initial-boundary value problem:

$$(1.12) \quad \begin{aligned} \frac{\partial u_\epsilon}{\partial t} &= L_\epsilon u_\epsilon \quad \text{if } x \in \Omega, \quad t > 0, \\ u_\epsilon(x, 0) &= 0 \quad \text{if } x \in \Omega, \\ u_\epsilon(x, t) &= 0 \quad \text{if } x \in \partial\Omega, \quad t > 0. \end{aligned}$$

By standard theory [3] there exists a unique solution. By Ito's formula,

$$(1.13) \quad u_\epsilon(x, t) = P_x\{\tau^\epsilon \leq t\}$$

where τ^ϵ is the exit time of $\xi^\epsilon(t)$ from Ω .

Let

$$I(t, x, \partial\Omega) = \inf_{\phi} I_t(\phi)$$

where ϕ varies over the set of all ϕ in C_t satisfying: $\phi(0) = x$, $\min_{0 \leq s \leq t} \rho(\phi(s), \partial\Omega) = 0$.

Theorem 1.3. *Let (A) hold. Then, for any $t > 0$, $x \in \Omega$,*

$$(1.14) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log u_\epsilon(x, t)] = -I(t, x, \partial\Omega).$$

This theorem was proved in [11]; the proof follows fairly easily from Theorem 1.1 and a special case of Theorem 1.2 (when $\Gamma = \mathbf{R}^n$, $\Delta = \phi$).

2. Behavior of fundamental solutions. In this section and in the following one we assume:

(A') The condition (A) holds and, in addition, the b_i are continuously differentiable and

$$\left| \frac{\partial b_i(x)}{\partial x_i} - \frac{\partial b_i(\bar{x})}{\partial x_i} \right| \leq M |x - \bar{x}|^\alpha.$$

Denote by $p_\epsilon(t, x, y)$ the fundamental solution of the Cauchy problem for the parabolic equation

$$(2.1) \quad \frac{\partial u}{\partial t} = L_\epsilon u \quad (x \in \mathbf{R}^n, t > 0),$$

where L_ϵ is given by (0.3). It was proved by Aronson [1] that (when (A') holds)

$$(2.2) \quad p_\epsilon(t, x, y) \leq \frac{A_0}{\epsilon^n t^{n/2}} \exp \left\{ -\frac{c |\phi(t, x) - y|^2}{\epsilon^2 t} \right\} \quad \text{if } t < T^*,$$

$$(2.3) \quad p_\epsilon(t, x, y) \geq \frac{A_1}{\epsilon^n t^{n/2}} \exp \left\{ -\frac{\gamma |\phi(t, x) - y|^2}{\epsilon^2 t} \right\} \\ - \frac{A_2}{\epsilon^{n-2\alpha} t^{n/2-\alpha}} \exp \left\{ -\frac{c |\phi(t, x) - y|^2}{\epsilon^2 t} \right\} \quad \text{if } t < t^*;$$

T^* is any positive number, t^* is a sufficiently small positive number, A_0, A_1, A_2, c, γ are positive constants, and $\phi(t, x)$ is the solution of

$$\frac{d\phi}{dt} = b(\phi), \quad \phi(0) = x.$$

Let

$$I_t(x, y) = \inf_{\phi} I_t(\phi)$$

where ϕ varies in C_t , $\phi(0) = x$, $\phi(s) = y$ for some $s \in [0, t]$.

We shall prove:

Theorem 2.1. *Let (A') hold. Then, for any x, y in $\mathbf{R}^n$ and $t > 0$,*

$$(2.4) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log p_{\epsilon}(t, x, y)] = -I_t(x, y).$$

This theorem was proved by Varadhan [7] in the special case where $b \equiv 0$. Our proof, for general b , employs ideas of [7] as well as the results of Section 1.

Proof. We first prove that

$$(2.5) \quad \overline{\lim}_{\epsilon \rightarrow 0} [2\epsilon^2 \log p_{\epsilon}(t, x, y)] \leq -I_t(x, y),$$

i.e., for any $h > 0$,

$$(2.6) \quad p_{\epsilon}(t, x, y) \leq \exp \left\{ \frac{-I_t(x, y) + h}{2\epsilon^2} \right\}$$

if ϵ is sufficiently small.

Suppose $d \equiv I_t(x, y) > 0$. For any $0 < \delta < d$, let

$$G_{\delta} = \{z \in \mathbf{R}^n; I_t(x, z) < d - \delta\}.$$

This is a bounded set (cf. [11]). Let G be an open set whose boundary ∂G consists of a finite number of C^2 hypersurfaces, such that $G_{\delta} \subset G \subset G_{\delta/2}$. Then there is a solution $u_{\epsilon}(z, s)$ of

$$L_{\epsilon} u_{\epsilon} - \frac{\partial u_{\epsilon}}{\partial s} = 0 \quad \text{if } z \in G, \quad s > 0,$$

$$u_{\epsilon}(z, 0) = 0 \quad \text{if } z \in G,$$

$$u_{\epsilon}(z, s) = 1 \quad \text{if } z \in \partial G, \quad s > 0.$$

Comparing $u_{\epsilon}(z, s)$ with the function $p_{\epsilon}(s, z, y)$ (for y fixed), by means of the maximum principle [3], we get

$$(2.7) \quad p_{\epsilon}(t, x, y) \leq u_{\epsilon}(x, t) \sup_{\substack{0 \leq s \leq t \\ z \in \partial G}} p_{\epsilon}(s, z, y).$$

By Theorem 1.3, for any $\mu > 0$

$$(2.8) \quad u_\epsilon(x, t) < \exp \left[-\frac{I(t, x, \partial G)}{2\epsilon^2} + \frac{\mu}{2\epsilon^2} \right] \leq \exp \left[-\frac{d}{2\epsilon^2} + \frac{\delta + \mu}{2\epsilon^2} \right]$$

provided ϵ is sufficiently small.

