A Fatou Theorem for the General One-Dimensional Parabolic Equation

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1. Introduction. Let R be a finite or infinite one-dimensional open interval. The purpose here is to find all positive weak solutions of the equation

$$(1.1) \quad \frac{\partial}{\partial t} u = \mathcal{L}u = a(x)u_{xx} + b(x)u_x + c(x)u \quad \text{in } R \times (0, T)$$

where $T \leq \infty$. Here $a(x) \geq 0$, $c(x)$ is bounded from above⁽¹⁾, and $1/a(x)$, $b(x)/a(x)$, and $c(x)/a(x)$ are locally integrable in R , but otherwise the coefficients are unrestricted. These results can be viewed either as a generalization of Fatou's theorem for positive harmonic functions on the unit disk, or as a uniqueness theorem for a Cauchy problem.

Our results extend characterizations of Widder ([23, 24]; see §2.1) for positive solutions of the heat equation. In particular, we find that all positive solutions of (1.1) are of the form

$$(1.2) \quad u(x, t) = \int_R p(t, x, y)F(dy),$$

where $p(t, x, y)$ is the fundamental solution of (1.1), if and only if the Green's function of (1.1) is not of trace class at either endpoint of R (see §2). While these results are one-dimensional, they are complete, and the techniques of proof suggest generalizations to equations in higher dimensions.

As an example, we find that all positive solutions of $u_t = u_{xx} - x^{2a}u$ in $(-\infty, \infty) \times (0, T)$ are of the form (1.2) for $a \leq 1$, but other solutions can occur if $a > 1$. Similarly, all positive solutions of $u_t = x^2u_{xx}$ in $(0, \infty) \times (0, T)$ are given by (1.2), while if $u_t = x^4u_{xx}$, a boundary term "at infinity" can appear (see §2).

Feller [7, 9] and Hille [12] studied a similar question: existence and uniqueness of semi-groups $\{T_t\}$ of bounded linear operators generated by $\mathcal{L}$ in either $C(\bar{R})$ or $L^1(R)$. One always has existence, but different criteria for uniqueness in $C(\bar{R})$ and $L^1(R)$. The limit point-limit circle theory of Weyl [4, 22] is the

same question in $L^2(R)$ and gives intermediate criteria for uniqueness. If $c(x) = 0$, our conditions are the same as Feller–Hille’s for $L^1(R)$, although the problem considered is quite different (for example a non-negative $u(x, t)$ needn’t be in $L^1(R)$ for any t). In probabilistic terminology (see §2), this criterion is that neither endpoint be accessible or entrance.

Equation (1.1) can always be transformed into a similar equation with $c(x) = 0$; assume for the moment $c = 0$ in (1.1). Then by a change of variables we can write (1.1) in “Feller form”

$$(1.3) \quad \frac{\partial}{\partial t} u = \mathcal{L}u = \frac{d}{dm(x)} \frac{d}{ds(x)} u \quad \text{in } R \times (0, T)$$

where $(df/ds)(x) = \lim_{y \rightarrow x} (f(y) - f(x))/(s(y) - s(x))$ and $m(x)$ and $s(x)$ are strictly increasing in R . We can also consider equation (1.3) in $R \times (0, T)$ for an arbitrary continuous strictly increasing function $s(x)$ (a “scale”) on R and Borel measure $m(dx)$ which is positive on open sets. Then $\mathcal{L}$ becomes the infinitesimal generator of the general non-singular Markov diffusion process r_t in R which terminates at the endpoints. (More exactly, the semigroup $T_t f(x) = E(f(r_t) \mid r_0 = x)$ has $\mathcal{L}$ as its infinitesimal generator; see Breiman [3] or Ito-McKean [13].) Indeed, in terms of the process r_t , $u(x, t)$ turns out to be a weak solution of (1.3) iff

$$(1.4) \quad u(r_t, T - t) \text{ is a local martingale} \quad 0 < t < T.$$

(See Remark (3) in the Appendix.) In this context, the problem is to determine when all local martingales of the form (1.4) are actually given by (1.2), and to classify the ones that are not. Such representation theorems have had applications in proving theorems of iterated logarithm type in probability (see §2.1 and [16–18]). In particular we would like to thank Prof. David Siegmund for access to the manuscript of [18] and for many helpful conversations.

While the results of this article were motivated by probability (specifically whether all positive solutions of (1.1) could be interpreted as in §3), the use of probability in the proofs is mostly heuristic and the arguments are easily transformed into analytic ones. The key in proofs via probability theory seems to be that probabilities are always non-negative and less than one; these observations usually occur at points where maximum principles apply in analytic proofs.

In the following, §2 contains precise statements of our results and some historical remarks; the Appendix contains definitions and some preliminary results. In §3 we gave a probabilistic interpretation of the terms appearing in Theorem 2.1 below. The proofs of the theorems of §2 take up §4–§11; in §12 we give some remarks about when $u(x, t) \in L^1(R)$ ($= L^1(R, dm)$). Usually if $u(x, t) \in L^1(R)$ for any one value of t then $u(x, t) \in L^1(R)$ for a.e. t , but if R contains a “pure exit” boundary for $\mathcal{L}$ (see §2) then $u(x, t)$ can pop in and out of $L^1(R)$ at will.

One of the key steps in the proofs below is in §7, which introduces a “duality” notion for operators $\mathcal{L}$ which reduces the number of boundary types we have to consider from four to two. This idea appears in Feller [9] but is not exploited.

Since the positive solutions of (1.3) are exactly the positive harmonic functions of the operator $\mathfrak{L} - \partial/\partial t$ (or of the space-time process $z_t = (r_t, T - t)$), our results are in the domain of the theory of Martin boundaries. In general this theory doesn't seem to be of any help in obtaining the specific representations we consider here. Woo [25], however, finds all positive solutions of $u_t = u_{xx}$ in $R \times (-\infty, 0)$, $R = (-\infty, \infty)$, by a Martin boundary approach; *i.e.* by first finding all extreme positive solutions in a certain cone and then proving all solutions are averages of these.

One application of our results, which follows from Theorem 2.1 and the arguments in §5 and §10, is that for any $\mathfrak{L}$ of the form (1.1) or (1.3) and $T > 0$, there exists a positive solution of (1.3) which cannot be extended as a continuous function to $R \times (0, T) \cup N[(x_0, T), \epsilon]$ for any $x_0 \in R$ and $\epsilon > 0$.

Note the similarity between (2.1) below and the usual Green's formula identity for solutions of parabolic equations. Indeed, if $\mathfrak{L}$ and $u(x, t)$ are smooth in $\bar{R} \times [0, T)$, equation (2.1) can be obtained directly (see equation (A11) in the Appendix).

Finally, we remark that for $\mathfrak{L}$ in (1.1),

$$(1.5) \quad m(dx) = a(x)^{-1} \exp \left(\int^x (b(t)/a(t)) dt \right) dx,$$

$$s(x) = \int^x \exp \left(- \int^y (b(t)/a(t)) dt \right) dy$$

which leads to the local integrability requirement for $1/a(x)$ and $b(x)/a(x)$. (See also equation (2.3) below.)

2. Statements of results. The positive solutions of (1.3) depend on the behavior of $\mathfrak{L}$ near the endpoints of R . Since the form of (1.1) or (1.3) is invariant under diffeomorphisms of R , it is sufficient to assume $R = (0, 1)$. If $s(0) > -\infty$ (*i.e.* if all solutions of $\mathfrak{L}u = 0$ are bounded from below), 0 is said to be an *accessible boundary* if $\int_0^{1/2} (s(x) - s(0))m(dx) < \infty$ and an *inaccessible exit (I.E.) boundary* if this integral diverges. If $s(0) = -\infty$, 0 is an *entrance boundary* if $\int_0^{1/2} |s(x)|m(dx) < \infty$ and is otherwise a *natural boundary*. Note that we have split what are usually called natural boundaries into two categories, natural and I.E. If 0 is accessible and $m[(0, \frac{1}{2}]] < \infty$, 0 is a *regular boundary*; if 0 is accessible and $m[(0, \frac{1}{2}]] = \infty$, a *pure exit boundary*. We make the same classifications at 1. See Breiman [13] or Ito-McKean [13] for a discussion of these notions in terms of the pathwise behavior of r_t .

We remark that equation (1.3) always has at least one fundamental solution with respect to $m(dx)$; let $p(t, x, y)$ be the minimal fundamental solution (which is symmetric in x and y). This corresponds to zero boundary conditions for $u(x, t)$ or $p(t, x, y)$ at accessible boundaries and $du/ds(a, t) = 0$ ($a = 0$ or 1) at entrance boundaries (Ito-McKean [13, p 149+]).

See the Appendix for the definition of a weak solution. In particular any C^2

solution of (1.1) is a weak solution, and any weak solution is essentially continuous.

Theorem 2.1. *Let $u(x, t)$ be a non-negative weak solution of equation (1.3) in $R \times (0, T)$. Then, there exist non-negative Borel measures $c(ds)$ and $\tilde{c}(ds)$ on $[0, T)$ and $F(dx)$ in R such that*

$$(2.1) \quad u(x, t) = \int_{0-}^t q_0(t-s, x)c(ds) + \int_{0-}^t q_1(t-s, x)\tilde{c}(ds) + \int_R p(t, x, y)F(dy)$$

for $m(dx) \times dt$ a.e. $(x, t) \in R \times (0, T)^{(2)}$, where

(i) If 0 is an accessible boundary,

$$(2.2) \quad q_0(t, x) = \lim_{a \rightarrow 0} \frac{p(t, a, x)}{s(a) - s(0)} = \frac{dp}{ds}(t, 0, x),$$

the limit existing weakly in t . (I.e., the indefinite integrals converge. If 0 is regular, $q_0(t, x)$ exists pointwise.) Similarly, if 1 is accessible, $q_1(t, x) = -(dp/ds)(t, 1, x)$.

(ii) If 0 is an entrance boundary, $q_0(t, x) = p(t, 0+, x)$ in the same sense as (i), and similarly $q_1(t, x) = p(t, 1-, x)$ if 1 is entrance.

(iii) In the remaining cases, i.e. if 0 is an inaccessible exit (I.E.) or natural boundary, no term for $q_0(t, x)$ appears in (2.1), and the kernel $q_0(t, x)$ of (2.2) (respectively $p(t, 0+, x)$ if 0 is natural) converges and is identically zero.

Hence all non-negative solutions of (1.3) are of the form $\int p(t, x, y)F(dy)$ iff both boundaries are natural or I.E.

Conversely, suppose we define a function $u(x, t)$ by (2.1) for given non-negative Borel measures $\tilde{c}(ds)$, $c(ds)$ on $[0, T)$ and $F(dy)$ on R . Then, if $u(x_0, t_0) < \infty$ for any (x_0, t_0) , u is a weak solution of $u_t = \mathcal{L}u$ in $R \times (0, t_0)$ and in fact can be redefined on a null set to be continuous in $R \times (0, t_0)$ (see §10).

