Fourier Transforms and Non-Compact Rotation Groups

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§1. Introduction. A well-known formula of Bochner [1] gives the Fourier transform of a radial function times a spherical harmonic as

$$(1) \quad [f(|x|)Y_k(x)]^\wedge(y) = i^k Y_k(y) \mathfrak{H}_{n/2+k}(f)(|y|)$$

where

$$\mathfrak{H}_{n/2+k}(f)(r) = (2\pi)^{n/2+k} \int_0^\infty \frac{J_{(n/2)+k-1}(rt)}{(rt)^{(n/2)+k-1}} f(t) t^{n+2k-1} dt.$$

Here k is the degree of the spherical harmonic, n is the dimension of the Euclidean space and J is the usual Bessel function. Not only does this formula provide an important computational method, it also reveals an interesting periodicity relationship since the transformation $\mathfrak{H}_{(n/2)+k}$ depends only on $(n/2) + k$.

Bochner's formula may be thought of as giving a decomposition of the Euclidean Fourier transform under the action of the rotation or orthogonal group ($SO(n)$ or $O(n)$). The radial functions $f(|x|)$ are invariant under the group action and the spherical harmonics transform according to irreducible representations of the group. From this point of view we are led naturally to the following general question:

Given a Lie group G and a finite dimensional real representation of G acting on $\mathbf{R}^n$, what is the Fourier transform of a function (or distribution) which is invariant under the action of G or which transforms under this action according to an irreducible representation of G ? In particular, can we find a formula analogous to (1.1) involving known special functions, and can we find any periodicity relations between different groups and representations?

In this paper we answer these questions in two special cases. In the first case (Section 2) we take the pseudo-orthogonal group $O(p, q)$ acting on $\mathbf{R}^{p+q}$ in the obvious way, and in the second case (Section 4) we take the complex orthogonal group $O(n, \mathbf{C})$ acting on $\mathbf{R}^{2n}$ (identified with $\mathbf{C}^n$). In both cases we obtain a

formula analogous to (1.1) with similar periodicity properties. The special functions involved are the J , K and Y Bessel functions in the first case, and the Meijer G -function in the second case. The analogues of spherical harmonics are homogeneous solutions of the corresponding group invariant second order partial differential equation. Because the groups are non-compact the corresponding representations are infinite dimensional and depend on a continuous parameter (the degree of homogeneity). Certain distribution theoretic problems arise because the groups are non-compact, but we avoid the worst of these at the cost of a slight loss of generality.

In addition to shedding light on the general problem (see also Gelbart [4]), we hope our results will be useful *per se*. We give two applications (Section 3a) to the Klein–Gordon equation. The first gives the result in “space-time” coordinates of the Fourier decomposition under the Lorentz group of the space of finite norm solutions usually given in “momentum-energy” coordinates. The second generalizes the basic *a priori* estimate for the wave equation in [9] to the Klein–Gordon equation.

As a third application (Section 3b) we solve a problem posed by Stein ([7a] Problem 6) concerning the L^2 boundedness of certain $O(p, q)$ invariant operators of “fractional integral” type.

Our proofs are elementary in nature and rely heavily on some basic computations in Gelfand–Shilov. We have omitted some of the more routine calculations involving trigonometric identities for the sake of brevity.

Finally we mention that a similar analysis is possible in the case of $SL(n, \mathbf{R})$ or $SL(n, \mathbf{C})$ acting on $n \times n$ real or complex matrix space. The basic computations are in Gelbart [3] and Stein [7]. In this regard see Stein [7a] Section 8.

§2. Pseudo-orthogonal groups. We fix positive integers p, q, n with $n = p + q$, and consider the quadratic form

$$Q(x, y) = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_n y_n$$

on $\mathbf{R}^n$. We denote by $O(p, q)$ the group of linear transformations preserving this quadratic form, and we choose a definition of the Fourier transform

$$\hat{f}(y) = \int_{\mathbf{R}^n} e^{iQ(x, y)} f(x) dx$$

which commutes with the action of $O(p, q)$. This formula extends to tempered distributions in the usual way.

Now if $f(t)$ is a function on $\mathbf{R}^1$ we consider the function $f(Q(x, x))$ on $\mathbf{R}^n$. Note that this is invariant under the group action, and if f satisfies additional conditions so that $f(Q(x, x))$ defines a tempered distribution on $\mathbf{R}^n$, then this distribution is also invariant. Thus $f(Q(x, x))$ will be the analogue of the radial function in Bochner’s formula. Since the group $O(p, q)$ acts transitively on the surfaces $Q(x, x) = \text{constant}$, the most general invariant function is of the form

$f(Q(x, x))$. The problem of describing the most general invariant distribution in similar terms will not be considered here.

Next we consider the functions $g(x)$ which are solutions (in the weak sense) of the equation

$$\square g(x) = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \right) g(x) = 0$$

and which are homogeneous of degree (σ, ϵ) for $\sigma \in \mathbf{C}$ and $\epsilon = 0$ or 1 , meaning

$$g(tx) = t^\sigma g(x) \quad \text{for } 0 < t < \infty$$

and

$$g(-x) = (-1)^\epsilon g(x).$$

We shall refer to these as *homogeneous waves* of degree (σ, ϵ) . These are the analogues of the spherical harmonics. If σ is not an integer the space of homogeneous waves of degree (σ, ϵ) realizes an irreducible representation of $O(p, q)$ (see [10]).

Now with suitable additional hypotheses $f(Q(x, x))g(x)$ is a well defined tempered distribution. Our goal is to show that

$$[f(Q(x, x))g(x)]^\wedge(y) = F(Q(y, y))g(y)$$

and to give the transformation $f \rightarrow F$ explicitly. We do this by computing the Fourier transform explicitly for certain families of functions which can be combined to synthesize the general case.

If F is any real-valued expression and $\lambda \in \mathbf{C}$ we will write

$$\begin{aligned} F_+^\lambda &= \begin{cases} F^\lambda & \text{if } F > 0 \\ 0 & \text{if } F \leq 0 \end{cases} \\ F_-^\lambda &= \begin{cases} 0 & \text{if } F \geq 0 \\ (-F)^\lambda & \text{if } F \leq 0 \end{cases} \\ |F|_0^\lambda &= |F|^\lambda \\ |F|_1^\lambda &= |F|^\lambda \operatorname{sgn} F. \end{aligned}$$

Now if $\operatorname{Re} \lambda > 0$, $\operatorname{Re} \sigma > -1$, $\epsilon = 0$ or 1 and $\xi \in \mathbf{R}^n$ satisfies $Q(\xi, \xi) = 0$ but $\xi \neq 0$, the functions

$$Q(x, x)_\pm^\lambda |Q(x, \xi)|_\epsilon^\sigma$$

are locally integrable and of polynomial growth so they define tempered distributions in the natural way. Note that these families of distributions depend analytically on λ and σ , so we use the same symbols to denote the analytic continuation to other values of λ and σ when it exists. Similarly we will analytically continue

$$|Q(x, x)|_\epsilon^\lambda, |Q(x, \xi)|_\epsilon^\sigma.$$

We clearly have

$$|Q(x, x)|_{\epsilon}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma} = Q(x, x)_{+}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma} + (-1)^{\epsilon'} Q(x, x)_{-}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}$$

whenever the right hand side exists; but the left hand side may exist even if the right hand side fails to exist.

Lemma 1. $Q(x, x)_{\pm}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}$ is analytically continuable to all values of λ, σ satisfying

$$(a) \sigma \neq -1 - \epsilon, -3 - \epsilon, -5 - \epsilon, \dots$$

$$(b) \lambda \neq -1, -2, -3, \dots$$

$$(c) \lambda + \sigma + n/2 \neq 0, -1, -2, \dots$$

and $|Q(x, x)|_{\epsilon}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}$ is in addition continuable to $\lambda = \epsilon' - 2, \epsilon' - 4, \epsilon' - 6, \dots$ provided (a) and (c) hold. Furthermore we have an estimate

$$(2.1) \quad (Q(x, x)_{\pm}^{\lambda+\epsilon'} |Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi) \leq c_N (1 + |\rho|)^{-N} \|\varphi\|_N$$

if λ and ρ are real, (a), (b) and (c) hold and $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Here N is an arbitrary positive integer and $\|\cdot\|_N$ is one of the seminorms defining the topology of $\mathcal{S}(\mathbf{R}^n)$. A similar estimate holds for $Q(x, x)_{\epsilon}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}$.

Proof. For $\text{Re } \lambda$ and $\text{Re } \sigma$ sufficiently large we compute directly that

$$\square |Q(x, \xi)|_{\epsilon}^{\sigma} = 0$$

$$\square Q(x, x)_{\pm}^{\lambda} = \pm 2\lambda(n + 2\lambda - 2)Q(x, x)_{\pm}^{\lambda-1} \quad \text{and}$$

$$\square (Q(x, x)_{\pm}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}) = \pm 2\lambda(n + 2\lambda + 2\sigma - 2)Q(x, x)_{\pm}^{\lambda-1} |Q(x, \xi)|_{\epsilon}^{\sigma}.$$

These identities continue to hold in the distribution sense by analytic continuation for all values of λ, σ for which both sides of the equations make sense. From the last identity we obtain by iteration

$$(2.2) \quad Q(x, x)_{\pm}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma} = a(k, \lambda, \sigma) \square^k (Q(x, x)_{\pm}^{\lambda+k} |Q(x, \xi)|_{\epsilon}^{\sigma})$$

where $a(k, \lambda, \sigma) = [(\pm 2)^k (\lambda + 1) \cdots (\lambda + k)(n + 2\lambda + 2\sigma) \cdots (n + 2\lambda + 2\sigma + 2k - 2)]^{-1}$. Now $Q(x, \xi)$ is a linear function of x , so the analytic continuation of $|Q(x, \xi)|_{\epsilon}^{\sigma}$ is equivalent to the analytic continuation of the one-dimensional distribution $|t|_{\epsilon}^{\sigma}$. But it is well-known that this is possible if (a) holds, and in fact we have

$$(|Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi) \leq c_{\sigma} \|\varphi\|_{\sigma} \quad \text{where} \quad \|\cdot\|_{\sigma}$$

is one of the $\mathcal{S}$ semi-norms. Note that for fixed λ, σ if k is large enough $Q(x, x)_{\pm}^{\lambda+k}$ is sufficiently smooth (and of polynomial growth) so that

$$\|Q(x, x)_{\pm}^{\lambda+k} \varphi(x)\|_{\sigma} \leq c \|\varphi\|_{\sigma, \lambda, k}$$

which enables us to extend $|Q(x, \xi)|_{\epsilon}^{\sigma}$ by continuity so as to define

$$(Q(x, x)_{\pm}^{\lambda+k} |Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi) = (|Q(x, \xi)|_{\epsilon}^{\sigma}, Q(x, x)_{\pm}^{\lambda+k} \varphi)$$

as a tempered distribution. A careful examination of this argument shows that

$$(2.3) \quad |(Q(x, x)_{\pm}^{\lambda+k} | Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi)| \leq p(\sigma, \lambda, k) \|\varphi\|_{\sigma, \lambda, k}$$

where $\|\cdot\|_{\sigma, \lambda, k}$ depends only on the real part of λ , and

$$(2.4) \quad |p(\sigma, \lambda, k)| \leq c_{k, \sigma} (1 + |\lambda|)^{N_{\sigma}}$$

where N_{σ} depends only on σ (and not on k).