Since $z \neq y$ for $z \in \partial G$, also $\phi(s, z) \neq y$ if s is sufficiently small, i.e., $|\phi(s, z) - y| \geq \gamma_0 > 0$ if $z \in \partial G$, $0 \leq s \leq s_0$, for some $s_0 \in (0, t)$. But then, by (2.2),

$$p_\epsilon(s, z, y) \leq C \text{ if } 0 < s \leq s_0, z \in \partial G \quad (C \text{ constant}).$$

If $s_0 \leq s \leq t$ then, by (2.2),

$$(2.9) \quad p_\epsilon(s, z, y) \leq \frac{C}{\epsilon^n}$$

with another constant C . Thus, (2.9) holds for all $0 < s \leq t$, $z \in \partial G$.

Substituting (2.8), (2.9) in (2.7), the assertion (2.6) follows in case $I_t(x, y) > 0$. Since, by (2.2),

$$(2.10) \quad p_\epsilon(t, x, y) \leq \frac{A_0}{\epsilon^n t^{n/2}} \leq \exp \left[\frac{h}{2\epsilon^2} \right]$$

for any $h > 0$, provided ϵ is sufficiently small, the assertion (2.6) in case $I_t(x, y) = 0$ is certainly true.

We next have to prove that

$$(2.11) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log p_\epsilon(t, x, y)] \geq -I_t(x, y),$$

i.e., for any $h_0 > 0$,

$$(2.12) \quad p_\epsilon(t, x, y) > \exp \left[\frac{-I_t(x, y) - h_0}{2\epsilon^2} \right]$$

provided ϵ is sufficiently small.

Let $\zeta \in \mathbf{R}^n$, $\theta > 0$, $|\zeta - y| \leq h$ where $0 < h < 1$, and consider the curve ψ :

$$\psi(s) = \phi(-s, y) \quad \text{for } 0 \leq s \leq \theta.$$

Let M be a positive number (to be determined later on). Suppose first that

$$(2.13) \quad |\zeta - \psi(\theta)| > M\epsilon.$$

Let $0 < \lambda < \theta$ and consider the function

$$w(z, s) = p_\epsilon(s + \lambda, z, y) \quad \text{for } z \in B_{\lambda+s}, 0 < s < \theta - \lambda$$

where $B_s = \{z; |z - \psi(s)| > M\epsilon\}$. If $z \in \partial B_{\lambda+s}$, $|z - \psi(s + \lambda)| = M\epsilon$; hence

$$|\phi(s + \lambda, z) - \phi(s + \lambda, \psi(s + \lambda))| \leq CM\epsilon \quad (C \text{ constant}).$$

But since

$$\phi(s + \lambda, \psi(s + \lambda)) = \phi(s + \lambda, \phi(-s - \lambda, y)) = \phi(0, y) = y,$$

we get

$$|\phi(s + \lambda, z) - y| \leq CM\epsilon.$$

Using (2.3) we conclude that

$$(2.14) \quad w(z, s) = p_\epsilon(s + \lambda, z, y) \geq \frac{C_\lambda}{\epsilon^n} \text{ if } z \in \partial B_{\lambda+s}, 0 < s < \theta - \lambda, \theta < t^*,$$

where C_λ is a positive constant depending on λ , provided ϵ is sufficiently small (depending on λ).

We also have

$$(2.15) \quad w(z, 0) > 0 \quad \text{if } z \in B_\lambda.$$

Let

$$G = G(M) = \bigcup_{0 \leq s \leq \theta - \lambda} B_{s+\lambda} \times \{s\}, \quad \partial_0 G = \partial_0 G(M) = \bigcup_{0 \leq s \leq \theta - \lambda} (\partial B_{s+\lambda} \times \{s\}).$$

Consider the solution $u_\epsilon(z, s)$ of

$$(2.16) \quad \begin{aligned} \frac{\partial u_\epsilon}{\partial t} &= L_\epsilon u_\epsilon \quad \text{if } z \in B_{s+\lambda}, \quad 0 < s < \theta - \lambda, \\ u_\epsilon(z, 0) &= 0 \quad \text{if } z \in B_\lambda, \\ u_\epsilon(z, s) &= 1 \quad \text{if } z \in \partial B_{\lambda+s}, \quad 0 < s < \theta - \lambda. \end{aligned}$$

Assume first that $\partial_0 G$ is in C^2 . Then (by [3]) there exists a unique solution u_ϵ of (2.16). By Ito's formula,

$$(2.17) \quad u_\epsilon(z, \theta - \lambda) = P_z(\tau < \theta - \lambda) \text{ if } z \in B_{(\theta - \lambda) + \lambda} = B_\theta,$$

where τ is the first time the path $(\theta - \lambda - s, \xi^\epsilon(s))$ hits the set $\partial_0 G(M)$. Notice that, by (2.13), $\zeta \in B_\theta$; hence (2.17) holds, in particular, for $z = \zeta$.

Let ψ_0 be a straight line in $C_{\theta-\lambda}$ connecting ζ to $\psi(\lambda)$, i.e., $\psi_0(0) = \zeta$, $\psi_0(\theta - \lambda) = \psi(\lambda)$. Since $|\zeta - \psi(\lambda)| \leq |\zeta - y| + |\psi(\lambda) - y| \leq h + \text{const. } \lambda$,

$$(2.18) \quad I_{\theta-\lambda}(\psi_0) \leq \frac{\tilde{c}\hat{h}^2}{\theta - \lambda} + \tilde{c}(\theta - \lambda) \quad (\tilde{c} \text{ positive constant, } \hat{h}^2 = h^2 + \lambda^2).$$

Clearly

$$\{\tau < \theta - \lambda\} \supset \{\rho_{\theta-\lambda}(\xi^\epsilon, \psi_0) < M\epsilon\}.$$