We now consider equation (1.1) with arbitrary zero-th order term $c(x) \leq M$. Since we can replace $u(x, t)$ by $e^{-Mt}u(x, t)$, it is sufficient to assume $c(x) \leq 0$. The analog of (1.3) for this situation is

$$(2.3) \quad \frac{\partial}{\partial t} u = \mathcal{L}_1 u = \frac{d}{dm} \left[\frac{du}{ds} - \int^x u(y)k(dy) \right]$$

where $k(dy)$ is a non-negative Borel measure on R [13, Ch. 4]. If $k(dx) = c(x)m(dx)$, this reduces to $\mathcal{L}_1 u = \mathcal{L}u - c(x)u$. The formalism, however, would also allow a delta function for $c(x)$.

Let $p(t, x, y)$ be the (symmetric) minimal fundamental solution of (2.3) as before, and let $h_+(x) \uparrow$, $h_-(x) \downarrow$ be two linearly independent positive solutions of

$$(2.4) \quad \mathcal{L}_1 h_+(x) = \mathcal{L}_1 h_-(x) = 0.$$

Such functions exist whenever $k(R) > 0$ (see §11; if $s(0)$ is finite we normalize by setting $h_+(0) = 0$).

It will turn out that the extra term in (1.1) or (2.3) can add additional (non-

zero) terms to the representation (2.1) but not remove them. Assume $k(R) > 0$ for definiteness. Then (see §11).

Theorem 2.2. *Let $u(x, t)$ be a non-negative weak solution of equation (2.3) in $R \times (0, T)$. Then, $u(x, t)$ also has the representation (2.1), where $q_0(t, x) \not\equiv 0$ (i.e. a term appears) iff*

$$(2.5) \quad \int_0^{1/2} h_+(x)h_-(x)m(dx) < \infty.$$

If (2.5) converges, $q_0(t, x) = \lim_{a \rightarrow 0} p(t, a, x)/h_+(a)$ as in Theorem 2.1, where

$$(2.6) \quad c(t-) = \int_{0-}^{t-} c(ds) = \lim_{a \rightarrow 0} \frac{K}{h_-(a)} \int_0^t u(a, s) ds, \quad 0 < t < T;$$

here K is the Wronskian of $h_+(x)$ and $h_-(x)$. The integral (2.5) always converges if 0 is accessible or entrance for $\mathcal{L}$. The same results hold mutatis mutandum at the endpoint 1.

If $k(dx) = c(x)m(dx)$ for bounded $c(x)$, (2.5) converges iff 0 is accessible or entrance for $\mathcal{L}$.

The analog of (2.6) for equation (1.3) is

Theorem 2.3. *For $u(x, t)$ as in Theorem 2.1, $u(x, t)m(dx) \rightarrow F(dx)$ weakly as $t \rightarrow 0$ on compact sets in R , and, if 0 is accessible,*

$$c(t-) = \lim_{a \rightarrow 0} \int_0^t u(a, s) ds < \infty, \quad 0 < t < T,$$

where $c(t) = \int_{0-}^t c(ds)$. If 0 is entrance,

$$c(t-) = \lim_{a \rightarrow 0} \frac{1}{|s(a)|} \int_0^t u(a, \theta) d\theta < \infty, \quad 0 < t < T.$$

If 0 is a natural boundary or inaccessible exit,

$$\lim_{a \rightarrow 0} \int_0^t \left[u(a, s) - \int_R p(s, a, y)F(dy) \right] ds = 0.$$

In particular, if $u(x, t)$ is a non-negative solution of (1.3) and 0 is entrance,

$$\int_0^t u(a, \theta) d\theta = o(|s(a)|) \quad \text{as } a \rightarrow 0, \quad 0 < t < T.$$

For $u(x, t)$ as in Theorem 2.2, the bound is $o(h_-(a))$.

Corollary 1. *A non-negative weak solution $u(x, t)$ is bounded in $R \times (0, T)$ iff there are no contributions from entrance boundaries, and the remaining boundary measures $c(ds)$, $\tilde{c}(ds)$, and $F(dx)$ are absolutely continuous with bounded derivatives with respect to Lebesgue measure and $m(dx)$ respectively.*

An examination of the proofs in §4–10 also yields

Corollary 2. *An arbitrary real weak solution $u(x, t)$ of (1.3) in $R \times (0, T)$ has the representation (2.1) (where now $c(ds)$, $\tilde{c}(ds)$, and $F(dx)$ are signed Borel measures) if and only if $|u(x, t)| \leq u_0(x, t)$, where $u_0(x, t)$ is a positive weak solution of (1.3). Necessary and sufficient conditions for this are*

$$\int_a^b |u(y, t)| m(dy) = o(1) \quad \text{as } t \rightarrow 0, \quad 0 < a < b < 1,$$

$$\int_0^t |u(x, \theta)| d\theta = o(1) \quad \text{at all accessible boundaries, } 0 < t < T,$$

$$\int_0^t |u(x, \theta)| d\theta = o(|s(x)|) \quad \text{at all entrance boundaries, } 0 < t < T,$$

and if $u(y, t_n)m(dy) \rightarrow F(dy)$ locally weakly in R for some sequence $t_n \rightarrow 0$,

$$\int p(\cdot, x, y) |F(dy)| < \infty, \quad \text{all } (x, t) \in R \times (0, T),$$

$$\int_0^t \left| u(x, \theta) - \int p(\theta, x, y) F(dy) \right| d\theta \rightarrow 0$$

at all natural and I.E. boundaries, $0 < t < T$.

Examples. (1) Let $\mathcal{L}_1 u = u_{xx} - (12/x^2)u$ in $(0, \infty)$. Then $h_-(x) = 1/x^3$, $h_+(x) = x^4$, $m(dx) = dx$, and $K = 7$. Hence (2.5) converges at zero, and $q_0(t, x) = \lim_{a \rightarrow 0} p(t, a, x)/a^4$. In particular $\int_0^t u(x, s) ds = o(1/x^3)$ as $x \rightarrow 0$ for a general positive solution of (2.3).

(2) Let

$$(2.7) \quad \mathcal{L}_1 u = u_{xx} - x^a u \quad \text{in } (0, \infty).$$

Then ∞ is a natural boundary for $\mathcal{L}$ ($\mathcal{L}u = u_{xx}$) but (2.5) converges at ∞ iff $a > 2$. Thus, if $a > 2$, the general positive solution of $u_t = \mathcal{L}_1 u$ in $(0, \infty) \times (0, T)$ has a boundary term at ∞ . Incidentally $\mathcal{L}_1$ is of limit point type at ∞ for all real a in the sense of Weyl.

2.1. Some historical remarks. The first results of this type were due to Widder [23, 24] and Hartman and Wintner [11], who obtained Theorem 2.1 for positive solutions of $u_t = u_{xx}$ for $R = (-\infty, \infty)$, $(0, \infty)$ (Widder) and $R = (a, b)$ for finite a, b . For this equation, the boundaries $\pm\infty$ are natural and finite boundaries are accessible. Later authors ([1, 2, 10, 21, 26, 27]) extended Widder's result to (1.1) with various conditions on $a(x)$, $b(x)$ and $c(x)$ (often in higher dimensions also) but always under conditions that exclude boundary terms at $\pm\infty$. In a different vein is a recent paper of Robbins and Siegmund, [18], who use probabilistic arguments to find all positive solutions of

$$(2.8) \quad \frac{\partial}{\partial t} u = u_{xx} + \frac{2}{x} u_x \quad \text{in } (0, \infty) \times (0, T).$$

Here ∞ is an I.E., but now 0 is an entrance boundary. This appears to be the first example of a Fatou representation (2.1) with a term for an inaccessible boundary, although the possibility is suggested by the work of Feller and Hille. Robbins and Siegmund were interested in positive solutions of (2.8) which are constant along certain curves $x = x(t)$; these can be interpreted as the probability that three-dimensional Brownian motion ever escapes (or alternately is overtaken by) an expanding sphere of radius $r = x(t)$. The unbounded term in (2.1) yields a law of iterated logarithm type for the minimum modulus of three-dimensional Brownian motion. By Theorem 2.1, these arguments can be extended to more general diffusion processes.

The comparison of Widder's results with the general solution of (2.8) led to the conjecture of Theorem 2.1. In the special case $u(x, t) = e^{\lambda t}g(x)$ (where $\mathcal{L}g = \lambda g$, $\lambda > 0$) this was proven by Tz. L. Lai [14] by estimating $g(x)$ from properties of the solutions of $\mathcal{L}^n h = 0$. He also obtained representation theorems of the form (1.2) for solutions $u(x, t)$ under various growth conditions at the boundaries.

3. A probabilistic interpretation. In Theorem (2.1), it is easy to interpret the terms involving $F(dy)$ and any accessible boundaries. For example, assume 0 is an accessible boundary, and $c(ds) = h(s)ds$, $F(dy) = f(y)m(dy)$. Now $\text{Prob}(\tau_0 \leq dt \mid r_0 = x) = q_0(t, x)dt$ (see Theorem 7.3 in §7) and by definition $p(t, x, y)m(dy) = \text{Prob}(r_t \in dy \mid r_0 = x)$. Hence

$$(3.1) \quad \int_0^t q_0(t-s, x)c(ds) + \int_R p(t, x, y)F(dy) \\ = E_x[h(t - \tau_0)\chi_{\{\tau_0 \leq t\}}] + E_x[f(r_t)\chi_{\{\tau_0 > t, r_t > t\}}]$$

where " E_x " refers to the probabilistic expectation for the process r_t beginning at x , χ is the characteristic function, and $\tau_a = \inf\{t : r_t = a\}$ ($\tau_a = \infty$ if $r_t \neq a$ for all t).

More generally, one just replaces h and f by measures. This is very similar to the Kakutani representation for harmonic functions of the Laplacian in terms of Brownian motion (see Doob [6]), and is a standard way of representing solutions of initial-boundary value problems for second order parabolic equations.