Now if (b) and (c) hold we avoid the poles of $a(k, \lambda, \sigma)$ and so we may use (2.2) with k sufficiently large to define the analytic continuation. From (2.3) we obtain

$$|(Q(x, x)_{\pm}^{\lambda} | Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi)| \leq |a(k, \lambda, \sigma)| p(\sigma, \lambda, k) \cdot \|\square^k \varphi\|_{\sigma, \lambda, k}.$$

If we fix σ and $\operatorname{Re} \lambda$ then $a(k, \lambda, \sigma)$ is $O(|\lambda|^{-2k})$ as $\operatorname{Im} \lambda \rightarrow \infty$. In view of (2.4) we may make $|a(k, \lambda, \sigma)|$ have arbitrary polynomial decay by taking k large, establishing (2.1).

Finally we must establish the analytic continuation of $|Q(x, x)|_{\epsilon'}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}$ to the additional points $\lambda = \epsilon' - 2, \epsilon' - 4, \dots$. For simplicity we treat only the case $\lambda = -1, \epsilon' = 1$ and $\operatorname{Re} \sigma \geq 0$. We need only show

$$(|Q(x, x)|_1^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi)$$

is continuous at $\lambda = -1$. But

$$\begin{aligned} (|Q(x, x)|_1^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}, \varphi) &= [2(\lambda + 1)(n + 2\lambda + 2\sigma)]^{-1} \\ &\quad \cdot (\square(|Q(x, x)|_0^{\lambda+1} |Q(x, \xi)|_{\epsilon}^{\sigma} - |Q(x, \xi)|_{\epsilon}^{\sigma}), \varphi) \end{aligned}$$

and

$$\lim_{\lambda \rightarrow -1} \left(\frac{(|Q(x, x)|_0^{\lambda+1} - 1)}{\lambda + 1} \right) |Q(x, \xi)|_{\epsilon}^{\sigma}, \square \varphi = ((\log |Q(x, x)|) |Q(x, \xi)|_{\epsilon}^{\sigma}, \square \varphi)$$

by the dominated convergence theorem. Since $n - 2 + 2\sigma \neq 0$ by (c) we obtain the desired continuity. The other cases are handled by similar but more complicated arguments.

Lemma 2. Assume σ, λ and $\lambda + \sigma + (n/2)$ are not integers and $Q(\xi, \xi) = 0$. Then

$$\begin{aligned} (2.5) \quad & [Q(x, x)_{+}^{\lambda} |Q(x, \xi)|_{\epsilon}^{\sigma}]^{\wedge}(y) \\ &= \gamma(\lambda, \sigma)(-i)^{\epsilon} |Q(y, \xi)|_{\epsilon}^{\sigma} \left(-\sin \frac{\pi}{2} (2\lambda + q + \sigma + \epsilon) Q(y, y)_{+}^{-\lambda - (n/2) - \sigma} \right. \\ &\quad \left. + \sin \frac{\pi}{2} (p + \sigma - \epsilon) Q(y, y)_{-}^{-\lambda - (n/2) - \sigma} \right) \end{aligned}$$

and

$$\begin{aligned}
(2.6) \quad [Q(x, x)_-^\lambda |Q(x, \xi)|_\epsilon^\sigma]^\wedge(y) \\
= \gamma(\lambda, \sigma)(-i)^\epsilon |Q(y, \xi)|_\epsilon^\sigma \left(\sin \frac{\pi}{2} (q + \sigma + \epsilon) Q(y, y)_+^{-\lambda - (n/2) - \sigma} \right. \\
\left. - \sin \frac{\pi}{2} (2\lambda + p + \sigma - \epsilon) Q(y, y)_-^{-\lambda - (n/2) - \sigma} \right)
\end{aligned}$$

where

$$(2.7) \quad \gamma(\lambda, \sigma) = 2^{2\lambda + n + \sigma} \pi^{(n/2) - 1} \Gamma(\lambda + 1) \Gamma\left(\lambda + \frac{n}{2} + \sigma\right).$$

Proof. The idea of the proof is that $(Q(x, x)_\pm^\lambda)^\wedge$ and $(|Q(x, \xi)|_\epsilon^\sigma)^\wedge$ are known, so the Fourier transform of the product is the convolution of the Fourier transforms which we can compute explicitly for some values of λ and σ . Note that since both sides of (2.5) and (2.6) are meromorphic in λ and σ , it suffices to establish these identities for some values of λ and σ to obtain the result by analytic continuation. Specifically we will assume

$$-1 < \operatorname{Re} \sigma < 0 \quad \text{and} \quad -\operatorname{Re} \sigma < \operatorname{Re} \lambda + \frac{n}{2} < 1.$$

Now Gelfand-Shilov [6] (p. 284) prove

$$\begin{aligned}
(2.8) \quad [Q(x, x)_+^\lambda]^\wedge(y) \\
= \gamma(\lambda, 0) \left[-\sin \pi \left(\lambda + \frac{q}{2} \right) Q(y, y)_+^{-\lambda - (n/2)} + \sin \pi \frac{p}{2} Q(y, y)^{-\lambda - (n/2)} \right]
\end{aligned}$$

and

$$\begin{aligned}
(2.9) \quad [Q(x, x)_-^\lambda]^\wedge(y) \\
= \gamma(\lambda, 0) \left[\sin \frac{\pi q}{2} Q(y, y)_+^{-\lambda - (n/2)} - \sin \pi \left(\lambda + \frac{p}{2} \right) Q(y, y)^{-\lambda - (n/2)} \right]
\end{aligned}$$

provided λ and $\lambda + (n/2)$ are not integers. This of course agrees with the result obtained by taking $\epsilon = 0$ and $\sigma = 0$ in (2.5) and (2.6). Note that with the restrictions on λ and σ , (2.8) and (2.9) define locally integrable functions. To compute the Fourier transform of $|Q(x, \xi)|_\epsilon^\sigma$ we proceed as follows:

Since $0(p, q)$ is transitive on the cone of non-zero solutions of $Q(\xi, \xi) = 0$ (see [10]) we may assume without loss of generality that

$$Q(x, \xi) = x_1 - x_{p+1}.$$

We make a unimodular change of variable by setting

$$x'_1 = (x_1 - x_{p+1})/2^{1/2}, \quad x'_{p+1} = (x_1 + x_{p+1})/2^{1/2}$$

and $x_i'' = x_i$ if $j \neq 1$ and $j \neq p + 1$. Then

$$Q(x, y) = x'_1 y'_{p+1} + x'_{p+1} y'_1 + R(x'', y'')$$

where

$$R(x'', y'') = x_2'' y_2'' + \cdots + x_p'' y_p'' - x_{p+2}'' y_{p+2}'' - \cdots - x_n'' y_n'',$$

and $|Q(x, \xi)|_\epsilon^\sigma = 2^{\sigma/2} |x_1'|_\epsilon^\sigma$. Now it is well-known ([6], p. 173) that

$$\int_{-\infty}^{\infty} e^{i x_1' y_{p+1}'} |x_1'|_\epsilon^\sigma dx_1' = -2(-i)^\epsilon \Gamma(\sigma + 1) \sin \frac{\sigma + \epsilon}{2} \cdot |y_{p+1}'|_\epsilon^{-\sigma-1}$$

in the distribution sense. Note again that our restriction on σ makes this locally integrable. Thus we have

$$\begin{aligned} (2.10) \quad & \frac{1}{(2\pi)^n} [Q(x, x)_\pm^\lambda]^\wedge * [|Q(x, \xi)|_\epsilon^\sigma]^\wedge \\ &= \frac{1}{2\pi} 2^{\sigma/2} \left(-2(-i)^\epsilon \Gamma(\sigma + 1) \sin \pi \frac{\sigma + \epsilon}{2} \right) \\ & \cdot \int_{-\infty}^{\infty} [Q(x, x)_\pm^\lambda]^\wedge(y_1', t, y'') |y_{p+1}' - t|_\epsilon^{-\sigma-1} dt. \end{aligned}$$

We next show that the integral in (2.10) is absolutely convergent for almost every y to a locally integrable function given by the right hand side of (2.5) or (2.6).

Now the basic computation we need is the integral

$$I(\lambda, \sigma, \epsilon, \pm) = \int_{-\infty}^{\infty} (2y_1' t + R(y'', y''))_\pm^{-\lambda-(n/2)} |y_{p+1}' - t|_\epsilon^{-\sigma-1} dt$$

We make the change of variable $t \rightarrow (Q(y, y)t - R(y'', y''))/2y_1'$ to obtain

$$\begin{aligned} I(\lambda, \sigma, \epsilon, \pm) &= 2^\sigma |y_1'|_\epsilon^\sigma |Q(y, y)|_\epsilon^{-\sigma} \int_{-\infty}^{\infty} (Q(y, y)t)_\pm^{-\lambda-(n/2)} |1 - t|_\epsilon^{-\sigma-1} dt \\ &= 2^{\sigma/2} |Q(y, \xi)|^\sigma (\operatorname{sgn} Q(y, \xi))^\epsilon. \end{aligned}$$

$$\begin{aligned} 2^{\sigma/2} |Q(y, \xi)|_\epsilon^\sigma &\left[(\pm 1)^\epsilon Q(y, y)_\pm^{-\lambda-(n/2)-\sigma} \int_0^\infty t^{-\lambda-(n/2)} |1 - t|_\epsilon^{-\sigma-1} dt \right. \\ &\quad \left. + (\mp 1)^\epsilon Q(y, y)_\mp^{-\lambda-(n/2)-\sigma} \int_{-\infty}^0 |t|^{-\lambda-(n/2)} |1 - t|_\epsilon^{-\sigma-1} dt \right]. \end{aligned}$$

But by using the beta integral

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$$

and the related integral (see [12] p. 242-3)

$$\int_0^\infty t^{\alpha-1} (1+t)^{\beta-1} dt = \Gamma(\alpha)\Gamma(1-\alpha-\beta)/\Gamma(1-\beta)$$

we obtain

$$\int_0^\infty t^{-\lambda-(n/2)} |1-t|_\epsilon^{-\sigma-1} dt$$

$$= \frac{\Gamma(-\sigma)\Gamma\left(-\lambda-\frac{n}{2}+1\right)}{\Gamma\left(-\lambda-\frac{n}{2}-\sigma+1\right)} + (-1)^\epsilon \frac{\Gamma(-\sigma)\Gamma\left(\lambda+\frac{n}{2}+\sigma\right)}{\Gamma\left(\lambda+\frac{n}{2}\right)}$$

and

$$\int_{-\infty}^0 |t|^{-\lambda-(n/2)} |1-t|^{-\sigma-1} dt = \frac{\Gamma\left(\sigma+\lambda+\frac{n}{2}\right)\Gamma\left(-\lambda-\frac{n}{2}+1\right)}{\Gamma(\sigma+1)}.$$