Hence, by Theorem 1.1 and (2.17) (with $z = \zeta$)

$$\begin{aligned} u_\epsilon(\zeta, \theta - \lambda) &\geq P_\zeta\{\rho_{\theta-\lambda}(\xi^\epsilon, \psi_0) < M\epsilon\} \\ &\geq (1/2) \exp \left\{ -\frac{I_{\theta-\lambda}(\psi_0)}{2\epsilon^2} - \frac{C\delta^\alpha}{2\epsilon^2} - \frac{(4CK)^{1/2}}{\epsilon} \right\} \end{aligned}$$

where $K = \int_0^{\theta-\lambda} (1 + |d\psi_0/ds|^2) ds$, $\delta = M\epsilon$, provided $C_0\epsilon^2\theta \leq \delta^2/4$. Notice that the last inequality means that $C_0\theta \leq M^2/4$. We now choose M so that this last inequality holds. Making use also of (2.18), we get

$$(2.19) \quad u_\epsilon(\zeta, \theta - \lambda) \geq \exp \left\{ -\frac{(\bar{c} + 1)(h^2 + \theta^2)}{(\theta - \lambda)\epsilon^2} \right\}$$

provided ϵ is sufficiently small.

We now apply the maximum principle in order to compare $u_\epsilon(z, s)$ with $p_\epsilon(s + \lambda, z, y)$ for $z \in B_{s+\lambda}$, $0 < s \leq \theta - \lambda$. Using (2.14), (2.15), we get

$$(2.20) \quad p_\epsilon(s + \lambda, z, y) \geq \frac{C_\lambda}{\epsilon^n} u_\epsilon(z, s).$$

Taking, in particular, $s = \theta - \lambda$, $z = \zeta$, $\lambda = \theta/2$ and using (2.19), we obtain

$$(2.21) \quad p_\epsilon(\theta, \zeta, y) \geq \frac{C(\theta)}{\epsilon^n} \exp \left\{ -\frac{c(h^2 + \theta^2)}{\theta\epsilon^2} \right\}, \quad (|\zeta - y| \leq h, \theta < t^*),$$

for all ϵ sufficiently small, where $C(\theta)$ is a positive constant depending on θ ($C(\theta) \rightarrow 0$ if $\theta \rightarrow 0$), and $c = 3(\bar{c} + 1)$.

We have proved (2.21) assuming that $\partial_0 G(M)$ is in C^2 . If this is not the case, we replace G by a region G' with $G(M') \subset G' \subset G(M)$ such that its lateral boundary $\partial_0 G'$ is in C^2 and $(\theta - \lambda, \zeta) \in G'$. We then proceed as before, replacing G and $\partial_0 G$ everywhere by G' and $\partial_0 G'$ respectively.

So far we have proved (2.21) under the restriction (2.13). If $|\zeta - \psi(\theta)| \leq M\epsilon$ then $|\phi(\theta, \zeta) - y| \leq CM\epsilon$ for some constant C . But then (2.21) follows immediately from (2.3), provided $0 < \theta < t^*$ and ϵ is sufficiently small (depending on θ).

In order to prove (2.11) we shall need, in addition to (2.21), the following result:

Let G be a ball of radius $|G|$ and center z . Then, for any $s > 0$, there is a number ν in $(0, s)$ such that

$$(2.22) \quad P_x \{ \xi^\epsilon(s - \nu) \in G \} \geq \exp \left\{ -\frac{I_s(x, z)}{2\epsilon^2} - \frac{h'}{2\epsilon^2} \right\}$$

for any $h' > 0$, provided ϵ is sufficiently small; ν depends on z, s, h' , but not on $|G|$.

To prove (2.22) let ψ be a curve in C_s such that

$$\psi(0) = x, \quad \psi(s - \nu) = z, \quad I_s(\psi) \leq I_s(x, z) + \frac{h'}{2} \quad \text{for some } \nu \in (0, s).$$

Clearly

$$\{ \xi^\epsilon(s - \nu) \in G \} \supset \{ \rho_{s-\nu}(\xi^\epsilon, \psi) < |G| \},$$

and (2.22) follows upon taking P_x of both sides and applying Theorem 1.1.

We proceed to prove (2.11), using the semi-group property

$$p_\epsilon(t, x, y) = \int_{\mathbf{R}^n} \cdots \int_{\mathbf{R}^n} p_\epsilon(t - \mu_1 - \cdots - \mu_k, x, z_1) \\ \cdot p_\epsilon(\mu_1, z_1, z_2) \cdots p_\epsilon(\mu_k, z_k, y) dz_1 \cdots dz_k.$$

This implies that

$$(2.23) \quad p_\epsilon(t, x, y) \geq \int_{G_1} \cdots \int_{G_k} p_\epsilon(t - \mu_1 - \cdots - \mu_k, x, z_1) \\ \cdot p_\epsilon(\mu_1, z_1, z_2) \cdots p_\epsilon(\mu_k, z_k, y) dz_1 \cdots dz_k;$$

the number k is chosen such that $\mu/2 \leq \mu_i \leq \mu < t^*$ for each i , and $G_i = \{z_i; |z_i - y| < h\}$, where μ, h are any given small positive numbers.

By (2.21),

$$p_\epsilon(\mu_1, z_1, z_2) \cdots p_\epsilon(\mu_k, z_k, y) \geq \beta(\mu) \exp \left\{ -\frac{\beta(h^2 + \mu^2)}{\epsilon^2 \mu} \right\}$$

provided ϵ is sufficiently small, where $\beta(\mu)$, β are positive constants independent of h, ϵ (and β is also independent of μ). Hence,

$$(2.24) \quad p_\epsilon(t, x, y) \geq \gamma_0 h^{k-1} \beta(\mu) \\ \cdot \exp \left\{ -\frac{\beta(h^2 + \mu^2)}{\epsilon^2 \mu} \right\} P_x \{ \xi^\epsilon(t - \mu_1 - \cdots - \mu_k) \in G_1 \},$$

where γ_0 is a positive constant.