However, the terms from entrance boundaries in (2.1) cannot be interpreted as in (3.1), since an entrance boundary is by (our) definition inaccessible. There is another approach, though, which does predict the entrance boundary terms. In general, let r_t be a Markov process in a state space X with transition probabilities $P(t, x, A)$, and let $n(t)$ be a Poisson process with rate $c(dt)$. (See e.g. Doob [5, p. 398+].) Let $\{\beta_n\}$ be the points of increase of $n(t)$ (with repetitions according to multiplicity), and, at each time β_n , begin a copy $r_n(t)$ of the process $r(t)$ at some fixed point $x_0 \in X$. Here the processes $r_n(t)$ are independent of one another and of $n(t)$. Finally, set

$$(3.2) \quad N(A, t) = \text{card}\{n : \beta_n \leq t, r_n(t - \beta_n) \in A\}.$$

Thus $N(A, t)$ is the number of processes r_n in the set $A \subseteq X$ at a time $t > 0$. It turns out that $N(A, t)$ is itself a Poisson random variable for each A and $t > 0$, and has expectation

$$(3.3) \quad E[N(A, t)] = \int_0^t P(t-s, x_0, A)c(ds).$$

Deferring (two) proofs of this somewhat well-known fact to §3.1, suppose 0 is an entrance boundary for $R = (0, 1)$, and apply this construction with $X = [0, 1)$ and $x_0 = 0$. By the definition of entrance boundary, each process $r_n(t)$ at time β_n exists immediately away from 0 into the interior of R and never returns, and the “heat distribution”

$$E[N(A, t)] = \int_0^t P(t-s, 0, A)c(ds) = \int_A \int_0^t p(t-s, 0, y)c(ds)m(dy)$$

has a “temperature” with respect to $m(dx)$ given by

$$(3.4) \quad u(x, t) = \int_0^t p(t-s, 0, x)c(ds).$$

Since $p(t, x, y) = p(t, y, x)$, it seems reasonable that this “temperature” $u(x, t)$ is a solution of $u_t = \mathcal{L}u$, and in fact can be proven to be a weak solution. The result (Theorem 2.3) that $\int_0^t u(x, s) ds \rightarrow \infty$ as $x \rightarrow 0$ whenever $c(t) > 0$ can be viewed in terms of the fact that $m(dx)$ becomes very sparse near an entrance boundary.

The term in (2.1) involving $F(dy)$ can also be given a “Poisson random field” interpretation. For, if $r_n(t)$ are independent copies of a Markov process $r(t)$ in a state space X , begun at points $\{r_n(0)\}$ which are Poisson distributed in X with density $F(dy)$, then for each $t > 0$

$$M(A, t) = \text{card } \{n : r_n(t) \in A\}$$

is also Poisson, and has mean $Q(A, t) = E[M(A, t)] = \int P(t-s, y, A)F(dy)$. This version of (3.3) is sometimes called Doob’s theorem; here (using $p(t, x, y) = p(t, y, x)$ again) $Q(A, t) = \int_A u(x, t)m(dx)$ for $u(x, t)$ as in (1.2). Similarly, if 0 is a regular boundary (i.e. iff $r(t)$ can be extended beyond τ_0 in $[0, 1)$ as a process with continuous paths in such a way to terminate only at τ_1), this procedure also makes sense, where in (3.4) we take the fundamental solution $P_r(t, x, y)$ of (1.3) in R with reflecting boundary condition at 0 (i.e. $(d/ds) p_r(t, 0, y) = 0$). However, if 0 is a pure exit boundary (i.e. accessible but with $m([0, \frac{1}{2}]) = \infty$), then it is impossible for the process $r(t)$ to leave 0 continuously, and a Poisson random field-interpretation of the corresponding boundary term in (2.1) is not possible. Thus there does not appear to be a interpretation of (2.1) from either the Kakutani or P.R.F. point of view which makes sense in all cases, unless one is willing to view a pure-exit boundary in terms of the extraction of negative heat from R .

Note that our heuristic verification of (3.4) as a solution of $u_t = \mathfrak{L}u$ depends on the symmetry of $p(t, x, y)$ in x and y . For example, if $\mathfrak{L}$ were given by (1.1) with $c = 0$, and $p(t, x, y)$ was the fundamental solution of (1.1) with respect to Lebesgue measure in R , then $u(x, t)$ would be a solution not of $u_t = \mathfrak{L}u$ but of the Fokker-Planck equation

$$\frac{\partial}{\partial t} u = (a(x)u)_{xx} - (b(x)u)_x.$$

Hence this method of producing solutions of (1.3) or (1.4) from an entrance boundary depends on the existence of an invariant measure for the process. This, in turn, seems to depend essentially on the fact $P_a(\tau_b < \infty) > 0$ for all a, b , in R .

3.1. Here we give two proofs of the statements about $N(A, t)$. First, we remark that for fixed $t > 0$, $N(A, t)$ defined by (3.2) is a random measure in A . Since the characteristic function of a random variable determines its distribution, the statements about $N(A, t)$ in §3 will follow from the identity

$$(3.5) \quad w(t) = E\left(\exp\left(\int h(x)N(dx, t)\right)\right) = \exp\left(\int (e^{h(x)} - 1)c_t(dx)\right)$$

for all bounded Borel functions $h(x)$ on X , where $c_t(A)$ is the measure defined by the right-hand side of (3.3). Indeed (3.5) not only implies that $N(A, t)$ is a Poisson random variable with mean $c_t(A)$ for fixed A , but also that $N(A, t)$ and $N(B, t)$ are independent for disjoint sets A and B . By the definition of $N(A, t)$,

$$\begin{aligned} w(t) &= E\left[\prod_{\{\beta_n \leq t\}} \exp(h(r_n(t - \beta_n)))\right] \\ &= \sum_0^\infty E(\chi_{\{\beta_n \leq t < \beta_{n+1}\}} \prod_1^n \exp(h(r_k(t - \beta_k)))) \\ &= \sum_0^\infty \frac{c(t)^n}{n!} \exp(-c(t)) E\left(\prod_1^n \exp(h(r_k(t - \beta_k))) | \beta_n \leq t < \beta_{n+1}\right). \end{aligned}$$

Now, given $\beta_n \leq t < \beta_{n+1}$, *i.e.* that $n(t) = n$, it is a standard result (see [5, p398]) that the n jumps of $n(t)$ are independently and identically distributed in $[0, t]$ with distribution $c(ds)/c(t)$. Hence

$$\begin{aligned} w(t) &= \sum \frac{1}{n!} \exp(-c(t)) \left[\int_0^t E(\exp(h(r(t-s)))c(ds) \right]^n \\ &= e^{-c(t)} \exp\left(\int_0^t \int \exp(h(y))P(t-s, x_0, dy)c(ds)\right) \end{aligned}$$

which is (3.5).

If $c(dt) = c dt$, there is also a proof of (3.5) using branching diffusion processes (see [20]). First, extend the process $r(t)$ to $D = X \cup \{e\}$, where $e \notin X$ and $P_e(r(t) \equiv e) = 1$. We now consider the branching process in D which results from $r(t)$

branching according to the following rules. A process $r(t)$ in X never branches; if $r(t) = e$, it branches in time dt with probability $c dt$. When a process at e does branch, it is replaced by two daughter processes, one at e and one at x_0 in X . All daughter processes evolve independently of one another and of pre-existing processes.

Then, in the terminology of [20], the branching rate is of "Kac type" with kernel $V(x)$ given by

$$V(x) = 0 \quad \text{in } X, \quad V(e) = c.$$

Assume that a function $f(x)$ is in the domain of the infinitesimal generator A of the process $r(t)$ in D and $|f(x)| \leq d < 1$. Let

$$u(x, t) = E_x[\hat{f}(x_t)] = E_x\left[\prod_{i=1}^{N(t)} f(r^{(n)}(t))\right],$$

where $\{r^{(n)}\}$ are the processes extant at time t , and " E_x " refers to the expectation beginning with one process at time zero at the point $x \in D$. Then, by the Corollary to Theorem 2.3 in [20], $u(x, t)$ is a solution of the non-linear equation

$$(3.6) \quad \frac{\partial}{\partial t} u = Bu; \quad B = A + V(L - 1); \quad Lf(e) \equiv f(e)f(x_0).$$

Since e is an absorbing point for the process in D , $Ag(e) = 0$ for every function $g(x)$ in the domain of A , and the system (3.6) reduces to

$$(3.7) \quad \begin{aligned} \frac{\partial}{\partial t} u(x, t) &= Au(x, t) \quad \text{for } x \neq X, \\ \frac{\partial}{\partial t} u(e, t) &= V(L - I)u(e, t) = cu(e, t)(u(x_0, t) - 1). \end{aligned}$$

Integrating the second equation in (3.7),

$$(3.8) \quad \begin{aligned} u(e, t) &= u(e, 0) \exp \left[c \int_0^t (u(x_0, s) - 1) ds \right] \\ &= f(e) \exp \left(c \int_0^t (u(x_0, s) - 1) ds \right). \end{aligned}$$

Now $u(x, t) = E_x(f(r(t))) = \int f(y)P(t, x, dy)$ by construction, and hence

$$\begin{aligned} u(e, t) &= f(e)E\left[\prod_{\{\beta_n \leq t\}} f(r_n(t - \beta_n))\right] \quad (\text{by definition}) \\ &= f(e) \exp \left(c \int (f(y) - 1)c_t(dy) \right) \end{aligned}$$

which is (3.5) for $h(x) = \log f(x)$. All that remains is to remark that the domain of A is dense in the set of all bounded Borel functions in the topology of uniformly bounded pointwise convergence.

4. The initial boundary. Let $u(x, t)$ be a non-negative weak solution of equation (1.3) in $R \times (0, T)$. Then, according to Theorem A2 in the Appendix (and Corollary A1)

$$(4.1) \quad \begin{aligned} u(x, t) = & \int_0^{t-c} u(a, t - \theta) P_x[\tau_a \leq \theta, \tau_a < \tau_b] \\ & + \int_0^{t-c} u(b, t - \theta) P_x[\tau_b \leq \theta, \tau_b < \tau_a] \\ & + \int_a^b p_{ab}(t - c, x, y) u(y, c) m(dy) \end{aligned}$$

identically for $0 < a < x < b < 1$ and $0 < c < t < T$. Here $p_{ab}(t, x, y)$ is the fundamental solution of (1.3) in (a, b) with zero boundary conditions, $\tau_a = \min \{t : r_t = a\}$ for r_t as in §1 (if $r_t \neq a$ for all t , e.g. if $\tau_0 < \tau_a$, then $\tau_a = \infty$), and “ P_x ” means probability given $r_0 = x$. The purpose of this section is to prove

Theorem 4.1. *There exists a measure $F(dy) \geq 0$ on R such that*

$$(4.2) \quad v(x, t) = u(x, t) - \int p(t, x, y) F(dy) \geq 0$$

and for all $t \in (0, T)$, $0 < a < x < b < 1$,

$$(4.3) \quad \begin{aligned} \int_0^t v(x, \theta) d\theta = & \int_0^{t-} v(a, \theta) P_x[\tau_a \leq t - \theta, \tau_a < \tau_b] d\theta \\ & + \int_0^{t-} v(b, \theta) P_x[\tau_b \leq t - \theta, \tau_b < \tau_a] d\theta. \end{aligned}$$

Proof. Since $u(x, t) \geq 0$,

$$(4.4) \quad u(x, t + c) \geq \int_a^b p_{ab}(t, x, y) u(y, c) m(dy)$$

by (4.1), for $0 < t + c < T$. Now $p_{ab}(t, x, y) > 0$ for $t > 0$, $x, y \in (a, b)$, ([13, Ch. 4]) and $p_{ab}(t, x, y) \in C[(0, \infty) \times (a, b)^2]$. Thus

$$(4.5) \quad \int_a^b u(y, c) m(dy) \leq C_{ab}, \quad 0 < c < c_0, \quad a, b \in R.$$