(Note that the integrals converge by our assumptions on λ and σ .) Substituting in these results and using the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ to simplify, we obtain

$$(2.11) \quad I(\lambda, \sigma, \epsilon, +) = \frac{\pi 2^{\sigma/2} \Gamma\left(\lambda + \frac{n}{2} + \sigma\right)}{\Gamma\left(\lambda + \frac{n}{2}\right) \Gamma(\sigma + 1)} |Q(y, \xi)|_\epsilon^\sigma$$

$$\cdot \left[\frac{\left[-\sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) - (-1)^\epsilon \sin \pi \left(\lambda + \frac{n}{2} \right) \right]}{\sin \pi \left(\lambda + \frac{n}{2} \right) \sin \pi \sigma} \right]$$

$$\cdot \left[Q(y, y)_+^{-\lambda-(n/2)-\sigma} + \frac{(-1)^\epsilon Q(y, y)_-^{-\lambda-(n/2)-\sigma}}{\sin \pi \left(\lambda + \frac{n}{2} \right)} \right]$$

and

$$(2.12) \quad I(\lambda, \sigma, \epsilon, -)$$

$$= \frac{\pi 2^{\sigma/2} \Gamma\left(\lambda + \frac{n}{2} + \sigma\right)}{\Gamma\left(\lambda + \frac{n}{2}\right) \Gamma(\sigma + 1)} |Q(y, \xi)|_\epsilon^\sigma \cdot \left[\frac{Q(y, y)_+^{-\lambda-(n/2)-\sigma}}{\sin \pi \left(\lambda + \frac{n}{2} \right)} \right.$$

$$\left. + \frac{\left[-(-1)^\epsilon \sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) - \sin \pi \left(\lambda + \frac{n}{2} \right) \right]}{\sin \pi \left(\lambda + \frac{n}{2} \right) \sin \pi \sigma} \right] Q(y, y)_-^{-\lambda-(n/2)-\sigma}.$$

Now from (2.8) we obtain that (2.10) equals

$$-(-i)^{\epsilon} \pi^{-1} 2^{\sigma/2} \Gamma(\sigma + 1) \sin \pi \frac{\sigma + \epsilon}{2} \cdot \gamma(\lambda, 0) \\ \left[-\sin \pi \left(\lambda + \frac{q}{2} \right) I(\lambda, \sigma, \epsilon, +) + \sin \frac{\pi p}{2} I(\lambda, \sigma, \epsilon, -) \right].$$

We substitute in (2.11) and (2.12) and simplify to obtain

$$[Q(x, x)_{+}^{\lambda} |Q(x, \xi)|_{+}^{\sigma}]^{\wedge}(y) = \gamma(\lambda, \sigma) (-i)^{\epsilon} \frac{\sin \pi \frac{\sigma + \epsilon}{2}}{\sin \pi \sigma} |Q(y, \xi)|_{+}^{\sigma} \\ \cdot \left[\left(\frac{-\sin \pi \left(\lambda + \frac{q}{2} \right) \sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) - \sin \pi \frac{p}{2} \sin \pi \sigma}{\sin \pi \left(\lambda + \frac{n}{2} \right)} - (-1)^{\epsilon} \sin \pi \left(\lambda + \frac{q}{2} \right) \right) \right. \\ \left. Q(y, y)_{+}^{-\lambda - (n/2) - \sigma} + \left(\sin \pi \frac{p}{2} + (-1)^{\epsilon} \right. \right. \\ \left. \left. \cdot \frac{\sin \pi \frac{p}{2} \sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) + \sin \pi \left(\lambda + \frac{q}{2} \right) \sin \pi \sigma}{\sin \pi \left(\lambda + \frac{n}{2} \right)} \right) Q(y, y)_{-}^{-\lambda - (n/2) - \sigma} \right].$$

To obtain the right hand side of (2.5) from this it remains only to verify

$$\frac{\sin \pi \left(\lambda + \frac{q}{2} \right) \sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) - \sin \pi \frac{p}{2} \sin \pi \sigma}{\sin \pi \left(\lambda + \frac{n}{2} \right)} = \sin \pi \left(\lambda + \frac{q}{2} + \sigma \right)$$

and

$$\frac{\sin \pi \frac{p}{2} \sin \pi \left(\lambda + \frac{n}{2} + \sigma \right) + \sin \pi \left(\lambda + \frac{q}{2} \right) \sin \pi \sigma}{\sin \pi \left(\lambda + \frac{n}{2} \right)} = \sin \pi \left(\frac{p}{2} + \sigma \right),$$

and further simplify by the addition formula

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}.$$

We obtain the right hand side of (2.6) similarly. Note that the above computations also show that the integral in (2.10) is absolutely convergent almost everywhere.

To complete the proof it suffices to show

$$(Q(x, x)_\pm^\lambda |Q(x, \xi)|_e^\sigma)^\wedge = \frac{1}{(2\pi)^n} (Q(x, x)_\pm^\lambda)^\wedge * (|Q(x, \xi)|_e^\sigma)^\wedge.$$

We do this by approximating $|Q(x, \xi)|_e^\sigma$ by smooth functions. Let $\psi(t) \in C^\infty(\mathbf{R}^1)$ be non-negative, with $\psi \equiv 0$ for $|t| < 1$ and $\psi \equiv 1$ for $|t| > 2$, and define $g_\delta \in \mathcal{S}(\mathbf{R}^1)$ by

$$\hat{g}_\delta(t) = e^{-\delta|t|^\sigma} \psi(t/\delta) \cdot \left[-2(-i)^\sigma \Gamma(\sigma + 1) \sin \pi \frac{\sigma + \epsilon}{2} |t|_e^{-\sigma-1} \right].$$

Now it is easy to see that $g_\delta(t) \rightarrow |t|_e^\sigma$ in $L^1 + L^\infty$ so that

$$a(k, \lambda, \sigma)(\square^k(Q(x, x)_\pm^{\lambda+k} g_\delta(Q(x, \xi))), \varphi) \rightarrow (Q(x, x)_\pm^\lambda |Q(x, \xi)|_e^\sigma, \varphi)$$

as $\delta \rightarrow 0$ for k sufficiently large. But we may compute, as in the proof of Lemma 1, that

$$\square^k(Q(x, x)_\pm^{\lambda+k} g_\delta(Q(x, \xi))) = Q(x, x)_\pm^\lambda g_{\delta,k}(Q(x, \xi))$$

where

$$g_{\delta,k}(t) = (\pm 2)^k (\lambda + 1) \cdots (\lambda + k) \cdot \left(n + 2\lambda + 2t \frac{d}{dt} \right) \cdots \left(n + 2\lambda + 2k - 2 + 2t \frac{d}{dt} \right) \cdot g_\delta(t).$$

Now $g_{\delta,k} \in \mathcal{S}(\mathbf{R}^1)$ so multiplication by $g_{\delta,k}(Q(x, \xi))$ and convolution with its Fourier transform are defined for all tempered distributions. Thus

$$[Q(x, x)_\pm^\lambda g_{\delta,k}(Q(x, \xi))]^\wedge = \frac{1}{(2\pi)^n} [Q(x, x)_\pm^\lambda]^\wedge * [g_{\delta,k}(Q(x, \xi))]^\wedge$$

so

$$([Q(x, x)_\pm^\lambda |Q(x, \xi)|_e^\sigma]^\wedge, \varphi) = \lim_{\delta \rightarrow 0} \left(\frac{1}{(2\pi)^n} [Q(x, x)_\pm^\lambda]^\wedge * [a(k, \lambda, \sigma) g_{\delta,k}(Q(x, \xi))]^\wedge, \varphi \right).$$

To complete the proof we need to show

$$a(k, \lambda, \sigma) \hat{g}_{\delta,k}(t) \rightarrow (|x|_e^\sigma)^\wedge(t)$$

almost everywhere and

$$|a(k, \lambda, \sigma) \hat{g}_{\delta,k}(t)| \leq c (|x|_e^\sigma)^\wedge(t)$$

so that we may apply the dominated convergence theorem. But we easily compute

$$\left(-\frac{d}{dt} t \right)^i \hat{g}_\delta(t) \rightarrow \sigma^i (|x|_e^\sigma)^\wedge(t)$$

and

$$\left| \left(-\frac{d}{dt} t \right)^i g_i(t) \right| \leq c(|x|^\sigma)^\wedge(t)$$

so the desired result follows by adding up these estimates.

Remarks. If σ is an integer and $\sigma \neq -1 - \epsilon, -3 - \epsilon, \dots$ then the lemma still holds by continuity.

By taking linear combinations of (2.5) and (2.6) and simplyfying we also obtain

$$\begin{aligned} [|Q(x, x)|_0^\lambda |Q(x, \xi)|_\epsilon^\sigma]^\wedge(y) &= -2\gamma(\lambda, \sigma)(-i)^\epsilon |Q(y, \xi)|_\epsilon^\sigma \\ &\quad \cdot \sin \frac{\pi}{2} \lambda \left(\cos \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon \right) \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma \right) |Q(y, y)|_0^{-\lambda - (n/2) - \sigma} \right. \\ &\quad \left. + \sin \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon \right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma \right) |Q(y, y)|_1^{-\lambda - (n/2) - \sigma} \right) \end{aligned}$$

and

$$\begin{aligned} [|Q(x, x)|_1^\lambda |Q(x, \xi)|_\epsilon^\sigma]^\wedge(y) &= -2\gamma(\lambda, \sigma)(-i)^\epsilon |Q(y, \xi)|_\epsilon^\sigma \\ &\quad \cdot \cos \frac{\pi}{2} \lambda \left(\sin \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon \right) \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma \right) |Q(y, y)|_0^{-\lambda - (n/2) - \sigma} \right. \\ &\quad \left. + \cos \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon \right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma \right) |Q(y, y)|_1^{-\lambda - (n/2) - \sigma} \right). \end{aligned}$$

Theorem 1. Let $f \in \mathcal{S}(\mathbf{R}^1)$, and let g be a homogeneous wave of degree (σ, ϵ) which is C^∞ in the complement of $\{x : Q(x, x) = 0\}$. Then

$$(2.13) \quad [f(|Q(x, x)|_1^{1/2})g(x)]^\wedge(y) = g(y)\mathcal{H}_{p,q,\sigma,\epsilon}(f)(|Q(y, y)|_1^{1/2}),$$

$$\begin{aligned} (2.14) \quad \mathcal{H}_{p,q,\sigma,\epsilon}(f)(r) &= (i)^\epsilon (2\pi)^{n/2} \int_0^\infty (rt)^{-\alpha} \left(\cos \frac{\pi}{2} (q + \sigma + \epsilon) J_\alpha(rt) \right. \\ &\quad \left. - \sin \frac{\pi}{2} (q + \sigma + \epsilon) Y_\alpha(rt) \right) f(t) t^{n+2\sigma-1} dt + (-i)^\epsilon (2\pi)^{n/2} (2/\pi) \\ &\quad \cdot \int_0^\infty (rt)^{-\alpha} \sin \frac{\pi}{2} (q + \sigma + \epsilon) K_\alpha(rt) f(-t) t^{n+2\sigma-1} dt \end{aligned}$$

and

$$\begin{aligned} (2.15) \quad \mathcal{H}_{p,q,\sigma,\epsilon}(f)(-r) &= (-i)^\epsilon (2\pi)^{n/2} (2/\pi) \int_0^\infty (rt)^{-\alpha} \sin \frac{\pi}{2} (p + \sigma - \epsilon) K_\alpha(rt) f(t) t^{n+2\sigma-1} dt \\ &\quad + (-i)^\epsilon (2\pi)^{n/2} \int_0^\infty (rt)^{-\alpha} \left(\cos \frac{\pi}{2} (p + \sigma - \epsilon) J_\alpha(rt) \right. \\ &\quad \left. - \sin \frac{\pi}{2} (p + \sigma - \epsilon) Y_\alpha(rt) \right) f(-t) t^{n+2\sigma-1} dt \end{aligned}$$

where

$$\alpha = \frac{n-2}{2} + \sigma, \quad \text{and} \quad r > 0.$$

Remarks. For some values of σ the functions

$$\frac{J_\alpha(t |Q(x, x)|_1^{1/2})}{|Q(x, x)|_1^{\alpha/2}} g(x)$$

and similar expressions with Y and K may not be locally integrable functions, so (2.13) must be suitably interpreted. We do this by expanding J , Y and K in power series and using Lemma 1. If σ is an integer some of the singular terms in the Y and K series may not be meaningful separately, but taken together they are covered by the second part of Lemma 1.