By (2.22) with $s = t$, there is a number $\theta \in (0, t)$ (independent of h) such that

$$P_x \{ \xi^\epsilon(t - \theta) \in G_1 \} \geq \exp \left\{ -\frac{I_\epsilon(x, y) + h'}{2\epsilon^2} \right\}$$

if ϵ is sufficiently small. Taking $\mu_1 + \cdots + \mu_k = \theta$ (this is possible if $2\mu < \theta$) in (2.24), we get

$$\lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log p_\epsilon(t, x, y)] \geq -\frac{\beta(h^2 + \mu^2)}{\mu} - I_\epsilon(x, y) - h'.$$

Notice that μ is independent of the parameter h , and $\mu < \theta/2$. Taking first $h \rightarrow 0$, then $\mu \rightarrow 0$, and finally $h' \rightarrow 0$, the assertion (2.11) follows.

3. Behavior of Green's function. Let Ω be a bounded domain with C^2 boundary. Then (see [3]) there exists Green's function $q_\epsilon(t, x, y)$ of (2.1) in the cylinder $\Omega \times (0, \infty)$, i.e., a fundamental solution in $\Omega \times (0, \infty)$ such that

$$q_\epsilon(t, x, y) \rightarrow 0 \quad \text{if} \quad \rho(x, \partial\Omega) \rightarrow 0, \quad t > 0.$$

It is easy to verify that for any measurable subset A of Ω ,

$$q_\epsilon(t, x, A) = P_x \{ \xi^\epsilon(s) \in A \quad \text{for} \quad 0 \leq s \leq t, \xi^\epsilon(t) \in A \}$$

where

$$q_\epsilon(t, x, A) = \int_A q_\epsilon(t, x, y) dy.$$

Notice, by the maximum principle, that

$$(3.1) \quad 0 \leq q_*(t, x, y) \leq p_*(t, x, y).$$

For any x, y in Ω , let

$$I_t^0(x, y) = \inf_{\phi} I_t(\phi)$$

where ϕ varies over the functions in C_t satisfying: $\phi(0) = x$, and there is an $s_0 \in (0, t]$ such that $\phi(s) \in \Omega$ for $0 \leq s \leq s_0$, $\phi(s_0) = y$. Notice that if we replace $I_t(\phi)$ by $I_{s_0}(\phi)$, the definition of $I_t^0(x, y)$ does not change.

Theorem 3.1. *Let (A') hold. Then, for any x, y in Ω and $t > 0$,*

$$(3.2) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log q_*(t, x, y)] = -I_t^0(x, y).$$

For $b \equiv 0$ this result was previously proved by Varadhan [8].

Proof. We first establish the inequality

$$(3.3) \quad \overline{\lim}_{\epsilon \rightarrow 0} [2\epsilon^2 \log q_*(t, x, y)] \leq -I_t^0(x, y),$$

i.e., for any $h > 0$,

$$(3.4) \quad q_*(t, x, y) \leq \exp \left\{ \frac{-I_t^0(x, y) + h}{2\epsilon^2} \right\}$$

if ϵ is sufficiently small.

For any $0 < \delta < \rho(y, \partial\Omega)$, let S_δ be a sphere of radius δ and center y , τ = first time the set S_δ is reached, given $\xi^*(0) = x$. By the strong Markov property,

$$(3.5) \quad q_*(t, x, y) = \int_0^t \int_{S_\delta} q_*(s, x, \eta) q_*(t-s, \eta, y) dS^n P(\tau \leq ds)$$

where dS^n is the surface element on S_δ and $P(\tau \leq ds)$ is the probability measure of τ .

Since $|\eta - y| = \delta$, we get from (2.2) (cf. the paragraph containing (2.9))

$$p_*(t-s, \eta, y) \leq \frac{C}{\epsilon^n}.$$

Hence, by (3.1),

$$(3.6) \quad q_*(t-s, \eta, y) \leq \frac{C}{\epsilon^n}.$$

For any $\Lambda > 0$, define

$$\Phi_\delta = \{ \text{all } \phi \text{ in } C_t ; \phi(0) = x, \phi(s) \text{ intersects } S_\delta ; \text{ if}$$

$$s_0 = \min \{ s ; \phi(s) \in S_\delta \}, \text{ then } \phi(s) \in \Omega \text{ for all } 0 \leq s \leq s_0 \},$$

$$\Phi_{\delta, \Lambda} = \{ \text{all } \phi \text{ in } \Phi_\delta \text{ with } I_t(\phi) \leq I_t^0(x, y) - \Lambda \}.$$

If $\phi \in \Phi_{\delta, \Lambda}$ and if $\phi(s)$ intersects S_δ for the first time at time s_0 , then we can modify ϕ into ψ (cf. the proof of Theorem 1.2) such that $\psi \in C_t$, $\psi(0) = x$, $\psi(s_1) = y$ for some $s_1 \in (0, t)$, $\psi(s) \in \Omega$ for $0 \leq s \leq s_1$, and $I_t(\psi) \leq I_t(\phi) + C_1\delta$; C_1 is a positive constant independent of Λ . Hence, if $\Lambda = 2C_1\delta$ then

$$I_t(\psi) < I_t^\Omega(x, y),$$

but this contradicts the definition of $I_t^\Omega(x, y)$. Consequently, there can be no ϕ in $\Phi_{\delta, \Lambda}$, i.e., $\Phi_{\delta, \Lambda}$ is empty if $\Lambda = 2C_1\delta$.