Hence by Helly's Theorem, there exists a sequence $c_n \downarrow 0$ and a Borel measure $F(dy)$ on R such that

$$(4.6) \quad u(y, c_n) m(dy) \rightarrow F(dy)$$

weakly on compact sets in R . Also, for $t > 0$,

$$p_{ab}(t - \epsilon, x, y) \rightarrow p_{ab}(t, x, y) \quad \text{as } \epsilon \rightarrow 0$$

uniformly in $[a, b] \times [a, b]$ (see the Appendix); hence by (4.1) and (4.6)

$$(4.7) \quad u(x, t) = \int_a^b p_{ab}(t, x, y)F(dy) + \int_0^{t-} u(a, t - \theta)P_x[\tau_a \leq d\theta, \tau_a < \tau_b] \\ + \int_0^{t-} u(b, t - \theta)P_x[\tau_b \leq d\theta, \tau_b < \tau_a].$$

In particular

$$\int_a^b p_{ab}(t, x, y)F(dy) \leq u(x, t)$$

for all $a, b \in R$. However $p_{ab}(t, x, y) \uparrow p(t, x, y)$ as $(a, b) \uparrow R$. (Proof.

$$\int_A p_{ab}(t, x, y)m(dy) = P_x(r_t \in A, \tau_a \wedge \tau_b > t) \uparrow P_x(r_t \in A, \tau_0 \wedge \tau_1 > t) \\ = \int_A p(t, x, y)m(dy), \text{ and } p(t, x, y) \in C[(0, \infty) \times (a, b)^2].$$

Hence

$$(4.8) \quad \int p(t, x, y)F(dy) \leq u(x, t), \quad \text{all}(x, t).$$

For fixed $z \in R$, $u(x, t) = p(t, x, z)$ is a “classical” solution of $u_t = \mathcal{L}u$ ([13]) and hence a weak solution (Theorem A1). Thus by (4.6)

$$(4.9) \quad p(t, x, z) = p_{ab}(t, x, z)\chi_{(a,b)}(z) + \int_0^{t-} p(t - \theta, a, z)P_x[\tau_a \leq d\theta, \tau_a < \tau_b] \\ + \int_0^{t-} p(t - \theta, b, z)P_x[\tau_b \leq d\theta, \tau_b < \tau_a].$$

Evidently $p(t, x, z) \in L^1(R, dF)$ for all (t, x) by (4.8). If we integrate the above equation with respect to $F(dz)$ and subtract it from (4.7), we obtain

$$(4.10) \quad v(x, t) = \int_0^{t-} v(a, t - \theta)P_x[\tau_a \leq d\theta, \tau_a < \tau_b] \\ + \int_0^{t-} v(b, t - \theta)P_x[\tau_b \leq d\theta, \tau_b < \tau_a].$$

Finally, we remark (see Lemma A2 in the Appendix)

$$\int_0^t \int_0^s v(a, s - \theta)P_x[\tau_a \leq d\theta, \tau_a < \tau_b] ds \\ = \int_0^t v(a, t - \theta)P_x[\tau_a \leq t - \theta, \tau_a < \tau_b] d\theta$$

and (4.3) (*i.e.*, Theorem 4.1) results from integration in (4.10).

Corollary 4.1. $u(x, t)m(dx) \rightarrow F(dx)$ as $t \rightarrow 0$ weakly on compact sets in R .

Proof. By (4.5), the measures $\{u(x, t)m(dx) : t \leq t_0 < T\}$ are weakly compact on compact sets in R ; all that remains is to show that the limiting measure $F(dy)$ is unique. However, this follows from (4.7) since

$$\int_a^b g(x) \int_a^b p_{ab}(t, x, y) F(dy) m(dx) \rightarrow \int_a^b g(y) F(dy) \text{ as } t \rightarrow 0$$

for any continuous function $g(x)$ of compact support in (a, b) .

5. Accessible boundaries.

Theorem 5.1. *Assume 0 is an accessible boundary, and $u(x, t)$, $v(x, t)$ are as in §4. Then there exists a measure $c(ds) \geq 0$ on $[0, T)$ such that*

$$(5.1) \quad \int_0^t v(x, \theta) d\theta = \int_0^t P_x[\tau_0 \leq t - \theta, \tau_0 < \tau_b] c(d\theta) \\ + \int_0^t v(b, \theta) P_x[\tau_b \leq t - \theta, \tau_b < \tau_0] d\theta$$

for all $t \in (0, T)$ and $0 < x < b$.

The proof will depend on

Lemma 5.1. *If x is sufficiently close to an accessible boundary 0,*

$$P_x[\tau_0 \leq \delta, \tau_0 < \tau_b] = \lim_{a \rightarrow 0} P_x[\tau_a \leq \delta, \tau_a < \tau_b] = \eta > 0$$

for all $\delta > 0$. Moreover, for fixed $\delta > 0$, the constant $\eta = \eta(x) \rightarrow 1$ as $x \rightarrow 0$.

Proof. For any $\lambda > 0$ and $a < x < b$

$$(5.2) \quad P_x[\tau_a \leq \delta, \tau_a < \tau_b] \geq E_x[e^{-\lambda\tau_a} \chi_{\{\tau_a < \tau_b\}}] - e^{-\lambda\delta}$$

Now the function

$$g(x) = E_x[e^{-\lambda\tau_a} \chi_{\{\tau_a < \tau_b\}}] \in \mathfrak{D}(\mathfrak{L}) \quad \text{in } (a, b)$$

and satisfies $\mathfrak{L}g = \lambda g$. Also, the space of all solutions of $\mathfrak{L}u = \lambda u$ in (a, b) is spanned by two particular solutions $\varphi_+(x, \lambda) \uparrow$, $\varphi_-(x, \lambda) \downarrow$ in R which, if 0 is accessible, can be assumed to satisfy (see [3, 13])

$$\varphi_+(0+, \lambda) = 0, \quad \varphi_-(0+, \lambda) < \infty.$$

By construction $g(a+) = 1$, $g(b-) = 0$, so

$$g(x) = \left[\varphi_-(x, \lambda) - \frac{\varphi_-(b, \lambda)}{\varphi_+(b, \lambda)} \varphi_+(x, \lambda) \right] / \left[\varphi_-(a, \lambda) - \frac{\varphi_-(b, \lambda)}{\varphi_+(b, \lambda)} \varphi_+(a, \lambda) \right]$$

from which follows

$$\lim_{x \rightarrow 0} \lim_{a \rightarrow 0} E_x[e^{-\lambda\tau_a} \chi_{\{\tau_a < \tau_b\}}] = 1$$

and by (5.2)

$$\liminf_{x \rightarrow 0} \lim_{a \rightarrow 0} P_x[\tau_a \leq \delta, \tau_a < \tau_b] \geq 1 - e^{-\lambda \delta}$$

for all $\lambda > 0$.

Remark. The argument can be extended to prove $p_x[\tau_0 \leq \delta, \tau_0 < \tau_b] > 0$ for all $x \in (0, b)$, but this will not be needed.

Proof of Theorem 5.1. The proof is now more or less standard from (4.3). Fix x close to 0. Then

$$\begin{aligned} \int_0^{t-\delta} v(a, \theta) P_x[\tau_a \leq \delta, \tau_a < \tau_b] d\theta \\ \leq \int_0^t v(a, \theta) P_x[\tau_a \leq t - \theta, \tau_a < \tau_b] d\theta \leq \int_0^t v(x, \theta) d\theta \end{aligned}$$

and by Lemma 5.1

$$(5.3) \quad \int_0^t v(a, \theta) d\theta \leq \frac{2}{\eta} \int_0^{t+\delta} v(x, \theta) d\theta < \infty$$

for $a \leq a_0 < x$ and $t + \delta < T$. Hence by Helly's theorem there exists a sequence $a_n \rightarrow 0$ and a measure $c(ds) \geq 0$ such that

$$v(a_n, \theta) d\theta \rightarrow c(d\theta)$$

weakly on compact sets in $[0, T]$. On the other hand

$$\begin{aligned} P_x[\tau_0 \leq t, \tau_0 < \tau_b] &\leq P_x[\tau_a \leq t, \tau_a < \tau_b] \\ &\leq P_x[\tau_0 \leq t + \epsilon, \tau_0 < \tau_b] + P_x[\tau_a \leq t, E] \end{aligned}$$

where E is the event that the process r_t , starting at time τ_a , does not hit the boundary 0 before time $\tau_a + \epsilon$, before hitting the point b . Hence by the strong Markov property (estimating $P_x(\tau_a \leq t) \leq 1$)

$$\begin{aligned} P_x[\tau_a \leq t, \tau_a < \tau_b] - P_x[\tau_0 \leq t, \tau_0 < \tau_b] \\ \leq P_x[t < \tau_0 \leq t + \epsilon, \tau_0 < \tau_b] + 1 - P_x[\tau_0 \leq \epsilon, \tau_0 < \tau_b]. \end{aligned}$$

However

$$\begin{aligned} (5.4) \quad P_x[t < \tau_0 \leq t + \epsilon, \tau_0 < \tau_b] &= E_x[\chi_{\{\tau_a \wedge \tau_b > t\}} P_{r_t}(\tau_0 \leq \epsilon, \tau_0 < \tau_b)] \\ &= \int_0^b p_{0b}(t, x, y) P_y[\tau_0 \leq \epsilon, \tau_0 < \tau_b] m(dy) \end{aligned}$$

where $p_{0b}(t, x, y)$ is the fundamental solution of $u_t = \mathcal{L}u$ in the interval $(0, b)$ with zero boundary conditions at the endpoints. Thus (see the Appendix) the expression in (5.4) is a continuous function of $t > 0$. On the other hand, if x is fixed, $P_x(\tau_a \leq \delta)$ is uniformly small for small a and δ . Hence by Lemma 5.1 and Dini's theorem

$$(5.5) \quad P_x[\tau_a \leq t, \tau_a < \tau_b] \rightarrow P_x[\tau_0 \leq t, \tau_0 < \tau_b]$$

uniformly for bounded t as $a \rightarrow 0$. Now, given (4.3), the formula (5.1) follows from weak convergence and (5.5) in the first term and monotone convergence in the second.

Corollary 5.1. For $v(x, t)$ as in (5.1),

$$(5.6) \quad \lim_{x \rightarrow 0} \int_0^t v(x, s) ds = c(t-) = \int_{0-}^{t-} c(ds), \quad 0 < t < T.$$

Proof. Holding b fixed, let $x \rightarrow 0$ in (5.1).

Corollary 5.2. $\int_0^t u(x, s) ds < \infty$, all $(x, t) \in R \times (0, T)$.

Proof. By (4.4),

$$\int_c^{t+c} u(x, s) ds \geq \int_a^b p_{ab}(c, x, y) \int_0^t u(y, s) dsm(dy)$$

and $\int_0^t u(y, s) ds < \infty$ for a.e. y . On the other hand

$$(5.7) \quad \int_0^\infty p_{ab}(t, x, y) dt = g_{ab}(x, y) \leq s(b) - s(a) < \infty$$

(see the Appendix). Hence the Corollary follows from (4.7) and Lemma A2.