Proof. First assume σ is not an integer. Then $g(x)$ is the inverse Fourier transform of a homogeneous distribution of degree $(-n - \sigma, \epsilon)$ with support on the cone $\{y : Q(y, y) = 0\}$. From this we obtain the representation

$$(2.16) \quad g(x) = \int_{\Sigma} |Q(x, \xi')|_*^\sigma \varphi(\xi') d\xi'$$

where

$$\Sigma = \{\xi' \in \mathbf{R}^n : Q(\xi', \xi') = 0 \quad \text{and} \quad |\xi'|^2 = 2\}$$

($\Sigma = S^{p-1} \times S^{q-1}$) and φ is a C^∞ function on Σ (see [10]).

Thus if σ is not an integer it suffices to prove the theorem for

$$g(x) = |Q(x, \xi)|_*^\sigma \quad \text{with} \quad Q(\xi, \xi) = 0,$$

and since the result is analytic in σ we may also take $\text{Re } \sigma \geq 0$.

Now if $f \in \mathcal{S}(\mathbf{R}^1)$ we define the Mellin transforms

$$F_\pm(z) = \frac{1}{2\pi} \int_0^\infty t^{z-1} f(\pm t) dt \quad \text{for} \quad \text{Re } z > 0$$

and recall the inversion formula

$$f(\pm t) = \int_{-\infty}^\infty t^{-b-i\rho} F_\pm(b+i\rho) d\rho \quad \text{for} \quad t > 0$$

provided $b > 0$. If we take $\text{Re } \sigma \geq 0$ and $b < 1$ then we obtain

$$(f(|Q(x, x)|_1^{1/2})g(x), \varphi) = \sum_{\pm} \int_{-\infty}^\infty (Q(x, x)_\pm^{-(b+i\rho)/2} g(x), \varphi) F_\pm(b+i\rho) d\rho$$

by interchanging the order of integration. Thus

$$\begin{aligned}
([f(|Q(x, x)|_1^{1/2})g(x)]^\wedge, \varphi) &= \sum_{\pm} \int_{-\infty}^{\infty} (Q(x, x)_{\pm}^{-(b+i\rho)/2} g(x), \varphi) F_{\pm}(b+i\rho) d\rho \\
&\quad \cdot (-i)^{\epsilon} \int_{-\infty}^{\infty} \gamma((-b+i\rho)/2, \sigma) \left[\cos \frac{\pi}{2} (q + \sigma + \epsilon) \right. \\
&\quad \cdot \sin \pi(1 - (b+i\rho)/2) (Q(x, x)_{+}^{(b+i\rho-n-2\sigma)/2} g(x), \varphi) \\
&\quad + \sin \frac{\pi}{2} (q + \sigma + \epsilon) \cos \pi(1 - (b+i\rho)/2) (Q(x, x)_{+}^{(b+i\rho-n-2\sigma)/2} g(x), \varphi) \\
&\quad \left. + \sin \frac{\pi}{2} (p + \sigma - \epsilon) (Q(x, x)_{-}^{(b+i\rho-n-2\sigma)/2} g(x), \varphi) \right] \\
&\quad \cdot F_{+}(b+i\rho) d\rho + \text{similar terms involving } F_{-}
\end{aligned}$$

by Lemma 2 and the $\sin(a+b)$ identity.

Now $\operatorname{Re}(b+i\rho-n-2\sigma)/2 < 0$ so we must use Lemma 1 to write

$$\begin{aligned}
(Q(x, x)_{\pm}^{(b+i\rho-n-2\sigma)/2} g(x), \varphi) \\
= a(k, (b+i\rho-n-2\sigma)/2, \sigma) (Q(x, x)_{\pm}^{(2k+b+i\rho-n-2\sigma)/2} g(x), \square^k \varphi)
\end{aligned}$$

for k sufficiently large so the last inner product is an absolutely convergent integral. We easily compute

$$a(k, (b+i\rho-n-2\sigma)/2, \sigma) \gamma(-(b+i\rho)/2, \sigma) = (\pm 1)^k \gamma(-(2k+b+i\rho)/2, \sigma)$$

hence

$$\begin{aligned}
([f(|Q(x, x)|_1^{1/2})g(x)]^\wedge, \varphi) &= (-1)^k (-1)^{\epsilon} \square^k \\
&\quad \cdot \cos \frac{\pi}{2} (q + \sigma + \epsilon) \int_{-\infty}^{\infty} \gamma(-(2k+b+i\rho)/2, \sigma) \\
&\quad \cdot \sin \pi(1 - (2k+b+i\rho)/2) Q(x, x)_{+}^{(2k+b+i\rho-n-2\sigma)/2} g(x) F_{+}(b+i\rho) d\rho \\
&\quad + \sin \frac{\pi}{2} (q + \epsilon + \sigma) \int_{-\infty}^{\infty} \gamma(-(2k+b+i\rho)/2, \sigma) \\
&\quad \cdot \cos \pi(1 - (2k+b+i\rho)/2) Q(x, x)_{+}^{(2k+b+i\rho-n-2\sigma)/2} g(x) F_{+}(b+i\rho) d\rho \\
&\quad + \sin \frac{\pi}{2} (p + \sigma - \epsilon) \int_{-\infty}^{\infty} \gamma(-(2k+b+i\rho)/2, \sigma) \\
&\quad \cdot Q(x, x)_{-}^{(2k+b+i\rho-n-2\sigma)/2} g(x) F_{+}(b+i\rho) d\rho + \text{similar terms involving } F_{-}.
\end{aligned}$$

The interchange of integrals is justified by the fact that $F_{\pm}(b+i\rho)$ decreases rapidly as $\rho \rightarrow \infty$, and the distributions involved are locally integrable functions.

Next we substitute in the definition of $F_{\pm}$ and interchange the t and ρ integrations. We may do this provided we take k large enough so that the $\gamma(-(2k+b+i\rho)/2, \sigma) \sin \pi(1 - (2k+b+i\rho)/2)$ factor and related factors

vanish sufficiently rapidly as $\rho \rightarrow \infty$ (we shall make this estimate below). We are then left with the task of evaluating the following integrals:

$$I_1 = \int_{-\infty}^{\infty} 2^{-(2k+b+i\rho)+n+\sigma-1} \pi^{(n/2)-2} \Gamma(1 - (2k+b+i\rho)/2) \\ \cdot \Gamma((n+2\sigma - (2k+b+i\rho))/2) \sin \pi(1 - (2k+b+i\rho)/2) \\ \cdot Q(x, x)_+^{(2k+b+i\rho-n-2\sigma)/2} g(x) t^{b+i\rho-1} d\rho. \\ I_2 = \text{same integral with cos in place of sin.} \\ I_3 = \text{same integral without the sin factor.}$$

There are Barnes–Mellin type integrals and they may be evaluated explicitly in terms of the J Bessel function, the Y Neumann function and the K MacDonald function (note that in [6] and [12] Y is denoted by N).

Indeed by Stirling's formula the integrands are

$$O(|\rho|^{(n/2)+\sigma-2k-b}) \quad \text{as } \rho \rightarrow \infty$$

(for I_3 we get even exponential decay). Thus if we choose b to avoid the singularities of Γ when $\rho = 0$ and choose k so that $\text{Re } (n/2) + \sigma - 2k - b < -1$ the integrals are absolutely convergent. Again using estimates based on Stirling's formula we may close the contour of integration around the right half-plane and evaluate by the calculus of residues (see Watson [13] p. 191) to obtain a power series expansion. Comparing these with the series for J , Y , K (see Watson [13] Chapter 3 or Vilenkin [12] p. 269–271; note however that the integrals in Vilenkin are not absolutely convergent and must be appropriately obtained in a limiting process) we obtain

$$I_1 = (2\pi)^{n/2} \sum_{j>A} \frac{(-1)^j}{j! \Gamma((n/2) + \sigma + j)} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{2j} g(x) t^{-2k+n+2\sigma-1}$$

where

$$A = (2k + b - n - 2 \text{Re } \sigma)/4$$

and

$$\square^k I_1 = (-1)^k (2\pi)^{n/2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma((n/2) + \sigma + j)} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{2j} g(x) t^{n+2\sigma-1} \\ = (-1)^k (2\pi)^{n/2} \frac{J_{\alpha}(tQ(x, x)_+^{1/2})}{(tQ(x, x)_+^{1/2})^{\alpha}} g(x) t^{n+2\sigma-1}$$

for $\alpha = (n-2)/2 + \sigma$ since $A \leq k$ and

$$\square(Q(x, x)_+^i g(x)) = 4j \binom{n}{2} + \sigma + j - 1 \Big) Q(x, x)_+^{i-1} g(x),$$

and the convergence of the series of distributions is easily established.

Similarly we find

$$I_2 = -(2\pi)^{n/2} \left[-\frac{\cos \lambda \pi}{\pi} \sum_{j>A} \frac{\Gamma(-j-\alpha)}{j!} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{2j} - \frac{1}{\pi} \sum_{j>B} \frac{\Gamma(\alpha-j)}{j!} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{-2\alpha+2j} \right] g(x) t^{n+2\sigma-1-2k}$$

where $B = (2k + b - 2)/4$ and

$$\square^k I_2 = -(-1)^k (2\pi)^{n/2} \frac{Y_\alpha(tQ(x, x)_+^{1/2})}{(tQ(x, x)_+^{1/2})^\alpha} g(x) t^{n+2\sigma-1}$$

and

$$I_3 = (2\pi)^{n/2} \frac{2}{\pi} \left(\frac{1}{2} \sum_{j>A} \frac{(-1)^j \Gamma(-j-\alpha)}{j!} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{2j} + \frac{1}{2} \sum_{j>B} \frac{(-1)^j \Gamma(\alpha-j)}{j!} \left(\frac{tQ(x, x)_+^{1/2}}{2} \right)^{-2\alpha+2j} \right) g(x) t^{n+2\sigma-1-2k}$$

and

$$\square^k I_3 = (-1)^k (2\pi)^{n/2} \frac{2}{\pi} \frac{K_\alpha(tQ(x, x)_+^{1/2})}{(tQ(x, x)_+^{1/2})^\alpha} g(x) t^{n+2\sigma-1}.$$

This completes the proof when σ is not an integer.