Taking $\Lambda = 2C_1\delta$ we then, trivially, have

$$(3.7) \quad \begin{aligned} & \{\xi^\epsilon(s) \in \Omega \text{ for } 0 \leq s \leq \tau, \tau \leq t\} \\ & \subset \{d_t(\xi^\epsilon, \Phi_{\delta, \Lambda}) > \beta; \xi^\epsilon(s) \in \Omega \text{ for } 0 \leq s \leq \tau, \tau \leq t\}, \end{aligned}$$

for any $\beta > 0$. Applying Theorem 1.2 (with $\Gamma = S_\delta$, $\Delta = \mathbb{R}^n \setminus \Omega$), we get

$$(3.8) \quad \begin{aligned} & \int_0^t \int_{S_\delta} q_\epsilon(s, x, \eta) dS_\eta P(\tau \in ds) = P_x\{\xi^\epsilon(s) \in \Omega \text{ for } 0 \leq s \leq \tau, \tau \leq t\} \\ & \leq \exp \left\{ -\frac{I_t^\Omega(x, y) - 2C_1\delta - h'}{2\epsilon^2} \right\} \text{ for any } h' > 0, \end{aligned}$$

provided ϵ is sufficiently small.

Substituting (3.6), (3.8) into (3.5), the assertion (3.4) follows.

It remains to prove that

$$(3.9) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log q_\epsilon(t, x, y)] \geq -I_t^\Omega(x, y),$$

i.e., for any $h > 0$,

$$(3.10) \quad q_\epsilon(t, x, y) \geq \exp \left\{ -\frac{I_t^\Omega(x, y) + h}{2\epsilon^2} \right\}$$

if ϵ is sufficiently small.

We begin with a lemma.

Lemma 3.2. *Let $\rho(y, \partial\Omega) \geq \delta_0 > 0$. If $z \in \Omega$, $|z - y| < \delta$ and δ, t are positive and sufficiently small (depending on δ_0), then*

$$(3.11) \quad q_\epsilon(t, z, y) \geq \beta_0(t) \exp \left[-\frac{\beta(\delta^2 + t^2)}{\epsilon^2 t} \right]$$

for all ϵ sufficiently small (depending on δ, t), where $\beta_0(t)$ is a positive constant depending on t but not on δ, ϵ , and β is a positive constant independent of t, δ, ϵ .

Proof. Denote by $\hat{\tau}$ the exit time from Ω . By the strong Markov property, if $z \in \Omega$, then

$$(3.12) \quad q_\epsilon(t, z, y) = p_\epsilon(t, z, y) - \int_0^t \int_{\partial\Omega} p_\epsilon(t-s, \eta, y) p_\epsilon(s, z, \eta) dS_\eta P(\hat{\tau} \in ds)$$

where dS_η is the surface element on $\partial\Omega$. Since, for $\eta \in \partial\Omega$, $\rho(y, \eta) \geq \delta_0$, $\rho(z, \eta) > \delta_0/2$ (if $\delta < \delta_0/2$), we have

$$|\phi(t - s, \eta) - y| > \delta_0/2,$$

$$|\phi(s, z) - \eta| > \delta_0/4$$

if t is sufficiently small, say $t \in (0, t_0)$. But then, by (2.2),

$$p_\epsilon(t - s, \eta, y)p_\epsilon(s, z, \eta) \leq C \exp \left[-\frac{k}{\epsilon^2 t} \right]$$

where C, k are positive constants depending on δ_0 . Using also (2.21), we deduce from (3.12) that, if $t_0 < t^*$,

$$q_\epsilon(t, z, y) \geq \frac{C(t)}{\epsilon^n} \exp \left[-\frac{c(\delta^2 + t^2)}{\epsilon^2 t} \right] - C_1 \exp \left[-\frac{k}{\epsilon^2 t} \right]$$

where C_1 is a positive constant. Thus, if $c(\delta^2 + t^2) < k/2$ and ϵ is sufficiently small then (3.11) follows.

The next fact we need is the following: Let A be a ball with center ζ and radius $|A|$, contained in Ω . Then, for any $h' > 0$ and $s \in (0, t]$ there exists a number ν in $(0, s)$, depending on s, ζ, h' but not on $|A|$, such that

$$(3.13) \quad q_\epsilon(s - \nu, x, A) \geq \exp \left\{ -\frac{I_s^0(x, \zeta) + h'}{2\epsilon^2} \right\}$$

for all ϵ sufficiently small.

To prove it, let ψ be a curve in C_s such that $\psi(0) = x$, $\psi(s - \nu) = \zeta$, $\psi(\lambda) \in \Omega$ for $0 \leq \lambda \leq s - \nu$, and

$$I_s(\psi) \leq I_s^0(x, \zeta) + \frac{1}{2}h'.$$

Then, $\rho(\psi(\lambda), \partial\Omega) \geq c_0 > 0$ if $0 \leq \lambda \leq s - \nu$, where c_0 is some positive constant. Hence,

$$\{\xi^\epsilon(\lambda) \in \Omega \text{ for } 0 \leq \lambda \leq s - \nu, \quad \xi^\epsilon(s - \nu) \in A\} \supset \{\rho_{s-\nu}(\xi^\epsilon, \psi) < \delta_1\}$$

provided $\delta_1 < \min(c_0, |A|)$. Using Theorem 1.1, (3.13) follows.

We shall now prove (3.9). By the semi-group property of q_ϵ ,

$$(3.14) \quad \begin{aligned} q_\epsilon(t, x, y) &= \int_\Omega \cdots \int_\Omega q_\epsilon(t - \mu_1 - \cdots - \mu_k, x, z_1) \\ &\quad \cdot q_\epsilon(\mu_1, z_1, z_2) \cdots q_\epsilon(\mu_k, z_k, y) dz_1 \cdots dz_k \\ &\geq \int_{G_1} \cdots \int_{G_k} q_\epsilon(t - \mu_1 - \cdots - \mu_k, x, z_1) \\ &\quad \cdot q_\epsilon(\mu_1, z_1, z_2) \cdots q_\epsilon(\mu_k, z_k, y) dz_1 \cdots dz_k \end{aligned}$$

where $G_i = \{z_i; |z_i - y| < \delta\}$; the number k is chosen such that $\mu/2 \leq \mu_i \leq \mu$ for all i , and μ, δ are any small positive numbers. By (3.11),

$$(3.15) \quad q_\epsilon(\mu_1, z_1, z_2) \cdots q_\epsilon(\mu_k, z_k, y) \geq \beta_0(\mu) \exp \left[-\frac{\tilde{\beta}(\delta^2 + \mu^2)}{\epsilon^2 \mu} \right]$$

provided μ and δ are sufficiently small, say $\mu \leq \mu^*$, $\delta < \delta^*$, and ϵ is sufficiently small, and $\tilde{\beta}_0(\mu)$, $\tilde{\beta}$ are positive constants.