Lemma 5.2. If 0 is accessible and $F(dy)$ is as above,

$$(5.8) \quad \lim_{x \rightarrow 0} \int_0^t \int_R p(s, x, y) F(dy) ds = 0, \quad 0 < t < T.$$

Proof. Let $q(x, t) = \int p(t, x, y) F(dy) \leq u(x, t)$. Then by (4.9)

$$(5.9) \quad q(x, t) = \int_0^t q(a, t-s) P_x(\tau_a \leq s, \tau_a < \tau_b) \\ + \int_0^t q(b, t-s) P_x(\tau_b \leq s, \tau_b < \tau_a) + \int_a^b p_{ab}(t, x, y) F(dy).$$

Since $p_{ab}(t, x, y) \uparrow p(t, x, y)$ for all (t, x, y) as $(a, b) \uparrow R$,

$$\int_0^t \int_a^b p_{ab}(s, x, y) F(dy) ds \uparrow \int_0^t q(x, s) ds < \infty$$

by Corollary 5.2. Hence by (5.9) and Lemma A2

$$(5.10) \quad \int_0^t \int_0^{t-s} q(a, \theta) d\theta P_x(\tau_a \leq s, \tau_a < \tau_b) \rightarrow 0$$

as $(a, b) \rightarrow R$. However by arguing as in Lemma 5.1

$$P_x(\tau_a \leq \delta, \tau_a < \tau_b) \rightarrow P_x(\tau_0 \leq \delta, \tau_0 < \tau_1) > 0$$

as $(a, b) \rightarrow R$ for $\delta > 0$ and x close to 0. Hence by (5.10)

$$\int_0^t q(x, s) ds \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Theorem 5.2. For 0 accessible and $u(x, t)$ as in §1,

$$\lim_{x \rightarrow 0} \int_0^t u(x, s) ds = c(t-) = \int_{0-}^{t-} c(ds), \quad 0 < t < T.$$

Proof. By the definition of $v(x, t)$ (see (4.2)), (5, 6), and (5.8).

Remark. If the boundaries 0, 1 are accessible or entrance, the measures $F(dy)$ which enter can be explicitly characterized. In particular (see Theorem 10.2 in §10) if 0 is accessible

$$\int_0^a (s(y) - s(0))F(dy) < \infty, \quad \text{all } a \in R.$$

6. Inaccessible exits. This section is devoted to proving

Theorem 6.1. Assume 1 is an inaccessible exit boundary (see §2), and let $u(x, t)$, $v(x, t)$ be as in §4. Then

$$\lim_{x \rightarrow 1} \int_0^t v(x, s) ds = 0, \quad 0 < t < T.$$

The proof will be broken up into a series of lemmas; the first will be used to substitute for explicit knowledge of the Green's function at the lateral boundaries. Equation (6.1) is a kind of reflection principle, in that it states that the probability of r_t going out and then returning is comparable to its staying out.

Lemma 6.1. For every $M > 0$ and $b_0 \in R$, there exist numbers ℓ, A, B , such that $b_0 < \ell < A < B < 1$ and, for all $x \in (0, b_0)$,

$$(6.1) \quad P_x(\tau_B \leq t, \tau_B < \tau_0) \leq \frac{2M}{M-t} P_x(r_t > A), \quad 0 < t < M,$$

$$(6.2) \quad P_A(\tau_B \leq t_0, \tau_B < \tau_1) \geq \frac{1}{2} - M/t_0, \quad t_0 > 2M.$$

Proof. Since $s(1) < \infty$ and the form of equation (1.3) is invariant under homeomorphisms, it is sufficient to assume $s(x) \equiv x$ and $R = (-\infty, 1)$ or $R = (0, 1)$. Since 1 is an I.E.,

$$(6.3) \quad \int_b^1 (1-y)m(dy) = \infty, \quad b \in R.$$

In general, if $a < x < b$ ([3, 13])

$$\begin{aligned} \int_a^b g_{ab}(x, y)m(dy) &= \int_0^\infty \int_a^b p_{ab}(t, x, y)m(dy) dt = \int_0^\infty P_x(\tau_a \wedge \tau_b > t) dt \\ &= E_x(\tau_a \wedge \tau_b) \\ &= \frac{b-x}{b-a} \int_a^x (y-a)m(dy) + \frac{x-a}{b-a} \int_x^b (b-y)m(dy) \end{aligned}$$

and $E_x(\tau_a \wedge \tau_b)$ is jointly continuous in (a, x, b) . Hence for $\delta > 0$

$$\varphi_\delta(x) = E_x(\tau_{x-\delta} \wedge \tau_{x+\delta}) \in C[(\delta, 1 - \delta)]$$

and satisfies $\lim_{x \rightarrow 1} \varphi_\delta(x - \delta) = \infty$ by (6.3). Thus

$$(6.4) \quad B = B(\delta) = \inf \{x : x \geq \tfrac{1}{2}, \varphi_\delta(x) = M\}$$

is attained for sufficiently small δ . Since $\varphi_\delta(x) \downarrow 0$ as $\delta \downarrow 0$ (and $\varphi_\delta(x)$ is continuous)

$$\lim_{\delta \rightarrow 0} B(\delta) = 1$$

and $B(\delta)$, $B(\delta) - 2\delta$ can be made arbitrarily close to 1. Now define numbers ℓ, A, B, C by

$$(\ell, A, B, C) = (B - 2\delta, B - \delta, B, B + \delta).$$

Clearly $C < 1$ since $\varphi_\delta(B) < \infty$; also, by (6.4)

$$E_A(\tau_B \wedge \tau_t) = \varphi_\delta(B - \delta) < M.$$

In particular, since $P_x(\tau_a > \tau_b) = [s(x) - s(a)]/[s(b) - s(a)]$ ([3, 13])

$$\begin{aligned} 1 - P_A(\tau_B \leq t, \tau_B < \tau_t) &= P_A(\tau_t < \tau_B) + P_A(\tau_B > t, \tau_B < \tau_t) \\ &\leq \tfrac{1}{2} + (1/t)E_A(\tau_B \wedge \tau_t) \leq \tfrac{1}{2} + M/t \end{aligned}$$

which implies (6.2). For (6.1), note that by the strong Markov property

$$\begin{aligned} P_x(\tau_B \leq t) &= P_x(\tau_B \leq t, r_t > A) + P_x(\tau_B \leq t, r_t \leq A) \\ &\leq P_x(r_t > A) + \int_0^t P_B(r_{t-s} \leq A)P_x(\tau_B \in ds) \\ &\leq P_x(r_t > A) + [\sup_{0 \leq s \leq t} P_B(r_s \leq A)]P_x(\tau_B \leq t). \end{aligned}$$

Now

$$\sup_{0 \leq s \leq t} P_B(r_s \leq A) \leq P_B(\tau_A \wedge \tau_C \leq t) = \eta(t) \leq 1$$

and we conclude

$$(6.5) \quad P_x(\tau_B \leq t) \leq \frac{1}{1 - \eta(t)} P_x(r_t > A).$$

It only remains to estimate $\eta(t)$. First we remark

$$g_{AC}(y, z) \leq 2g_{AC}(B, z), \quad y, x \in (A, C)$$

which follows from the definition of g_{AC} (see the Appendix). Hence

$$\begin{aligned} E_y(\tau_A \wedge \tau_C) &= \int_A^C g_{AC}(y, z)m(dz) \\ &\leq 2 \int_A^C g_{AC}(B, z)m(dz) = 2E_B(\tau_A \wedge \tau_C) = 2M. \end{aligned}$$

By equation (5.7) and the identity

$$p_{ab}(t + s, x, z) = \int_a^b p_{ab}(s, x, y)p_{ab}(t, y, z)m(dy)$$

we have

$$\begin{aligned} M &= \int_0^t \int_A p_{Ac}(s, B, y)m(dy) ds + \int_A^C \int_A p_{Ac}(t, B, z)g_{Ac}(z, y)m(dz)m(dy) \\ &\leq t + 2M \int_A^C p_{Ac}(t, B, z)m(dz) = t + 2MP_B(\tau_A \wedge \tau_C > t). \end{aligned}$$

Hence $P_B[\tau_A \wedge \tau_C > t] \geq (M - t)/2M$ and

$$\eta(t) = P_B(\tau_A \wedge \tau_C \leq t) \leq 1 - (M - t)/2M$$

which in (6.5) implies (6.1).

Lemma 6.2. For M, ℓ, A, B, C as before

$$(6.6) \quad \int_0^t v(B, \theta) d\theta \leq (1/2 - M/t_0)^{-1} \int_0^{t+t_0} v(A, \theta) d\theta$$

for $t_0 > 2M, 0 < t + t_0 < T$.

Proof. Let $s = t + t_0$. Then by (4.3) and (6.2)

$$\begin{aligned} \int_0^s v(A, \theta) d\theta &= \int_0^s v(\ell, \theta)P_A(\tau_\ell \leq s - \theta, \tau_\ell < \tau_B) d\theta \\ &\quad + \int_0^s v(B, \theta)P_A(\tau_B \leq s - \theta, \tau_B < \tau_\ell) d\theta \\ &\geq (1/2 - M/t_0) \int_0^{s-t_0} v(B, \theta) d\theta. \end{aligned}$$

Lemma 6.3.

$$\lim_{x \rightarrow 1} \int_0^t v(x, \theta) d\theta = 0, \quad 0 < t < T/4.$$

Proof. First, given T , choose M arbitrarily large if T is infinite, and set $M = T/4, t_0 = 3T/4$, if T is finite. Also, set

$$(6.7) \quad w(x, t) = \int_0^t v(x, s) ds = \int_0^t \left[u(x, s) - \int p(s, x, y)F(dy) \right] ds.$$

Then by Lemma 6.1, Lemma 6.2, and (6.7), for all $x < b_0, t < M$,

$$\begin{aligned} (6.8) \quad h_B(t) &= \int_0^t w(B, s)P_x(\tau_B \leq t - s) ds \\ &\leq \frac{2M}{M - t} (1/2 - M/t_0)^{-1} \int_0^t w(A, s + t_0)P_x(r_{t-s} > A) ds. \end{aligned}$$

Now by Theorem 4.1 (since $P_x(\cdots) \leq 1$)

$$(6.9) \quad w(A, t) \leq w(x, t) + w(y, t), \quad 0 < x < A < y < 1$$

and

$$\begin{aligned} h_B(t) &\leq \frac{12M}{M-t} \int_0^t w(x, s+t_0) P_x(r_{t-s} > A) ds \\ &\quad + \frac{12M}{M-t} \int_0^t \int_A^1 w(y, s+t_0) p(t-s, x, y) m(dy) ds. \end{aligned}$$

By (4.4) and (6.7)

$$\int_0^t \int_R w(y, s+t_0) p(t-s, x, y) m(dy) ds \leq tw(x, t+t_0) < \infty$$

for $t < T/4$, and by dominated convergence

$$(6.10) \quad \lim_{\delta \rightarrow 0} h_{B(\delta)}(t) = 0, \quad 0 < t < T/4.$$

Now, choose $a \in (0, x)$. By (4.3) and Lemma A2

$$\begin{aligned} \int_0^t w(x, s) ds &= \int_0^t w(a, s) P_x(\tau_a \leq t-s, \tau_a < \tau_B) ds \\ &\quad + \int_0^t w(B, s) P_x(\tau_B \leq t-s, \tau_B < \tau_a) ds \end{aligned}$$

and so by (6.10) ($0 < a < x < 1$)

$$\begin{aligned} (6.11) \quad \int_0^t w(x, s) ds &\equiv \int_0^t w(a, s) P_x(\tau_a \leq t-s) ds \\ &\leq P_x(\tau_a < \infty) \int_0^t w(a, s) ds = \frac{1-x}{1-a} \int_0^t w(a, s) ds \end{aligned}$$

since $P_x(\tau_a < \infty) = \lim_{b \rightarrow 1} P_x(\tau_a < \tau_b)$ for an I.E. boundary. Thus if $0 < t < T/4$, $t + \epsilon < T/4$,

$$\lim_{x \rightarrow 1} w(x, t) \leq \lim_{x \rightarrow 1} (1/\epsilon) \int_t^{t+\epsilon} w(x, s) ds = 0$$

since $w(x, t)$ is non-negative and increasing in t .