If σ is an integer we still have the representation (2.16) with certain orthogonality conditions on φ if σ is negative. With φ fixed we may regard (2.16) as defining an analytic family in σ . Since (2.13), (2.14) and (2.15) hold for all non-integral σ it follows that they hold also for σ an integer.

Corollary 1. Suppose $(n/2) + \sigma = (n'/2) + \sigma'$ and $p - p', q - q'$ and $\epsilon - \epsilon' + (q - q' - p + p')/2$ are even integers. Then

$$\mathcal{H}_{p', q', \sigma', \epsilon'} = (-i)^{\epsilon - \epsilon'} (-1)^{(q - q' + \sigma - \sigma' + \epsilon - \epsilon')/2} (2\pi)^{(n' - n)/2} \mathcal{H}_{p, q, \sigma, \epsilon}.$$

Corollary 2. Suppose p and q are even. Then

$$\mathcal{H}_{p, q, 0, 0}(f)(r) = \begin{cases} (-1)^{q/2} \mathcal{H}_{n/2}(f)(r) & \text{if } r > 0 \\ (-1)^{p/2} \mathcal{H}_{n/2}(f)(r) & \text{if } r < 0. \end{cases}$$

Remarks. (1) In particular if p and q are even we have

$$(e^{-iQ(x, x)^{1/2}})^{\wedge}(y) = (2\pi)^{n/2} (-1)^{q/2} e^{-iQ(y, y)^{1/2}} (\text{sgn } Q(y, y))^{n/2}.$$

(2) The condition $f \in \mathcal{S}(\mathbf{R}^1)$ and the smoothness condition on g in the hypotheses of Theorem 1 may be considerably relaxed by approximating f and g and passing to the limit.

§3. Applications. a. The Klein-Gordon equation. We set $q = 1$ so that $n = p + 1$, and consider the Klein-Gordon equation

$$(3.1) \quad \square u = m^2 u \quad m > 0$$

with mass m . The case of physical interest is $p = 3$, but the general case is mathematically just as simple. The solutions of this equation are most easily described in terms of their Fourier transforms (or “energy-momentum coordinates” in physical parlance) which must be supported on the hyperboloid of two sheets

$$(3.2) \quad H_m = \{y : Q(y, y) = -m^2\}.$$

The so-called “normalizable” solutions are given by

$$(3.3) \quad u(x) = (2\pi)^{-n} \int_{H_m} e^{iQ(x, y)} v(y) d\mu(y)$$

where

$$(3.4) \quad \int_{H_m} |v(y)|^2 d\mu(y) = \|u\|^2 < \infty$$

and $d\mu$ is the group-invariant measure on H_m . Since the analysis of $v(y)$ into homogeneous waves is known (see [10] or [5], [11], [12]) it is possible to use Theorem 1 to compute (3.3) more explicitly.

Indeed we have

(3.5)

$$v(y) = \frac{1}{2\pi} \int_0^\infty \int_\Sigma [|Q(y, \xi')|_0^{-\beta+i\rho} \psi_0(\xi', \rho) + |Q(y, \xi')|_1^{-\beta+i\rho} \psi_1(\xi', \rho)] d\xi' a(\rho)^2 d\rho$$

with

$$\|u\|^2 = \frac{1}{2\pi} \int_0^\infty \int_\Sigma [|\psi_0(\xi', \rho)|^2 + |\psi_1(\xi', \rho)|^2] d\xi' a(\rho)^2 d\rho$$

where $\beta = (n - 2)/2$ and $a(\rho) = 2(2\pi)^{-\beta} |\Gamma(\beta + i\rho)/\Gamma(i\rho)|$ and ψ_0 and ψ_1 are even and odd on Σ but otherwise are arbitrary, measurable complex valued functions which yield a finite norm ([10], p. 245). The so-called “positive-energy” solutions (those with $v(y)$ supported on the $y_n > 0$ sheet of H_m) correspond to taking $\psi_0 \equiv \psi_1$.

Now if we take ψ_0 and ψ_1 smooth and rapidly decreasing in ρ and $f \in \mathcal{S}(\mathbf{R}^n)$ approximating the δ function at m we may apply Theorem 1 to compute

$$\begin{aligned} (2\pi)^{-n} [v(y)f(|Q(y, y)|_1^{1/2})]^\wedge(x) \\ = (2\pi)^{-n-1} \int_0^\infty \int_\Sigma [(f(|Q(y, y)|_1^{1/2}) |Q(y, \xi)|_0^{-\beta+i\rho})^\wedge(x) \psi_0(\xi', \rho) \\ + (f(|Q(y, y)|_1^{1/2}) |Q(y, \xi')|_1^{-\beta+i\rho})^\wedge(x) \psi_1(\xi', \rho)] d\xi' a(\rho)^2 d\rho. \end{aligned}$$

Then letting f approach the δ function at m we obtain

$$\begin{aligned}
(3.6) \quad u(x) = & \sum_{\epsilon} \int_0^{\infty} \int_{\Sigma} 4(-i)^{\epsilon} (2\pi)^{-(3n+2)/2} \frac{|\Gamma(\beta + i\rho)|^2}{|\Gamma(i\rho)|^2} \cdot |Q(x, \xi')|_{\epsilon}^{-\beta+i\rho} \\
& \cdot \left[(mQ(x, x)_+^{1/2})^{-i\rho} \left\{ -\sin \frac{\pi}{2} (-\beta + i\rho + \epsilon) J_{i\rho}(mQ(x, x)_+^{1/2}) \right. \right. \\
& \left. \left. - \cos \frac{\pi}{2} (-\beta + i\rho + \epsilon) Y_{i\rho}(mQ(x, x)_+^{1/2}) \right\} \right. \\
& \left. + \frac{2}{\pi} (mQ(x, x)_-^{1/2})^{-i\rho} \cos \frac{\pi}{2} (-\beta - i\rho + \epsilon) \right. \\
& \left. \cdot K_{i\rho}(mQ(x, x)_-^{1/2}) \right] \psi_{\epsilon}(\xi', \rho) d\xi' d\rho.
\end{aligned}$$

This representation continues to hold for general ψ_{ϵ} of parity ϵ on Σ provided we interpret the integral suitably. It has the advantage of transforming simply under the action of the homogeneous Lorentz group, but the disadvantages of not transforming simply under translations, and it does not give a simple expression for the real-valued solutions. We hope it will prove useful in the scattering theory of the Klein-Gordon equation.

As a second application we will show how to obtain powers of the Klein-Gordon operator as explicit convolution kernels. This will enable us to conclude that the a priori inequation for the wave operator $\square$ used in [9] also holds for the Klein-Gordon operator $\square - m^2$, so essentially all the results of [9] hold also for the Klein-Gordon equation.

It is well known that there exist two distinguished fundamental solutions for the operator $\square - m^2$ supported in the forward and backward light cone, respectively. More generally, for any $\alpha \in \mathbf{C}$ we may define

$$(3.7) \quad P_{\pm}^{\alpha} \varphi = \lim_{\sigma \rightarrow 0^{\pm}} \mathfrak{F}^{-1}((Q(y, y) + \sigma^2 + m^2 - 2i\sigma y_n)^{\alpha} \phi(y)).$$

Here we choose the usual definition of z^{α} in the plane cut along the negative real axis. It is not hard to show that $P_{\pm}^{\alpha}$ is well-defined and is a convolution operator with kernel having support in the forward or backward light cone (by the Paley-Weiner theorem). Also $P_{\pm}^{\alpha}$ has the group property and $P_{\pm}^{-1} = \square - m^2$ so $P_{\pm}^{-1}$ are fundamental solutions of $\square - m^2$.

We will find an explicit expression for the convolution kernel of $P_{\pm}^{\alpha}$ (this is well known for $\alpha = -1$), and it will be clear from this expression that the kernel is bounded when $\text{Re } \alpha = -n/2$. To do this we may assume $-1 < \text{Re } \alpha < 0$ since $P_{\pm}^{\alpha}$ depends analytically on α . We then deduce easily that $P_{\pm}^{\alpha}$ is convolution with the kernel

$$\begin{aligned}
(3.8) \quad \mathfrak{F}^{-1}[(m^2 + Q(y, y)_+)^{\alpha} + (m^2 - Q(y, y)_-)^{\alpha} \\
+ \cos \pi \alpha (m^2 - Q(y, y)_-)^{\alpha} \mp i \sin \pi \alpha \text{sgn } y_n (m^2 - Q(y, y)_-)^{\alpha}].
\end{aligned}$$

But in this case $\text{sgn } y_n = |Q(y, \xi)|_1^0$ when $Q(y, y) < 0$ for any non-zero ξ with $Q(\xi, \xi) = 0$, so we may compute (3.8) using the results of Section 2.

Rather than use Theorem 1 directly it is simpler to use Lemma 2. From the Mellin inversion formula we obtain (for $b > 0$)

$$(m^2 + Q(y, y)_+)^{\alpha} = m^{2\alpha}(2\pi)^{-1} \int_{-\infty}^{\infty} \left(\frac{Q(y, y)}{m^2} \right)_+^{-b-i\rho} \frac{\Gamma(b+i\rho)\Gamma(-b-\alpha-i\rho)}{\Gamma(-\alpha)} d\rho,$$

$$(m^2 - Q(y, y)_-)^{\alpha} = m^{2\alpha}(2\pi)^{-1} \int_{-\infty}^{\infty} \left(\frac{Q(y, y)}{m^2} \right)_-^{-b-i\rho} \frac{\Gamma(b+i\rho)\Gamma(\alpha+1)}{\Gamma(b+i\rho+\alpha+1)} d\rho$$

and

$$(m^2 - Q(y, y)_-)^{\alpha} = m^{2\alpha}(2\pi)^{-1} \int_{-\infty}^{\infty} \left(\frac{Q(y, y)}{m^2} \right)_-^{-b-i\rho} \frac{\Gamma(-b-i\rho-\alpha)\Gamma(\alpha+1)}{\Gamma(1-b-i\rho)} d\rho.$$

After some simplifying we obtain

$$\begin{aligned} \mathcal{F}^{-1} \left[\frac{m^{2\alpha}}{2\pi} \int_{-\infty}^{\infty} \left(\frac{Q(y, y)}{m^2} \right)_+^{-b-i\rho} + \cos \pi(b+i\rho) \right. \\ \left. \cdot \left(\frac{Q(y, y)}{m^2} \right)_-^{-b-i\rho} \frac{\Gamma(b+i\rho)\Gamma(-b-\alpha-i\rho)}{\Gamma(-\alpha)} d\rho \right. \\ \left. \mp i \sin \pi \alpha \frac{m^{\alpha}}{2\pi} \int_{-\infty}^{\infty} \left(\frac{Q(y, y)}{m^2} \right)_-^{-b-i\rho} |Q(y, \xi)|_1^0 \frac{\Gamma(-b-i\rho-\alpha)\Gamma(\alpha+1)}{\Gamma(1-b-i\rho)} d\rho \right]. \end{aligned}$$