By (3.13) with $s = t$

$$(3.16) \quad \int_{G_1} q_\epsilon(t - \nu, x, z_1) dz_1 = q_\epsilon(t - \nu, x, G_1) \geq \exp \left\{ -\frac{I_t^{\Omega}(x, y) + h'}{2\epsilon^2} \right\}$$

for some $\nu \in (0, t)$; ν is independent of h . If $2\mu < \nu$ then ν can be written as $\mu_1 + \cdots + \mu_k$; consequently, (3.15) holds. Using (3.15), (3.16) in (3.14), we get

$$q_\epsilon(t, x, y) \geq C(t, \delta, h') \exp \left\{ -\frac{I_t^{\Omega}(x, y) + h' + c\delta^2/\mu + c\mu}{2\epsilon^2} \right\}$$

where c is a positive constant (independent of δ, μ). Hence,

$$\lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log q_\epsilon(t, x, y)] \geq -I_t^{\Omega}(x, y) - h' - \frac{c\delta^2}{\mu} - c\mu.$$

Taking $\delta \rightarrow 0$, then $\mu \rightarrow 0$, and finally $h' \rightarrow 0$, the assertion (3.9) follows.

Let

$$I_t^{\tilde{\Omega}}(x, y) = \inf_{\phi} I_t(\phi)$$

where ϕ varies over the subset $\tilde{\Phi}$ of C_t of functions ϕ satisfying: $\phi(0) = x$, $\phi(\tilde{t}) = y$ for some $\tilde{t} \in [0, t]$ and $\phi(s) \in \tilde{\Omega}$ for $0 \leq s \leq \tilde{t}$. Clearly

$$I_t^{\tilde{\Omega}}(x, y) \leq I_t^{\Omega}(x, y).$$

Since, however, $\partial\Omega$ is in C^2 , it is easily seen that

$$I_t^{\tilde{\Omega}}(x, y) = I_t^{\Omega}(x, y)$$

(cf. the construction of ψ^* in the proof of Theorem 1.2). Notice that there actually exists a function $\bar{\phi}$ in $\tilde{\Phi}$ such that

$$I_t^{\tilde{\Omega}}(x, y) = I_t(\bar{\phi}).$$

Theorem 3.3. (i) Let (A') hold. If $I_t(x, y) < I_t^{\Omega}(x, y)$, then

$$(3.17) \quad \frac{q_\epsilon(t, x, y)}{p_\epsilon(t, x, y)} \rightarrow 0 \quad \text{if } \epsilon \rightarrow 0.$$

(ii) Let (A') hold. If $I_t(x, y) = I_t^{\tilde{\Omega}}(x, y)$ and for every $\bar{\phi}$ in $\tilde{\Phi}$ for which $I_t(x, y) = I_t(\bar{\phi})$ we have $\bar{\phi}(s) \in \Omega$ for all $0 \leq s \leq \tilde{t}$ where $\tilde{t}$ is the first time when $\bar{\phi}(\tilde{t}) = y$, then

$$(3.18) \quad \frac{q_\epsilon(t, x, y)}{p_\epsilon(t, x, y)} \rightarrow 1 \quad \text{if } \epsilon \rightarrow 0.$$

Proof. The assertion (i) is a consequence of Theorems 2.1, 3.1. To prove (ii), denote by τ the first time $\xi^\epsilon(s)$ reaches the sphere S_δ (with center y and radius δ) after hitting $\partial\Omega$ at some previous time. By the strong Markov property,

$$(3.19) \quad p_\epsilon(t, x, y) - q_\epsilon(t, x, y) = \int_0^t \int_{S_\delta} p_\epsilon(s, x, \eta) p_\epsilon(t-s, \eta, y) dS^n P(\tau \leq s).$$

Since $|\eta - y| = \delta > 0$,

$$(3.20) \quad p_\epsilon(t-s, \eta, y) \leq \frac{C}{\epsilon^n}.$$

Denote by Φ the set of all ϕ in C_t with $\phi(0) = x$, such that ϕ intersects S_δ and, if $s'' = \min \{s; \phi(s) \in S_\delta\}$, then $\phi(s') \in \partial\Omega$ for some $0 < s' < s'' \leq t$. If $\phi \in \Phi$ then, by the assumptions of (ii),

$$I_t(\phi) > I_t(x, y) + 2c$$

for all δ sufficiently small, where c is a positive constant (independent of δ). Hence, by Theorem 1.2' (cf. the derivation of (3.8)),

$$\int_0^t \int_{S_\delta} p_\epsilon(s, x, \eta) dS^n P(\tau \leq s) = P_x\{\tau \leq t\} \leq \exp \left\{ -\frac{I_t(x, y) + c}{2\epsilon^2} \right\}.$$

Using this and (3.20) in (3.19), we get

$$\frac{p_\epsilon(t, x, y)}{q_\epsilon(t, x, y)} - 1 \leq \frac{C}{\epsilon^n} \exp \left\{ -\frac{I_t(x, y) + c}{2\epsilon^2} \right\} \frac{1}{q_\epsilon(t, x, y)}.$$

Since, by Theorem 2.1, the right-hand side converges to zero as $\epsilon \rightarrow 0$, the proof of (3.18) is complete.