Corollary 6.1. $\int_0^t v(x, s) ds = 0(s(1) - s(x))$ as $x \rightarrow 1$, $0 < t < T/4$.

If we consider the weak solution $u(x, t)$ in the subdomain $(a, 1) \times (0, T)$, the function $v(x, t)$ of §4 becomes

$$v_a(x, t) = u(x, t) - \int_a^1 p_{a1}(t, x, y) F(dy)$$

where $p_{a1}(t, x, y)$ is the fundamental solution of $u_t = \mathcal{L}u$ in $(a, 1)$. In particular

$$(6.12) \quad 0 \leq \int_0^t v(x, s) ds \leq \int_0^t v_a(x, s) ds, \quad \text{all } a \in (0, x).$$

Now the boundary a is a regular boundary for $\mathcal{E}$ on $(a, 1)$; hence, by (6.12), it is sufficient to prove Theorem 6.1 under the extra assumption that 0 is a regular boundary. This remark will be needed in the following lemma.

Lemma 6.4.

$$\lim_{x \rightarrow 1} \int_0^t v(x, \theta) d\theta = 0, \quad 0 < t < T/5 + T/5 = 2T/5.$$

Proof. Assume $0 < t < T/5$. Then by (6.7)

$$\begin{aligned} w(x, 2t) &= w(x, t) + \int_t^{2t} \left[u(x, s) - \int p(s, x, y) F(dy) \right] ds \\ &= w(x, t) + \int_0^t \left[u(x, s+t) - \int p(s, x, y) u(y, t) m(dy) \right] ds \\ &\quad + \int_0^t \int_R p(s, x, y) \left[u(y, t) - \int p(t, y, z) F(dz) \right] m(dy) ds. \end{aligned}$$

Given that $u(x, s)$ is a weak solution in $R \times (0, T)$, $u_t(x, s) = u(x, s+t)$ is a weak solution in $R \times (0, T-t)$. The second term in the last expression above is the function $w_t(x, s)$ corresponding to $u_t(x, s)$, and $t < (1/4)(T-t)$ iff $t < T/5$. Hence it is sufficient to prove

$$\lim_{x \rightarrow 1} \int_0^t \int_R p(s, x, y) v(y, t) m(dy) ds = 0, \quad 0 < t < T/4.$$

Now $w(x, 2t) \rightarrow 0$ ($t < T/5$) iff $\int_t^{t+\epsilon} w(x, 2s) ds \rightarrow 0$ ($2\epsilon < T/5 - t$) by monotonicity, and it is sufficient to prove

$$(6.13) \quad \lim_{x \rightarrow 1} \int_0^t \int_R p(s, x, y) w(y, t) m(dy) ds = 0, \quad 0 < t < T/4.$$

By Lemma 6.3, $w(y, t) \rightarrow 0$ as $y \rightarrow 1$, while by Corollary 5.1 in §5

$$\lim_{y \rightarrow 0} w(y, t) = c(t-) < \infty, \quad 0 < t < T,$$

since we are assuming 0 is regular. Hence $w(y, t) \leq c(t-)$, $0 < y < 1$, by (6.9), and

$$\begin{aligned} &\int_0^t \int p(s, x, y) w(y, t) m(dy) ds \\ &\leq \int_0^t [c(t-)P_x(r_s \leq A) + \epsilon P_x(r_s > A)] ds \leq tc(t-)P_x(\tau_A < \infty) + \epsilon t \end{aligned}$$

if $w(y, t) \leq \epsilon$ for $y \geq A$. Since $P_x(\tau_A < \infty) \rightarrow 0$ as $x \rightarrow 1$ for fixed A , (6.13) follows.

Lemma 6.5.

$$\lim_{x \rightarrow 1} \int_0^t v(x, s) ds = 0, \quad 0 < t < T.$$

Proof. By arguing as in Lemma 6.4, we can always increase the range of t for which Theorem 6.1 holds by $2/5$ of the remainder of the interval $(0, T)$. Hence Lemma 6.5 (and a posteriori Theorem 6.1) follows after an infinite number of applications of Lemma 6.4.

7. A duality principle. Given $s(x)$, $m(dx)$ as in §1, assume $s(x) \neq 0$ for all $x \in R$. Then, for $\mathfrak{L}$ as in (1.3), define

$$\mathfrak{L}^* = \frac{1}{dm^*(x)} \frac{1}{ds^*(x)}$$

where

$$(7.1) \quad s^*(x) = -\frac{1}{s(x)}, \quad m^*(dx) = s(x)^2 m(dx).$$

(See the Appendix for the precise definition of $\mathfrak{L}^*$.) I claim

Theorem 7.1. For $\mathfrak{L}$, $\mathfrak{L}^*$ as above

$$(7.2) \quad \mathfrak{L}^* = \frac{1}{s(x)} \mathfrak{L} s(x)$$

in values and in domain. That is, $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$ if and only if $f(x)s(x) \in \mathfrak{D}(\mathfrak{L})$, in which case $\mathfrak{L}^*f(x)$ is given by (7.2).

The form of (7.2) is motivated by the fact that if $u(x, t)$ is a solution of (1.3), then $u^* = u/s$ should be a solution of $(u^*)_t = \mathfrak{L}^*(u^*)$.

It will turn out that $\mathfrak{L}^*$ has different boundary behavior than $\mathfrak{L}$ (see Theorem 7.2 below).

Note that $\mathfrak{L}^*$ depends not only on $\mathfrak{L}$ but also on the particular choice of scale (i.e., within an additive constant) for $\mathfrak{L}$. Using the scale $s^*(x)$ of (7.1), $\mathfrak{L}^{**} = \mathfrak{L}$ by either (7.1) or (7.2).

Proof of Theorem 7.1. (I) As shown in the Appendix, $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$ if and only if

$$(7.3) \quad f(x) = -\int_a^b g_{ab}^*(x, y) \ell(y) m^*(dy) + C_1 s^*(x) + C_2,$$

where $\mathfrak{L}^*f(x) = \ell(x) \in C(R)$, for all closed intervals $[a, b] \subseteq R$ and where C_1, C_2 depend on a, b . Here

$$\begin{aligned}
g_{ab}^*(x, y) &= \frac{(s^*(b) - s^*(y))(s^*(x) - s^*(a))}{s^*(b) - s^*(a)} \quad (a < x < y < b) \\
&= \frac{[1/s(y) - 1/s(b)][1/s(a) - 1/s(x)]}{1/s(a) - 1/s(b)} \\
&= \frac{[s(b) - s(y)][s(x) - s(a)]}{s(x)s(y)[s(b) - s(a)]}
\end{aligned}$$

and

$$(7.4) \quad g_{ab}^*(x, y) = \frac{g_{ab}(x, y)}{s(x)s(y)}.$$

Hence by (7.3), if $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$,

$$\begin{aligned}
s(x)f(x) &= - \int_a^b \frac{g_{ab}(x, y)}{s(y)} \ell(y)s(y)^2 m(dy) + C_1 s(x)s^*(x) + C_2 s(x) \\
&= - \int_a^b g_{ab}(x, y) \ell(y)s(y) m(dy) + C_2 s(x) - C_1
\end{aligned}$$

and (as before) $s(x)f(x) \in \mathfrak{D}(\mathfrak{L})$, $\mathfrak{L}(sf)(x) = \ell(x)s(x) = s(x)\mathfrak{L}^*f(x)$. Since the argument is reversible, (7.2) follows.

(II). We now give a second and perhaps more natural proof which is adapted from Feller [8]. In general

$$\frac{dg(x)}{ds^*(x)} = \lim_{y \rightarrow x} \frac{g(y) - g(x)}{s^*(y) - s^*(x)} = s(x)^2 \lim_{y \rightarrow x} \frac{g(y) - g(x)}{s(y) - s(x)}$$

and if $s(x)f(x) \in \mathfrak{D}(\mathfrak{L})$

$$\begin{aligned}
\frac{d}{ds^*} f(x) &= s(x)^2 \frac{d}{ds} \frac{s(x)f(x)}{s(x)} = s(x)^2 \left[\frac{1}{s(x)} \frac{d}{ds} sf + s(x)f(x) \frac{d}{ds} \frac{1}{s(x)} \right] \\
&= s(x) \frac{d}{ds} s(x)f(x) - s(x)f(x).
\end{aligned}$$

Hence by Lemma A1 in the Appendix

$$\begin{aligned}
\frac{d}{ds^*} f(b) - \frac{d}{ds^*} f(a) &= \left[s(y) \frac{d}{ds} sf - s(y)f(y) \frac{d}{ds} s \right]_a^b \\
&= \int_a^b [s(y)\mathfrak{L}(sf)(y) - s(y)f(y)\mathfrak{L}s(y)] m(dy) \\
&= \int_a^b \frac{1}{s(y)} \mathfrak{L}(sf)(y) m^*(dy)
\end{aligned}$$

and $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$, $\mathfrak{L}^*f = (1/s)\mathfrak{L}(sf)$, etc. Since $\mathfrak{L}^{**} = \mathfrak{L}$, we are done.

Lemma 7.1. For all t and $0 < x < b < 1$

$$(7.5) \quad P_x^*[\tau_b \leq t, \tau_b < \tau_0] = \frac{s(b)}{s(x)} P_x[\tau_b \leq t, \tau_b < \tau_0]$$

where P_x^* , E_x^* refer to probabilities based on the process r_t^* generated by $\mathfrak{L}^*$.