By Lemma 2 this is

$$\begin{aligned} (2\pi)^{-n-1} m^{2\alpha+n} \int_{-\infty}^{\infty} (m^2 Q(x, x))_-^{b+i\rho-(n/2)} \gamma(-b-i\rho, 0) \\ \cdot \left[\sin \pi \frac{p}{2} + \cos \pi(b+i\rho) \sin \pi \left(b+i\rho - \frac{p}{2} \right) \right] \frac{\Gamma(b+i\rho)\Gamma(-b-\alpha-i\rho)}{\Gamma(-\alpha)} d\rho \\ \mp \sin \pi \alpha (2\pi)^{-n-1} m^{2\alpha+n} \int_{-\infty}^{\infty} (m^2 Q(x, x))_-^{b+i\rho-(n/2)} |Q(x, \xi)|_1^0 \\ \cdot \gamma(-b-i\rho, 0) \cos \pi \left(-b-i\rho + \frac{p}{2} \right) \frac{\Gamma(-b-i\rho-\alpha)\Gamma(\alpha+1)}{\Gamma(1-b-i\rho)} d\rho \end{aligned}$$

which simplifies to

$$\begin{aligned} \pi^{-1-(n/2)} m^{2\alpha+n} \int_{-\infty}^{\infty} \left(\frac{m^2}{4} Q(x, x) \right)_-^{b+i\rho-(n/2)} (1 \pm |Q(x, \xi)|_1^0) \\ (3.9) \quad \cdot \frac{\Gamma(-b-i\rho-\alpha)\Gamma(-b-i\rho+\frac{n}{2})}{\Gamma(-\alpha)} \sin \left(-b-i\rho + \frac{n}{2} \right) d\rho \\ = 2\pi^{1-(n/2)} m^{2\alpha+n} \frac{J_{-n/2-\alpha}(mQ(x, x)_-^{1/2})}{\Gamma(-\alpha)(mQ(x, x))_-^{\alpha+(n/2)}} (1 \pm \operatorname{sgn} x_n) \end{aligned}$$

provided $\alpha \neq 0, 1, \dots$.

Note that for $\operatorname{Re} \alpha = -n/2$, (3.9) is a bounded function with supremum of

exponential growth as $\text{Im } \alpha \rightarrow \infty$. Thus we may use the Stein interpolation theorem (as in [8]) to deduce the *a priori* estimate

$$(3.10) \quad \|u\|_q \leq c \|(\square - m^2)u\|_p$$

provided u has zero Cauchy data on a surface $x_n = \text{constant}$. Here $p = 2n/(n+2)$, $q = 2n/(n-2)$, $n \geq 3$ and c depends on n but not on m .

b. L^2 Boundedness of certain operators. We now let p and q be arbitrary positive integers, and let

$$(3.11) \quad B(s)u(x) = |Q(x, x)|_0^{-ns/2} u(x)$$

for $s \in \mathbf{C}$ and

$$(3.12) \quad A(s)u(x) = \mathcal{F}^{-1}(B(s)\hat{u})$$

(we could replace $|Q(x, x)|_0$ by $|Q(x, x)|_1$ in the definition of A and B without materially changing what follows). More explicitly, if p and q have the same parity (n even) then

$$(3.13) \quad A(s)u = \pm (2\pi)^{-n} \gamma(ns/2, 0) \sin \frac{\pi ns}{2} \int_{\mathbf{R}^n} u(x-y) |Q(y, y)|_{\epsilon}^{-n(1-s)/2} dy$$

where ϵ is the parity of $(q-p)/2$, and if p and q have opposite parity (n odd) then

$$(3.14) \quad A(s)u(x) = 2^{1/2} (2\pi)^{-n} \gamma\left(-\frac{ns}{2}, 0\right) \sin \frac{\pi ns}{4} \\ \cdot \int_{\mathbf{R}^n} u(x-y) \left[\pm \cos \frac{\pi n(1-s)}{4} |Q(y, y)|_0^{-n(1-s)/2} \right. \\ \left. \pm \sin \frac{\pi n(1-s)}{4} |Q(y, y)|_1^{-n(1-s)/2} \right] dy$$

by the remarks following Lemma 2 (the exact disposition of the $\pm$ signs is not very interesting).

Now it is obvious from (3.11) and (3.12) that $B(s)$ and $A(s)$ are unitary operators on $L^2(\mathbf{R}^n)$ if $\text{Re } s = 0$, hence so is their product $A(s)B(s)$. Stein ([7a] Problem 6) has proposed studying the boundedness of the product $A(s)B(s)$ on $L^2(\mathbf{R}^n)$ for s in a strip $0 \leq \text{Re } (s) < c$ (actually we have slightly reformulated the original problem when n is odd). The result is expected to have applications to the representation theory of the pseudo-orthogonal groups. Here we prove the result for $c = 1/n$, and it will be clear from the proof that this is best possible.

Theorem 2. *The operators $A(s)B(s)$ are bounded for s in the strip $0 \leq \text{Re } (s) < 1/n$.*

Proof. The idea of the proof is to decompose $L^2(\mathbf{R}^n)$ under the action of $O(p, q)$ and the dilation group. Since $A(s)B(s)$ commutes with these actions it

will appear as a multiplier transformation with respect to the decomposition. We will use Lemma 2 to compute the multiplier explicitly, and then we will be able to see that it remains bounded when $0 \leq \operatorname{Re}(s) < 1/n$.

Now the decomposition of $0(p, q)$ acting on L^2 of the hyperboloid $Q(x, x) = \text{constant}$ was given in [10] (a special case is (3.5)). There is a continuous expansion in terms of $|Q(x, \xi)|_\epsilon^{1-(n/2)+i\rho}$ for $\epsilon = 0, 1$ and a discrete expansion in terms of homogeneous waves $g(x)$ of degree (d, ϵ) where d is an integer, $d > 1 - (n/2)$ and either $g(x) = 0$ when $Q(x, x) < 0$ and ϵ is the parity of $q + d$, or else $g(x) = 0$ when $Q(x, x) > 0$ and ϵ is the parity of $p + d$. For our purposes it is not necessary to know the normalization of these functions or the exact Plancherel measure.

It follows from the above and the Mellin inversion formula that an arbitrary function in $L^2(\mathbf{R}^n)$ has a decomposition in terms of

$$(3.15) \quad |Q(x, x)|_\epsilon^{-1/2+i\epsilon} |Q(x, \xi)|_\epsilon^{1-(n/2)+i\rho}$$

and

$$(3.16) \quad |Q(x, x)|_\epsilon^{-(n/4)-(d/2)+i\epsilon} g(x)$$

where $g(x)$ is as above.

Now we make our basic computation. If $g_{\sigma, \epsilon}(x)$ is any homogeneous wave of degree (σ, ϵ) then

$$\begin{aligned} A(s)B(s)(|Q(x, x)|_0^\lambda g_{\sigma, \epsilon}(x)) &= A(s)(|Q(x, x)|_0^{\lambda-(n\epsilon/2)} g_{\sigma, \epsilon}(x)) \\ &= -2\gamma\left(\lambda - \frac{n\epsilon}{2}, \sigma\right)(-i)^\epsilon \sin \frac{\pi}{2} \left(\lambda - \frac{n\epsilon}{2}\right) \\ &\quad \cdot \mathcal{F}^{-1}\left(\cos \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right) \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2}(1-s) + \sigma\right) g_{\sigma, \epsilon}(y)\right. \\ &\quad \cdot |Q(y, y)|_0^{-\lambda-(n/2)-\sigma} + \sin \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right) \\ &\quad \cdot \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2}(1-s) + \sigma\right) g_{\sigma, \epsilon}(y) |Q(y, y)|_1^{\lambda-(n/2)-\sigma}) \\ &= (2\pi)^{-n} 4(-1)^\epsilon \gamma\left(\lambda - \frac{n\epsilon}{2}, \sigma\right) \gamma\left(-\lambda - \frac{n}{2} - \sigma, \sigma\right) \sin \frac{\pi}{2} \left(\lambda - \frac{n\epsilon}{2}\right) g_{\sigma, \epsilon}(x) \\ &\quad \cdot \left[\cos \frac{\pi}{2} \lambda \left(-\cos^2 \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right)\right. \right. \\ &\quad \cdot \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2}(1-s) + \sigma\right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma\right) \\ &\quad + \sin^2 \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2}(1-s) + \sigma\right) \\ &\quad \cdot \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma\right) \left. \right] |Q(x, x)|_0^\lambda \\ &\quad + \frac{1}{2} \sin \frac{\pi}{2} \lambda \sin \pi \left(\frac{q-p}{2} + \epsilon\right) \sin \frac{\pi n\epsilon}{4} |Q(x, x)|_1^\lambda \end{aligned}$$

and similarly

$$\begin{aligned}
 A(s)B(s)(|Q(x, x)|_1^\lambda g_{\sigma, \epsilon}(x)) \\
 = (2\pi)^{-n} 4(-1)^\epsilon \gamma\left(\lambda - \frac{ns}{2}, \sigma\right) \gamma\left(-\lambda - \frac{n}{2} - \sigma, \sigma\right) \cos \frac{\pi}{2} \left(\lambda - \frac{ns}{2}\right) g_{\sigma, \epsilon}(x) \\
 \cdot \left[-\frac{1}{2} \cos \frac{\pi}{2} \lambda \sin \pi \left(\frac{q-p}{2} - \epsilon\right) \sin \frac{\pi ns}{4} |Q(x, x)|_0^\lambda \right. \\
 + \sin \frac{\pi}{2} \lambda \left(\sin^2 \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right) \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2} (1-s) + \sigma\right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma\right) \right. \\
 - \cos^2 \frac{\pi}{2} \left(\frac{q-p}{2} + \epsilon\right) \sin \frac{\pi}{2} \left(\lambda + \frac{n}{2} (1-s) + \sigma\right) \\
 \left. \cdot \cos \frac{\pi}{2} \left(\lambda + \frac{n}{2} + \sigma\right) \right) |Q(x, x)|_1^\lambda \Big]
 \end{aligned}$$

by the remarks following Lemma 2.

This enables us to handle the continuous series (3.15) with $\lambda = -1/2 + it$ and $\sigma = 1 - (n/2) + i\rho$. For this we need to bound

$$\begin{aligned}
 (3.17) \quad & \Gamma\left(\frac{1}{2} - \frac{ns}{2} + it\right) \Gamma\left(\frac{1}{2} - \frac{ns}{2} + it + i\rho\right) \Gamma\left(\frac{1}{2} - it - i\rho\right) \Gamma\left(\frac{1}{2} - it\right) \\
 & \cdot \sin \frac{\pi}{2} \left(-\frac{1}{2} - \frac{ns}{2} + it\right) \cos \frac{\pi}{2} \left(\frac{1}{2} - \frac{ns}{2} + it + i\rho\right) \\
 & \cdot \sin \frac{\pi}{2} \left(-\frac{1}{2} - it - i\rho\right) \cos \frac{\pi}{2} \left(\frac{1}{2} - it\right)
 \end{aligned}$$

and related terms with some sines and cosines interchanged. But as a consequence of Stirling's formula we have

$$\Gamma(x + iy) = O(e^{\pi |y|/2} |y|^{x-(1/2)}) \quad \text{as } y \rightarrow \infty$$

which shows that (3.17) is bounded for $\text{Re } s \geq 0$ provided we avoid the poles of $\Gamma((1/2) - (ns/2) + it) \Gamma((1/2) - (ns/2) + it + i\rho)$ which occur for $\text{Re } s = 1/n, 3/n, 5/n, \dots$. It is clear that the bound is uniform in any strip $0 \leq \text{Re } s \leq c < 1/n$, and that a pole does result if $\text{Re } s = 1/n$.