Remark. Consider the space C_t of all continuous functions on $[0, t]$ (with values in $\mathbf{R}^n$), with the probabilities P_x^ϵ associated, in the usual way, with the process $\xi^\epsilon(s)$. Then,

$$(3.21) \quad \overline{\lim}_{\epsilon \rightarrow 0} [2\epsilon^2 \log P_x^\epsilon(A)] \leq - \inf_{\phi \in A_x} I_t(\phi) \quad \text{if } A \text{ is closed,}$$

$$(3.22) \quad \lim_{\epsilon \rightarrow 0} [2\epsilon^2 \log P_x^\epsilon(B)] \geq - \inf_{\phi \in B_x} I_t(\phi) \quad \text{if } B \text{ is open,}$$

where $A_x = \{\phi \in A, \phi(0) = x\}$, $B_x = \{\phi \in B, \phi(0) = x\}$. For $b \equiv 0$ this was proved by Varadhan [8]. For general b , the assertion (3.22) is an immediate consequence of Theorem 1.1. As for the proof of (3.21), it follows by slightly modifying the proof in [8] (for the case $b \equiv 0$), making use of Theorems 1.3, 3.1. Notice that Theorems 1.2, 1.2' are immediate consequences of a special case of (3.21); however, in proving (3.21) one makes use of Theorems 1.3, 3.1, which depend on Theorem 1.2.

4. The exit problem. Let Ω be a bounded domain with C^2 boundary $\partial\Omega$. Ventsel and Freidlin [11] have studied the exit problem in the case $b \cdot \nu < 0$ along $\partial\Omega$, where ν is the outward normal.

The last condition implies that every solution of the dynamical system

$$(4.1) \quad \frac{dx}{dt} = b(x)$$

with $x(0) \in \bar{\Omega}$ remains in Ω for all $t > 0$.

In this section we shall assume:

(B_1) Any solution of (4.1) with $x(0) \in \Omega$ remains in Ω for all $t \geq 0$.

(B_2) (i) The ω -limit set of each solution of (4.1) with $x(0) \in \Omega$ coincides with a given point ζ (independent of $x(0)$);

(ii) $\zeta \in \partial\Omega$;

(iii) ζ is an asymptotically stable equilibrium point of the system $dx/dt = b(x)$ in the sense that $b(\zeta) = 0$ and the eigenvalues of the matrix $[\partial b / \partial x]_{x=\zeta}$ all have negative real parts.

Theorem 4.1. *Let (A), (B_1), (B_2) hold. Then, for any $\delta > 0$,*

$$(4.2) \quad P_x^\epsilon \{ \rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \delta \} \rightarrow 0 \quad \text{if} \quad \epsilon \rightarrow 0.$$

Proof. For any $\rho > 0$, denote by Ω_ρ the ρ -neighborhood of ζ . Let $0 < \nu < \mu$; μ and ν are small numbers to be determined later. Given $x \in \Omega$, consider the solution $\xi^0(t)$ of (4.1) with $\xi^0(0) = x$. By (B_1), (B_2), $\xi^0(t)$ lies in Ω for all $t > 0$ and it intersects $\Omega_{\nu/2}$ at some first time t_ν .

Now, for any $T > 0$, $\delta_0 > 0$ (see [5]),

$$P_x \left\{ \sup_{0 \leq t \leq T} |\xi^\epsilon(t) - \xi^0(t)| > \delta_0 \right\} \rightarrow 0 \quad \text{if} \quad \epsilon \rightarrow 0.$$

Hence, for any $\eta_1 > 0$,

$$(4.3) \quad P_x \{ \xi^\epsilon(t) \text{ remains in } \Omega \text{ for } 0 \leq t \leq t_\nu; \xi^\epsilon(t) \text{ hits } \partial\Omega_\nu \text{ at some} \\ \text{first time } \tau_\nu \leq t_\nu \} > 1 - \eta_1,$$

provided ϵ is sufficiently small.

Denoting the exit time from Ω by τ^ϵ , and using (4.3) and the strong Markov property, we then have

$$\begin{aligned} P_x \{ \rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \mu \} \\ &= P_x \{ \tau^\epsilon < \tau_\nu, \rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \mu \} + P_x \{ \tau^\epsilon > \tau_\nu, \rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \mu \} \\ &< \eta_1 + E_x \{ \chi_{\tau^\epsilon > \tau_\nu} [E_x \chi_{\rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \mu}]_{x=\xi(\tau_\nu)} \}. \end{aligned}$$

If we prove that the function

$$(4.4) \quad w(x) = E_x \{ \chi_{\rho(\xi^\epsilon(\tau^\epsilon), \zeta) > \mu} \}$$

satisfies:

$$(4.5) \quad w(x) \leq \eta(\nu, \mu) \quad (x \in \partial\Omega_\nu \cap \Omega) \text{ where } \eta(\nu, \mu) \rightarrow 0 \text{ if } \nu \rightarrow 0,$$

then we conclude that

$$P_x\{\rho(\xi^*(\tau^*), \zeta) > \mu\} < \eta_1 + \eta(\nu, \mu) < 2\eta_1$$

if ν is sufficiently small (so that $\eta(\nu, \mu) < \eta_1$), and the proof of Theorem 4.1 is complete.

To prove (4.5) notice that $w(x)$ satisfies:

$$\begin{aligned} (4.6) \quad & L_\epsilon w(x) = 0 \quad \text{in } \Omega, \\ & w(x) = 0 \quad \text{if } x \in \partial\Omega, \rho(x, \zeta) < \mu, \\ & w(x) = 1 \quad \text{if } x \in \partial\Omega, \rho(x, \zeta) > \mu. \end{aligned}$$

For any $\rho > 0$, let $\Gamma_\rho = \partial\Omega \cap \partial\Omega_\rho$, $\Gamma'_\rho = \Omega \cap \partial\Omega_\rho$. Suppose $u(x)$ satisfies:

$$\begin{aligned} (4.7) \quad & L_\epsilon u \leq 0 \quad \text{in } \Omega_\mu, \\ & u \geq 0 \quad \text{on } \Gamma_\mu, \\ & u \geq 1 \quad \text{on } \Gamma'_\mu, \end{aligned}$$

and

$$(4.8) \quad u(x) \leq \eta(\nu, \mu) \quad \text{on } \Gamma'_\nu.$$

By the maximum principle, $w \leq 1$ in Ω and, in particular, on Γ'_μ . Comparing w with u , by means of the maximum principle, we conclude that

$$w(x) \leq u(x) \quad \text{if } x \in \partial\Omega_\nu \cap \Omega.$$

Thus (4.5) would follow from (4.8).