Proof. Let $\varphi^*(x) = E_x^*[e^{-\lambda\tau_b} \chi_{\{\tau_b < \tau_a\}}]$ where $0 < a < x < b$, $\lambda > 0$. Then ([3, 13]) $\varphi^*(x) \in \mathfrak{D}(\mathfrak{L}^*)$ in (a, b) and satisfies $\mathfrak{L}^*\varphi^* = \lambda\varphi^*$. Hence by (7.2), $s(x)\varphi^*(x) \in \mathfrak{D}(\mathfrak{L})$ and

$$\mathfrak{L}(s\varphi^*) = \lambda(s\varphi^*).$$

However if $u \in \mathfrak{D}(\mathfrak{L})$, $\mathfrak{L}u = \lambda u$, and $u(a) = u(b) = 0$, then $u(x) \equiv 0$; hence by subtraction

$$s(x)\varphi^*(x) = s(b)\varphi_+(x, \lambda)$$

where

$$\varphi_+(x, \lambda) = E_x(e^{-\lambda\tau_b} \chi_{\{\tau_b < \tau_a\}}).$$

Hence for $0 < a < x < b$

$$E_x^*[e^{-\lambda\tau_b} \chi_{\{\tau_b < \tau_a\}}] \equiv \frac{s(b)}{s(x)} E_x[e^{-\lambda\tau_b} \chi_{\{\tau_b < \tau_a\}}].$$

Let $a \rightarrow 0$. Then the above is (7.5) in Laplace transformation form.

Lemma 7.2. Let $p^*(t, x, y)$ be the fundamental solution of $u_t = \mathfrak{L}^*u$ in R , with zero boundary conditions where appropriate, (see §2) with respect to $m^*(dx)$. Then

$$(7.6) \quad p^*(t, x, y) \equiv \frac{p(t, x, y)}{s(x)s(y)} \quad \text{all } t > 0, \quad x, y \in R.$$

Proof. Let $p_{ab}^*(t, x, y)$ be the fundamental solution of $u_t = \mathfrak{L}^*u$ in (a, b) . Then

$$p_{ab}^*(t, x, y) = \sum e^{-\lambda_n t} \beta_n(x) \beta_n(y)$$

where $\{\beta_n(x)\}$ are the normalized eigenfunctions of the Fredholm equation

$$\beta_n(x) = \lambda_n \int_a^b g_{ab}^*(x, y) \beta_n(y) m^*(dy).$$

Now by (7.4) and (7.1)

$$(7.7) \quad s(x)\beta_n(x) = \lambda_n \int_a^b g_{ab}(x, y) s(y) \beta_n(y) m(dy)$$

and

$$\int_a^b (s(y)\beta_n(y))^2 m(dy) = \int_a^b \beta_n(x)^2 m^*(dx) = 1.$$

The density $p_{ab}(t, x, y)$ has a similar expansion in terms of the normalized eigenfunctions of (7.7). Hence by (7.7)

$$p_{ab}(t, x, y) = \sum e^{-\lambda_n t} \beta_n(x) s(x) \beta_n(y) s(y)$$

and

$$p_{ab}^*(t, s, y) = \frac{p_{ab}(t, x, y)}{s(x)s(y)}.$$

Finally, $p_{ab}^*(t, x, y) \uparrow p^*(t, x, y)$ as $(a, b) \uparrow R$ and (7.6) follows.

The advantage of this transformation is what it does at the endpoints.

Theorem 7.2. *Let $\mathcal{L}$, $\mathcal{L}^*$ be as before. Then (see §2 for definitions)*

(i) *If $s(0) = 0$ (so $s^*(0) = -\infty$), 0 is an accessible boundary for $\mathcal{L}$ iff it is an entrance boundary for $\mathcal{L}^*$, and an inaccessible exit for $\mathcal{L}$ iff a natural boundary for $\mathcal{L}^*$.*

(ii) *If $s(0) > 0$ (so $s^*(0)$ is finite), 0 is accessible for $\mathcal{L}$ iff it is accessible for $\mathcal{L}^*$, and I.E. for $\mathcal{L}$ iff I.E. for $\mathcal{L}^*$.*

(iii) *If $s(0) = -\infty$ (so $s^*(0) = 0$), 0 is entrance for $\mathcal{L}$ iff accessible for $\mathcal{L}^*$, and natural for $\mathcal{L}$ iff I.E. for $\mathcal{L}^*$.*

Proof. If $s(0) = 0$,

$$\int_0^a (s(x) - s(0))m(dx) = \int_0^a s(x)m(dx) = \int_0^a s(x)s(x)^{-2}m^*(dx) = \int_0^a |s^*(x)| m^*(dx),$$

and the two integrals are simultaneously finite or infinite. If $s(0) > 0$,

$$\int_0^a (s^*(x) - s^*(0))m^*(dx) = \int_0^a \frac{s(x)}{s(0)} (s(x) - s(0))m(dx)$$

where $s(0) > 0$. Case (iii) is similar.

Finally, as might be expected from the form of (7.2),

Lemma 7.3. *Assume $u(x, t)$ is locally integrable in $R \times (0, T)$ and $s(x)$ is a non-vanishing scale in R . Set*

$$u^*(x, t) = \frac{u(x, t)}{|s(x)|} \geq 0.$$

Then, u^ is a weak solution of $u_t = \mathcal{L}^*u$ if and only if u is a weak solution of $u_t = \mathcal{L}u$. In either case $u^*(x, t)m^*(dx) = u(x, t)|s(x)|m(dx) \rightarrow F^*(dx) = |s(x)|F(dx)$. and hence by (4.2)*

$$v^*(x, t) = u^*(x, t) - \int p^*(t, x, y)F^*(dy) = \frac{v(x, t)}{|s(x)|}.$$

Proof. By definition, $u^*(x, t)$ is a weak solution of $u_t = \mathcal{L}^*u$ iff

$$\int_{t_1}^{t_2} u^*(x, \theta) d\theta = - \int_a^b g_{ab}^*(x, y) [u^*(y, t_2) - u^*(y, t_1)] m^*(dy) \\ + \frac{s^*(b) - s^*(x)}{s^*(b) - s^*(a)} \int_{t_1}^{t_2} u^*(a, \theta) d\theta + \frac{s^*(x) - s^*(a)}{s^*(b) - s^*(a)} \int_{t_1}^{t_2} u^*(b, \theta) d\theta$$

for a.e. (a, x, b, t_1, t_2) (see the Appendix), from which the equivalence follows.

Example. If

$$\mathfrak{L}u = u'' + \frac{\alpha}{x} u' \quad \text{in } (0, \infty) \quad (\alpha \neq 1)$$

then $s(x) = \pm x^{1-\alpha}$ is a scale for $\mathfrak{L}$, with respect to which

$$\mathfrak{L}^*u = u'' + \frac{2-\alpha}{x} u'$$

and the transformation $\mathfrak{L} \rightarrow \mathfrak{L}^*$ reflects the class of Bessel differential operators about $\alpha = 1$. In particular the heat equation and the three-dimensional Bessel equation (2.8) are dual. If $\alpha = 1$, 0 is entrance and ∞ is natural, and there is no scale $s(x) \neq 0$ for all $x \in (0, \infty)$. Hence $\mathfrak{L}^*$ cannot be defined in $(0, \infty)$.

This transformation can also be used to give information about the distribution of $P_x(\tau_0 \leq t)$. Indeed

Lemma 7.4. If 0 is an accessible boundary or an I.E.,

$$(7.8) \quad P_x(\tau_0 \leq t, \tau_0 < \tau_1) = \lim_{a \rightarrow 0} \int_0^t \frac{p(\theta, a, x)}{s(a) - s(0)} d\theta$$

uniformly for $x \in [a_0, 1)$, any $a_0 > 0$.

Proof. From the identities

$$p_{ab}(t+s, x, z) = \int_a^b p_{ab}(t, x, y) p_{ab}(s, y, z) m(dy), \\ g_{ab}(x, y) = \int_0^\infty p_{ab}(t, x, y) dt$$

we conclude $(0 < \epsilon < a < x < b < 1)$

$$\int_0^t p_{\epsilon b}(\theta, a, x) d\theta = g_{\epsilon b}(a, x) - \int_\epsilon^b g_{\epsilon b}(a, y) p_{\epsilon b}(t, y, x) m(dy).$$

Now let $\epsilon \rightarrow 0$ and divide by $s(a) - s(0)$. By the definition of g_{ab}

$$\int_0^t \frac{p_{0b}(\theta, a, x)}{s(a) - s(0)} d\theta = \frac{s(b) - s(x)}{s(b) - s(0)} - \int_a^b p_{0b}(t, x, y) \frac{s(b) - s(y)}{s(b) - s(0)} m(dy) \\ - \int_0^a \frac{s(b) - s(a)}{s(b) - s(0)} \frac{s(y) - s(0)}{s(a) - s(0)} p_{0b}(t, x, y) m(dy).$$

Now let $b \rightarrow 1$. If $s(1) < \infty$, the equation has the same form; if $s(1) = \infty$, we make the convention $\infty/\infty = 1$. In either case

$$\begin{aligned} \lim_{a \rightarrow 0} \int_0^t \frac{p(\theta, a, x)}{s(a) - s(0)} d\theta &= \frac{s(1) - s(x)}{s(1) - s(0)} - \int p(t, x, y) \frac{s(1) - s(y)}{s(1) - s(0)} m(dy) \\ &= P_x(\tau_0 < \tau_1) - E_x[\chi_{\{\tau_0 \wedge \tau_1 > t\}} P_{r_t}(\tau_0 < \tau_1)] \\ &= P_x(\tau_0 < \tau_1) - P_x(\tau_0 \wedge \tau_1 > t; \text{ from time } t, \tau_0 < \tau_1) \\ &= P_x(\tau_0 \leq t, \tau_0 < \tau_1) \end{aligned}$$

(where of course $\tau_1 = \infty$ if 1 is inaccessible).

If 0 is accessible for $\mathfrak{L}$, then by Theorem 7.2 it is entrance for $\mathfrak{L}^*$. Hence the process r_t^* can be extended to $[0, 1)$ in such a way that, when started at 0, r_t^* immediately exists into the interior of R and never returns (since an entrance boundary is inaccessible). Moreover [3, 13]

$$\lim_{x \rightarrow 0} E_x^*(\varphi(r_t)) = E_0^*(\varphi(r_t))$$

for any $\varphi \in C(\bar{R})$. If $P^*(t, x, A) = P_x^*(r_t \in A)$ for sets A , by the Markov property

$$P(t, 0, A) = \iint_A p^*(s, y, z) m(dz) P^*(t - s, 0, dy)$$

and $p^*(t, 0, A) = \int_A p^*(t, 0, x) m(dx)$ for some density $p^*(t, 0, x)$. Then

Theorem 7.3. *If 0 is accessible and $s(0) = 0$,*

$$(7.9) \quad P_x(\tau_0 \leq t, \tau_0 < \tau_1) = \int_0^t s(x) p^*(\theta, 0, x) d\theta$$

for all $t > 0$ and a.e. $x \in R$. In this sense $(dp/ds)(t, 0, x) = s(x) p^*(t, 0, x)$ for a.e. (x, t) .