To handle the discrete series (3.16) we must make use of the observation that g vanishes when $\pm Q(x, x) < 0$. For then $|Q(x, x)|_{\epsilon'}^\lambda g(x) = (\pm 1)^\epsilon Q(x, x)_\pm^\lambda g(x)$ so that by Lemma 2 we have

$$\begin{aligned}
 (3.18) \quad & A(s)B(s)(Q(x, x)_+^{-(n/4)-(d/2)+it} g(x)) \\
 & = (2\pi)^{-n} (-1)^\epsilon \gamma\left(-\frac{n}{4} - \frac{d}{2} - \frac{ns}{2} + it, d\right) \gamma\left(-\frac{n}{4} - \frac{d}{2} - it, d\right) \\
 & \cdot \sin \pi \left(-\frac{n}{4} - \frac{d}{2} - \frac{ns}{2} + it + \frac{q+d+\epsilon}{2}\right) \\
 & \cdot \sin \pi \left(-\frac{n}{4} - \frac{d}{2} - it + \frac{q+d+\epsilon}{2}\right) \\
 & \cdot Q(x, x)_+^{-(n/4)-(d/2)+it} g(x)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.19) \quad & A(s)B(s)(Q(x, x)^{-(n/4) - (d/2) + it} g(x)) \\
 &= (2\pi)^{-n}(-1)^{\epsilon} \gamma\left(-\frac{n}{4} - \frac{d}{2} - \frac{ns}{2} + it, d\right) \gamma\left(-\frac{n}{4} - \frac{d}{2} - it, d\right) \\
 &\quad \cdot \sin \pi\left(-\frac{n}{4} - \frac{d}{2} - \frac{ns}{2} + it + \frac{p+d-\epsilon}{2}\right) \\
 &\quad \cdot \sin \pi\left(-\frac{n}{4} - \frac{d}{2} - it + \frac{p+d-\epsilon}{2}\right) \\
 &\quad \cdot Q(x, x)^{-(n/4) - (d/2) + it} g(x).
 \end{aligned}$$

Note that in the first case $q + d + \epsilon$ is even, and in the second case $p + d - \epsilon$ is even by the parity assumptions on g . Thus the coefficient in (3.18) or (3.19) is

$$\pm 2^{-ns} \frac{\Gamma\left(\frac{n}{4} + \frac{d}{2} - \frac{ns}{2} + it\right) \Gamma\left(\frac{n}{4} + \frac{d}{2} - it\right)}{\Gamma\left(\frac{n}{4} + \frac{d}{2} + \frac{ns}{2} - it\right) \Gamma\left(\frac{n}{4} + \frac{d}{2} + it\right)}.$$

Since $(n/4) + (d/2) > 1/2$ this is uniformly bounded in the strip $0 \leq \operatorname{Re} s \leq 1/n$.

Remark. The definite case ($q = 0$) which we have excluded, is different in character, not only because the size of the strip is independent of n (see [7a]), but also because there are related L^p results. In the case we consider, the operator $A(s)$ cannot be bounded on any $L^p(\mathbf{R}^n)$ for $p \neq 2$ and $\operatorname{Re} s = 0$ (but $s \neq 0$) if $n \geq 3$ according to an unpublished result of P. Tomas (*On radial Fourier multipliers*, Ph.D. thesis, Cornell University, 1974).

§4. $O(n, \mathbf{C})$. For $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ we write $z \cdot z = z_1^2 + \dots + z_n^2$ and denote by $O(n, \mathbf{C})$ the group of complex $n \times n$ matrices preserving this bi-linear form. We write

$$\begin{aligned}
 \frac{\partial}{\partial z_i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right) & \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right) \\
 dz_i &= dx_i + i dy_i & d\bar{z}_i &= dx_i - i dy_i
 \end{aligned}$$

and we define the Fourier transform on $\mathbf{C}^n$ (considered as $\mathbf{R}^{2n}$) by

$$(4.1) \quad \hat{f}(w) = \left(\frac{i}{2}\right)^n \int_{\mathbf{C}^n} f(z) e^{i \operatorname{Re} z \cdot w} dz d\bar{z}$$

so that it commutes with the action of $O(n, \mathbf{C})$. In terms of the underlying real variables x, y, u, v defined by $z = x + iy, w = u + iv$ we have

$$\hat{f}(u, v) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x, y) e^{i(x \cdot u - y \cdot v)} dx dy.$$

We wish to compute the Fourier transform of a product fg where f is invariant under $O(n, \mathbf{C})$ and g is a homogeneous solution of the equations

$$(4.2) \quad \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} g = 0 \quad \frac{\partial}{\partial \bar{z}} \cdot \frac{\partial}{\partial \bar{z}} g = 0.$$

(That these realize irreducible representations of $O(n, \mathbf{C})$ may be proved by techniques similar to those in [10].) Since the derivation parallels that given in Section 2 we will only indicate the modifications needed for the sake of brevity.

If F is any complex valued function and $\lambda, \mu \in \mathbf{C}$ have a real integer difference $\lambda - \mu$ we write

$$F^{\lambda, \mu} = F^\lambda \bar{F}^\mu = |F|^{\lambda, \mu} e^{i(\lambda - \mu) \arg F}.$$

As before, the basic idea is to compute the Fourier transform for special functions f and g from which the general functions may be synthesized. In this case they are

$$(z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b} \quad \text{where} \quad \zeta \in \mathbf{C}^n$$

satisfies $\zeta \cdot \zeta = 0$ but $\zeta \neq 0$. If $\operatorname{Re} \lambda + \mu \geq 0$ and $\operatorname{Re} a + b \geq 0$ these are locally integrable functions and define tempered distributions. They may be analytically continued as follows:

Lemma 3. *The family of distributions $(z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}$ may be analytically continued to all permissible values of λ, μ, a, b provided*

- (i) *either $a \neq -2, -3, -4, \dots$
or $b \neq -2, -3, -4, \dots$*

and

- (ii) *either $\lambda \neq -1, -2, \dots$ and $\lambda \neq -n/2 - a, -n/2 - a - 1, \dots$
or $\mu \neq -1, -2, \dots$ and $\mu \neq -n/2 - b, -n/2 - b - 1, \dots$.*

Furthermore, for any $\varphi \in \mathcal{S}$,

$$((z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}, \varphi)$$

is rapidly decreasing as $\operatorname{Im} \lambda \rightarrow \infty$ or $\operatorname{Im} \mu \rightarrow \infty$.

Proof. A simple computation shows

$$\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} ((z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}) = 4\lambda \left(\lambda - 1 + \frac{n}{2} + a \right) (z \cdot z)^{\lambda-1, \mu} (z \cdot \zeta)^{a, b}$$

and

$$\frac{\partial}{\partial \bar{z}} \cdot \frac{\partial}{\partial \bar{z}} ((z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}) = 4\mu \left(\mu - 1 + \frac{n}{2} + b \right) (z \cdot z)^{\lambda, \mu-1} (z \cdot \zeta)^{a, b}$$

provided $\operatorname{Re} (\lambda + \mu)$ and $\operatorname{Re} (a + b)$ are sufficiently large. Thus, if (ii) holds we may define

$$(4.3) \quad ((z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}, \varphi) = A(\lambda, \mu, a, b, k) \cdot \left((z \cdot \zeta)^{a, b}, (z \cdot z)^{\lambda+k, \mu} \left(\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} \right)^k \varphi \right)$$

or

$$(4.4) \quad ((z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}, \varphi) = B(\lambda, \mu, a, b, k) \cdot \left((z \cdot \zeta)^{a, b}, (z \cdot z)^{\lambda, \mu+k} \left(\frac{\partial}{\partial \bar{z}} \cdot \frac{\partial}{\partial \bar{z}} \right)^k \varphi \right)$$

where $A(\lambda, \mu, a, b, k) = (4^k(\lambda + 1) \cdots (\lambda + k)(\lambda + (n/2) + a) \cdots (\lambda + (n/2) + a + k - 1))^{-1}$ and $B(\lambda, \mu, a, b, k) = (4^k(\mu + 1) \cdots (\mu + k)(\mu + (n/2) + b) \cdots (\mu + (n/2) + b + k - 1))^{-1}$ and the analytic continuation of $(z \cdot \zeta)^{a, b}$ for a, b satisfying (i) is given in Gelfand-Shilov [6], p. 374. The rest of the proof is the same as for Lemma 1.

Lemma 4. Suppose that λ and μ are not both integers or half-integers (if n is odd). Then

$$(4.5) \quad [(z \cdot z)^{\lambda, \mu}]^{\wedge} = \gamma(\lambda, \mu) (w \cdot w)^{-\lambda - (n/2), -\mu - (n/2)}$$

where

$$(4.6) \quad \gamma(\lambda, \mu) = (2\pi)^n 4^{2(\lambda + \mu) + n} \frac{\Gamma(\lambda + 1) \Gamma\left(\lambda + \frac{n}{2}\right)}{\Gamma(-\mu) \Gamma\left(-\mu + 1 - \frac{n}{2}\right)}.$$

Proof. It is shown in [6], pp. 398–403 that

$$(4.7) \quad \left(\frac{i}{2}\right)^n \int (z \cdot z)^{\lambda, \lambda} \exp(-z \cdot \bar{z}) dz d\bar{z} = \pi^n 4^{\lambda} \Gamma(\lambda + 1) \Gamma\left(\lambda + \frac{n}{2}\right) / \Gamma\left(\frac{n}{2}\right).$$

Furthermore, $(z \cdot z)^{\lambda, \mu}$ is the unique (up to a constant multiple) distribution invariant under $O(n, \mathbf{C})$ with complex homogeneity (λ, μ) , so (4.5) must hold for some constant $\gamma(\lambda, \mu)$. To compute the constant we use (4.7) and Parseval's identity

$$\left(\frac{i}{2}\right)^n \int f(z) \overline{g(z)} dz d\bar{z} = \frac{1}{(2\pi)^{2n}} \left(\frac{i}{2}\right)^n \int \hat{f}(w) \overline{\hat{g}(w)} dw d\bar{w}.$$

We choose $g(z) = \exp(-z \cdot \bar{z})$ so that $\hat{g}(w) = \pi^n \exp(-w \cdot \bar{w}/4)$, and $f(z) = (z \cdot z)^{\lambda, \lambda}$. Thus we obtain

$$\begin{aligned} (4.8) \quad & \pi^n 4^{\lambda} \Gamma(\lambda + 1) \Gamma\left(\lambda + \frac{n}{2}\right) / \Gamma\left(\frac{n}{2}\right) \\ &= (2\pi)^{-2n} \pi^n \left(\frac{i}{2}\right)^n \int \gamma(\lambda, \lambda) (w \cdot w)^{-\lambda - (n/2), -\lambda - (n/2)} e^{-w \cdot \bar{w}/4} dw d\bar{w} \\ &= (2\pi)^{-2n} \pi^n 4^{-2\lambda} \pi^n 4^{-\lambda - (n/2)} \gamma(\lambda, \lambda) \Gamma\left(-\lambda - \frac{n}{2} + 1\right) \Gamma(-\lambda) / \Gamma\left(\frac{n}{2}\right) \end{aligned}$$

which gives (4.6) when $\lambda = \mu$.