Set $Mu = \sum a_{ij} u_{x_i x_j}$. If a function u satisfies

$$(4.9) \quad Mu \leq 0 \quad \text{in } \Omega_\mu,$$

$$(4.10) \quad b \cdot u_x \leq 0 \quad \text{in } \Omega_\mu,$$

$$(4.11) \quad u \geq 0 \text{ on } \Gamma_\mu, \quad u \geq 1 \text{ on } \Gamma'_\mu, \quad u(\zeta) = 0,$$

then u clearly satisfies (4.7), (4.8). Thus it remains to find a solution of (4.9)–(4.11).

For simplicity we shall now assume that $\zeta = 0$.

Suppose we perform a non-singular linear map $y = Px$. The stochastic system (1.1) becomes

$$\begin{aligned} d\eta^*(t) &= Pd\xi^*(t) = \epsilon P\sigma(\xi^*(t)) dw + Pb(\xi^*(t)) dt \\ &= \epsilon\sigma P(P^{-1}(\eta^*(t))) dw + Pb(P^{-1}(\eta^*(t))) dt \\ &\equiv \epsilon\tilde{\sigma}(\eta^*(t)) dw + \tilde{b}(\eta^*(t)) dt \end{aligned}$$

where

$$\tilde{b}(y) = PBP^{-1}y + o(|y|), \quad B = \left[\frac{\partial b}{\partial x} \right]_{x=\zeta=0}.$$

By assumption, $\operatorname{Re} \lambda_i < 0$, where λ_i are the eigenvalues of B .

Notice that the transformation $y = Px$ does not change the assumptions and assertions, except for changing Γ_μ , Γ'_μ , Ω_μ into $\tilde{\Gamma}_\mu$, $\tilde{\Gamma}'_\mu$, $\tilde{\Omega}_\mu$, respectively. Thus, it suffices to prove the assertions (4.9)–(4.11) in the y -space with Ω_μ replaced by $\tilde{\Omega}_\mu$. If $u > 0$ away from 0, then it is enough to consider the case where $\tilde{\Omega}_\mu$ is replaced by $\Omega_{\hat{\mu}}$ with a suitable $\hat{\mu}$; for simplicity we shall take $\hat{\mu} = \mu$.

We choose P so that $P^{-1}BP$ has the Jordan canonical form. Consider the function

$$\Phi(y) = \left\{ \sum_{i=1}^n \alpha_i y_i^2 \right\} \quad (\alpha_i \text{ are positive constants, } k > 0).$$

Set $\tilde{B} = PBP^{-1}$. We shall take later on $u(y) = f(\Phi(y))$ with $f(z)$ such that $f'(z) > 0$. Then, (4.10) holds (in the y -space) if

$$(4.12) \quad \tilde{B}y \cdot \Phi_y < 0 \quad \text{when} \quad y \neq 0, \text{ i.e., if } \sum \alpha_i \tilde{b}_{ii} y_i y_i < 0$$

when $y \neq 0$ ($\tilde{B} = (\tilde{b}_{ii})$).

Writing explicitly the $\tilde{b}_{ii}$, one can quickly determine how to choose the α_i so that (4.12) is satisfied. Thus, if the first $\ell \times \ell$ block is given by $\tilde{b}_{ii} = \lambda$, $\tilde{b}_{ij} = 1$ if $j = i + 1$, then we take, step by step, α_2/α_1 sufficiently large, α_3/α_2 sufficiently large, $\dots$, $\alpha_\ell/\alpha_{\ell-1}$ sufficiently large. If, on the other hand, the first $\ell \times \ell$ block is given by

$$\begin{pmatrix} \lambda & -\mu & 1 & 0 & & & & 0 \\ \mu & \lambda & 0 & 1 & & & & \\ & & \lambda & -\mu & 1 & 0 & & \\ & & \mu & \lambda & 0 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \lambda & -\mu \\ & 0 & & & & & \mu & \lambda \end{pmatrix}$$

then we take $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, $\dots$, $\alpha_{\ell-1} = \alpha_\ell$ and choose, step by step, α_3/α_1 sufficiently large, α_5/α_3 sufficiently large, etc.

As mentioned before, we shall take $u(y) = f(\Phi(y))$ with $f'(z) > 0$. If $f(0) = 0$ then (4.10) is satisfied. Thus, it remains then to verify (4.9), i.e.,

$$(4.13) \quad Mu \equiv f'(\Phi) \sum a_{ij} \Phi_{y_i} y_j + f''(\Phi) \sum a_{ij} \Phi_{y_i} \Phi_{y_j} \leq 0.$$

It is easily seen that (4.13) is a consequence of

$$(4.14) \quad \gamma f'(\Phi) + k f''(\Phi) \Phi = 0, \quad f'(\Phi) > 0$$

where γ is a suitable positive constant depending on the a_{ij} , α_i . A solution of (4.14) with $f(0) = 0$ is given by

$$f(z) = z^{1-(\gamma/k)}, \quad k > \gamma$$

Having thus completed the construction of u , the proof of Theorem 4.1 is complete.

Remark. The condition (B_2) can be relaxed so as to include cases where the ω -limit set of each solution of (4.1) with $x(0) \in \Omega$ consists of either ζ or a subset of one of a finite number of given compact subsets K_i of Ω . (The properties of K_i are as in [11]). One can then obtain a result which includes both Theorem 4.1 and Theorems 3.1, 9.1 of [11].

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