Suppose now $\lambda = \mu + k$ for k a positive integer. Then from (4.1) we obtain

$$((z \cdot z)^{\lambda, \mu})^{\wedge} = \left[-4 \left(\frac{\partial}{\partial w} \cdot \frac{\partial}{\partial w} \right) \right]^k ((z \cdot z)^{\mu, \mu})^{\wedge}(w).$$

Using (4.3) and simplifying we obtain (4.6), and similar reasoning yields the same result if $\lambda = \mu - k$.

Lemma 5. *Let $\zeta \in \mathbf{C}^n$ satisfy $\zeta \cdot \zeta = 0$ but $\zeta \neq 0$. Then*

$$(4.9) \quad [(z \cdot z)^{\lambda, \mu} (z \cdot \zeta)^{a, b}]^{\wedge}(w) = \gamma(a, b, \lambda, \mu) (w \cdot w)^{-\lambda - a - (n/2), -\mu - b - (n/2)} (w \cdot \zeta)^{a, b}$$

where

$$(4.10) \quad \gamma(a, b, \lambda, \mu) = i^{b-a} (2\pi)^n 4^{2(\lambda+\mu)+n+a+b} \frac{\Gamma(\lambda+1) \Gamma\left(\lambda + \frac{n}{2} + a\right)}{\Gamma(-\mu) \Gamma\left(-\mu - \frac{n}{2} - b - 1\right)}.$$

Proof. We present only the formal derivation. To justify the use convolution identity (4.13) requires an argument similar to that in Lemma 2.

Without loss of generality we may assume that $\zeta = (1, i, 0, \dots, 0)$. We introduce new variables $z_1' = (z_1 + iz_2)/2^{1/2}$, $z_2' = (z_1 - iz_2)/2^{1/2}$, $z'' = (z_3, \dots, z_n)$, and similarly for w . In terms of these variables the left-hand side of (4.9) becomes

$$(4.11) \quad \left(\frac{i}{2}\right)^n \iint (2z_1' z_2' + z'' \cdot z'')^{\lambda, \mu} 2^{(a+b)/2} (z_1')^{a, b} \cdot e^{i \operatorname{Re} (z_1' w_2' + z_2' w_1' + z'' \cdot w'')} dz_1' d\bar{z}_1' dz_2' d\bar{z}_2' dz'' d\bar{z}''.$$

But we have (see [6, p. 383]),

$$(4.12) \quad \left(\frac{i}{2}\right)^n \iint (z_1')^{a, b} e^{i \operatorname{Re} (z_1' w_2')} dz_1' d\bar{z}_1' = 2^{a+b+2} \pi i^{a-b} \frac{\Gamma(a+1)}{\Gamma(-b)} (w_2')^{-a-1, -b-1}.$$

Using this, together with the formula $(fg)^{\wedge} = (2\pi)^{-k} \hat{f} * \hat{g}$ for the k (real) dimensional Fourier transform, we simplify (4.11) to

$$2^{3(a+b)/2} \pi^{-1} i^{a-b} \frac{\Gamma(a+1)}{\Gamma(-b)} \left(\frac{i}{2}\right)^{n+1} \iint (2z_1' z_2' + z'' \cdot z'')^{\lambda, \mu} \cdot e^{i \operatorname{Re} (z_1' u + z_2' w_1' + z'' \cdot w'')} (w_2' - u)^{-a-1, -b-1} dz d\bar{z} du d\bar{u}.$$

Now we use Lemma 4 to evaluate the z -integration and obtain

$$(4.14) \quad c(a, b, \lambda, \mu) \left(\frac{i}{2}\right) \iint (2w_1' u + w'' \cdot w'')^{-\lambda - (n/2), -\mu - (n/2)} (w_2' - u)^{-a-1, -b-1} du d\bar{u}$$

where

$$(4.15) \quad c(a, b, \lambda, \mu) = (2\pi)^n 4^{2(\lambda+\mu)+n} 2^{3(a+b)/2} \frac{\pi^{-1} i^{a-b} \Gamma(a+1) \Gamma(\lambda+1) \Gamma\left(\lambda + \frac{n}{2}\right)}{\Gamma(-b) \Gamma(-\mu) \Gamma\left(-\mu + 1 - \frac{n}{2}\right)}.$$

To simplify the integral in (4.14) we make the change of variables $u \rightarrow ((w \cdot w)u - w'' \cdot w'')/2w_1'$ to obtain

$$(4.16) \quad c(a, b, \lambda, \mu)(w \cdot w)^{-\lambda - (n/2) - a, -\mu - (n/2) - b} 2^{(a+b)/2} (w \cdot \zeta)^{a, b} \\ \cdot \frac{i}{2} \iint u^{-\lambda - (n/2), -\mu - (n/2)} (1 - u)^{-a-1, -b-1} du d\bar{u}.$$

Note that this integral is absolutely convergent if $-2 < \operatorname{Re}(a + b) < 0$ and $-n - \operatorname{Re}(a + b) < \operatorname{Re}(\lambda + \mu) < -n + 2$. We evaluate it by using (4.12), (4.13) and the Fourier inversion formula. We set $f(u) = u^{-\lambda - (n/2), -\mu - (n/2)}$ and $g(u) = u^{-a-1, -b-1}$. We wish to compute $f * g(1) = (2\pi)^{-2} (\hat{f} \cdot \hat{g})^\wedge(-1)$. Repeated use of (4.12) shows that this is equal to

$$(-1)^{a-\lambda+b-a} \pi \frac{\Gamma\left(-\lambda - \frac{n}{2} + 1\right) \Gamma(-a) \Gamma\left(\lambda + \frac{n}{2} + a\right)}{\Gamma(b+1) \Gamma\left(\mu + \frac{n}{2}\right) \Gamma\left(-\mu - \frac{n}{2} - b + 1\right)}.$$

Substituting this back in (4.16) and using $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ to simplify we obtain (4.9) and (4.10).

Theorem 3. Let $f \in \mathcal{S}(\mathbf{C}^1)$ and let $g(z)$ be a solution of the equations (4.2) which is complex homogeneous of degree (a, b) , meaning

$$(4.17) \quad g(\lambda z) = \lambda^a \bar{\lambda}^b g(z) \quad \text{for } \lambda \in \mathbf{C} \quad \lambda \neq 0$$

and which is C^∞ on the complement of the cone $\{z \cdot z = 0\}$. Then

$$(4.18) \quad [f(z \cdot z)g(z)]^\wedge(w) = i^{b-a} (2\pi)^n \cdot \frac{i}{2} \int_{\mathbf{C}} \mathcal{L}_{n,a,b}(w \cdot w, \zeta) g(w) f(\zeta) d\zeta d\bar{\zeta}$$

where $\mathcal{L}_{n,a,b}$ is given in terms of the Meijer G -function ([2], p. 207) by

$$(4.19) \quad \mathcal{L}_{n,a,b}(\tau, \zeta) = (2\pi)^{-1} 4^{n+a+b} \sum_{k=-\infty}^{\infty} \tau^{(k/2) - (n/2) - a, -(k/2) - (n/2) - b} \zeta^{-(k/2) - 1, (k/2) - 1} \\ \cdot G_{0,4}^{2,0} \left(\frac{|\tau \zeta|}{16} \middle| \begin{matrix} \\ (k+2)/2, (k+n+2a)/2, (2-k)/2, (-k+n+2b)/2 \end{matrix} \right).$$

Note that $\mathcal{L}_{n,a,b}$ depends only on $(n/2) + a$ and $(n/2) + b$.

Proof. As in the proof of Theorem 1 it suffices to establish the result for $g(z) = (z \cdot \zeta)^{a,b}$ for $\operatorname{Re} a + b \geq 0$. Now for $\operatorname{Re} \tau > 0$ we define the complex Mellin transform

$$F(k, \tau) = \frac{1}{4\pi^2} \frac{i}{2} \int_{\mathbf{C}} f(\zeta) \zeta^{(\tau/2) - (k/2) - 1, (\tau/2) + (k/2) - 1} d\zeta d\bar{\zeta}$$

and from the inversion formula we obtain

$$(f(z \cdot z)g(z), \varphi) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} ((z \cdot z)^{(k-\sigma-i\rho)/2, (-k-\sigma-i\rho)/2} g(z), \varphi) F(k, \sigma + i\rho) d\rho$$

if say $0 < \sigma < 1/2$. Thus

$$\begin{aligned}
 & ([f(z \cdot z)g(z)]^\wedge, \varphi) \\
 &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma\left(a, b, \frac{k - \sigma - i\rho}{2}, \frac{-k - \sigma - i\rho}{2}\right) \\
 &\quad \cdot ((w \cdot w)^{(-k + \sigma + i\rho - 2a - n)/2, (k + \sigma + i\rho - 2b - n)/2} g(w), \varphi) F(k, \sigma + i\rho) d\rho \\
 &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} 4^j \gamma\left(a, b, \frac{k - \sigma - i\rho - 2j}{2}, \frac{-k - \sigma - i\rho}{2}\right) \\
 &\quad \cdot \left((w \cdot w)^{(-k + \sigma + i\rho - 2a - n + 2j)/2, (k + \sigma + i\rho - 2b - n)/2} g(w), \left(\frac{\partial}{\partial w} \cdot \frac{\partial}{\partial w}\right)^i \varphi\right) F(k, \sigma + i\rho) d\rho
 \end{aligned}$$

by Lemma 5 and (4.3) if j is sufficiently large.

But for j large enough the γ factor will decrease as $\rho \rightarrow \infty$ so that we may introduce the definition of $F(k, \sigma + i\rho)$ and rearrange the integrals to obtain

$$\begin{aligned}
 (4.20) \quad & [f(z \cdot z)g(z)]^\wedge(w) \\
 &= \left(4 \frac{\partial}{\partial w} \cdot \frac{\partial}{\partial w}\right)^i \frac{i}{2} \int_{\mathbf{C}} i^{b-a} (2\pi)^{n-1} \\
 &\quad \cdot 4^{n+a+b} \sum_{k=-\infty}^{\infty} (w \cdot w)^{(k/2) - (n/2) - a + j, (-k/2) - (n/2) - b} \zeta^{(-k/2) - 1, (k/2) - 1} g(w) \\
 &\quad \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{(w \cdot w)\zeta}{16} \right|^{\sigma + i\rho} \\
 &\quad \cdot \frac{\Gamma\left(\frac{k - \sigma - i\rho + 2 - 2j}{2}\right) \Gamma\left(\frac{k - \sigma - i\rho + n + 2a - 2j}{2}\right)}{\Gamma\left(\frac{\sigma + k + i\rho}{2}\right) \Gamma\left(\frac{\sigma + k + i\rho - n - 2b + 2}{2}\right)} d\rho f(\zeta) d\zeta d\bar{\zeta}.
 \end{aligned}$$

Now the inner integral is

$$G_{0,4}^{2,0} \left(\left| \frac{(w \cdot w)\zeta}{16} \right| \middle| \begin{matrix} \\ (k+2-2j)/2, (k+n+2a-2j)/2, (2-k)/2, (-k+n+2b)/2 \end{matrix} \right)$$

so that (4.20) reduces to (4.18) and (4.19) by the recursion relations for the G -function.

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