Proof. Since $s(0) = 0$

$$\int_0^t p^*(\theta, a, x) d\theta = \int_0^t \frac{p(\theta, a, x)}{s(a)s(x)} d\theta$$

and by Lemma 7.4

$$(7.10) \quad \lim_{a \rightarrow 0} \int_0^t p^*(\theta, a, x) d\theta$$

converges uniformly on compact sets in R . If $\varphi(x)$ is a continuous function of compact support in R , then as $a \rightarrow 0$

$$\begin{aligned} \int \varphi(x) \int_0^t p^*(\theta, a, x) d\theta m(dx) &= \int_0^t E_a^*(\varphi(r_\theta)) d\theta \\ &\rightarrow \int_0^t E_0^*(\varphi(r_\theta)) d\theta = \int \varphi(x) \int_0^t p^*(\theta, 0, x) d\theta m(dx) \end{aligned}$$

and hence the limit in (7.10) is $\int_0^t p^*(\theta, 0, x)d\theta$ at least for *a.e.* x . Since (7.10) converges uniformly on compact sets, the right-hand side of (7.9) can be redefined on a null set to be continuous in $R \times (0, \infty)$.

8. Theorem 2.1: Both boundaries accessible or I. E. First assume 0 is accessible and 1 is I.E. Then by Theorem 5.1

$$(8.1) \quad \int_0^t v(x, \theta) d\theta = \int_0^t P_x(\tau_0 \leq t - \theta, \tau_0 < \tau_b)c(d\theta) \\ + \int_0^t P_x(\tau_b \leq t - \theta, \tau_b < \tau_0)v(b, \theta) d\theta$$

for $0 < x < b < 1$ and some non-negative Borel measure $c(ds)$ on $[0, T]$. Hence by Theorem 6.1 (letting $b \rightarrow 1$)

$$(8.2) \quad \int_0^t v(x, \theta) d\theta = \int_0^t P_x(\tau_0 \leq t - \theta)c(d\theta) \\ = \int_0^t s(x) \int_0^{t-\theta} p^*(u, 0, x) du c(d\theta) \\ = \int_0^t \int_0^u s(x)p^*(u - \theta, 0, x)c(d\theta) du$$

by Theorem 7.3 and Lemma A2. Hence if $(dp/ds)(t, 0, x) = s(x)p^*(t, 0, x)$

$$v(x, t) \equiv u(x, t) - \int p(t, x, y)F(dy) = \int_0^t \frac{dp}{ds}(t - \theta, 0, x)c(d\theta)$$

for *a.e.* (x, t) , which is Theorem 2.1 in this case.

If both boundaries are I.E., $\int_0^t v(x, s)ds \equiv 0$ by Theorem 6.1 applied twice, and $u(x, t) = \int p(t, x, y)F(dy)$ for *a.e.* (x, t) by differentiation. If both boundaries are accessible, let $b \rightarrow 1$ in (8.1). As in §5, there exists a Borel measure $\tilde{c}(ds) \geq 0$ on $[0, T]$ such that

$$\int_0^t v(x, \theta) d\theta = \int_0^t P_x(\tau_0 \leq t - \theta, \tau_0 < \tau_1)c(d\theta) \\ + \int_0^t P_x(\tau_1 \leq t - \theta, \tau_1 < \tau_0)\tilde{c}(d\theta)$$

from which Theorem 2.1 follows as before.

9. Theorem 2.1: The remaining cases. First

Theorem 9.1. *If 1 is a natural boundary,*

$$\lim_{x \rightarrow 1} \int_0^t v(x, \theta) d\theta = 0, \quad 0 < t < T.$$

Proof. As in §6 (see (6.12)), it is sufficient to assume 0 is a regular boundary. If $s(x)$ is a scale for $\mathfrak{L}$, then $s(1) = \infty$; by adding a constant if necessary we can assume $s(0) > 0$. With this scale,

$$u^*(x, t) = \frac{u(x, t)}{s(x)}$$

is a weak solution of $u_t = \mathfrak{L}^*u$ in $R \times (0, T)$, and 1 is an I.E. for $\mathfrak{L}^*$ by Theorem 7.2. Hence by (8.2) as before

$$(9.1) \quad \int_0^t v^*(x, \theta) d\theta = \int_0^t v^*(a, \theta) P_x^*(\tau_a \leq t - \theta) d\theta \quad (0 < a < x < 1)$$

where $v^*(x, t)$ has the same relationship to $u^*(x, t)$ as v does to u . Now $P_x^*(\tau_a \leq t) = [s(a)/s(x)]P_x(\tau_a \leq t)$ by Lemma 7.1 and $v^*(a, t) = v(a, t)/s(a)$ by Lemma 7.3. Hence (9.1) is equivalent to

$$(9.2) \quad \int_0^t v(x, \theta) d\theta = \int_0^t v(a, \theta) P_x(\tau_a \leq t - \theta) d\theta.$$

However, if 1 is a natural boundary ($x > a$), $P_x(\tau_a \leq t) \leq e^{\lambda t} E_x(\exp(-\lambda\tau_a)) = e^{\lambda t} \varphi_-(x, \lambda) \rightarrow 0$ as $x \rightarrow 1$ ([13, p. 130]). Hence Theorem 9.1 follows from (9.2).

The extension of Theorem 2.1 to natural boundaries then goes as in §8. The case of one entrance boundary can be handled by the following adaption of Theorem 5.1:

Theorem 9.2. *If 0 is an entrance boundary, there exists a Borel measure $c(ds) \geq 0$ on $[0, T)$ such that for all $b \in (x, 1)$,*

$$(9.3) \quad \begin{aligned} \int_0^t v(x, \theta) d\theta &= \int_0^t \int_0^\theta p_{0b}(\theta - u, 0, x) c(du) d\theta \\ &\quad + \int_0^t v(b, \theta) P_x(\tau_b \leq t - \theta) d\theta. \end{aligned}$$

Proof. Let $s(x)$ be a scale for $\mathfrak{L}$. Since 0 is entrance, $s(0) = -\infty$; by subtracting a constant we can assume $s(b_1) \leq 0$ for $b_1 = (1 + b)/2$. Set $u^*(x, t) = u(x, t)/|s(x)|$, $v^*(x, t) = v(x, t)/|s(x)|$, and define $\mathfrak{L}^*$, p_{0b}^* , and p_{0b}^{**} as before (using the scale $s^*(x)$ of (7.1) for $\mathfrak{L}^*$). In particular $p_{0b}^{**} = p_{0b}$ by Lemma 7.2, and by Theorem 5.1 applied in $(0, b_1) \times (0, T)$ (since 0 is accessible for $\mathfrak{L}^*$ by Theorem 7.2)

$$\begin{aligned} \int_0^t v^*(x, \theta) d\theta &= \int_0^t P_x^*(\tau_0 \leq t - \theta, \tau_0 < \tau_b) c(d\theta) \\ &\quad + \int_0^t v^*(b, \theta) P_x^*(\tau_b \leq t - \theta, \tau_b < \tau_0) d\theta \\ &= \int_0^t s^*(x) \int_0^{t-\theta} p_{0b}(u, 0, x) du c(d\theta) \\ &\quad + \int_0^t \frac{v(b, \theta)}{|s(b)|} \frac{s(b)}{s(x)} P_x(\tau_b \leq t - \theta) d\theta \end{aligned}$$

by Theorem 7.3 and the inaccessibility of 0. Equation (9.3) follows from an application of Lemma A2 and multiplication by $|s(x)|$.

Corollary 9.1.

$$(9.4) \quad \lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u(x, \theta) d\theta = c(t-) = \int_0^{t-} c(ds), \quad 0 < t < T.$$

Proof. $\lim_{x \rightarrow 0} \int_0^t u^*(x, \theta) d\theta = c(t-)$ by Theorem 5.2.

If now 1 is natural or I.E., the second term in (9.3) vanishes as $b \rightarrow 1$, and

$$v(x, t) = u(x, t) - \int p(t, x, y) F(dy) = \int_0^t p(t-s, 0, x) c(ds)$$

for a.e. (x, t) . If 1 is accessible, we proceed as in §8.

If both boundaries are entrance, the situation is more difficult, since there is no scale $s(x) \neq 0$ in R , and we don't have the uniformity that we obtained for natural boundaries.

Borrowing a result from §10:

Lemma 9.1. (See §10.) Assume 0 is an entrance boundary, and set

$$u_0(x, t) = \int_0^t p(t-s, 0, x) \beta(ds) < \infty$$

where $\beta(ds)$ is a non-negative Borel measure on $[0, T)$. Then $u_0(x, t)$ is a weak solution of $u_t = \mathcal{L}u$ in $R \times (0, T)$. Moreover $u_0(x, t)m(dx) \rightarrow 0$ weakly on compact sets in R as $t \rightarrow 0$, and

$$(9.5) \quad \lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u_0(x, \theta) d\theta = \beta(t-) = \int_0^{t-} \beta(ds), \quad 0 < t < T.$$

Theorem 9.3. The representation (2.1) is valid if both boundaries are entrance.

Proof. By Theorem 9.2, there exists a measure $c(ds) \geq 0$ such that

$$v(x, t) \geq \int_0^t p_{0b}(t-s, 0, x) c(ds)$$

for all $b > x$. Let $b \rightarrow 1$. Then since $v(x, t) \leq u(x, t)$

$$u_1(x, t) = u(x, t) - \int_0^t p(t-s, 0, x) c(ds) \geq 0$$

and is a weak solution of $u_t = \mathcal{L}u$ by Lemma 9.1. Also by Lemma 9.1, $u_1(x, t)m(dx) - u(x, t)m(dx) \rightarrow 0$ weakly on compact sets, and hence the function $v_1(x, t)$ of §4 corresponding to $u_1(x, t)$ is

$$v_1(x, t) = v(x, t) - \int_0^t p(t-s, 0, x) c(ds).$$

By Corollary 9.1 and Lemma 9.1 applied a third time

$$\lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u_1(x, \theta) d\theta = 0 \quad 0 < t < T,$$

and the measure $c_1(ds)$ of Theorem 9.2 for $u_1(x, t)$ is identically zero. Hence by (9.3) ($0 < x < b$)

$$\int_0^t v_1(x, \theta) d\theta = \int_0^t v_1(b, \theta) P_x(\tau_b \leq t - \theta) d\theta = 0(1)$$

as $x \rightarrow 0$ (since $P_x(\tau_b \leq t) \leq 1$), and, since 0 is inaccessible in finite time,

$$(9.6) \quad \int_0^t v_1(x, \theta) P_b(\tau_x \leq t - \theta) d\theta \leq \int_0^t v_1(x, \theta) d\theta P_b(\tau_x \leq t) \rightarrow 0$$

as $x \rightarrow 0$.

Now, apply the analog of Theorem 9.2 to $u_1(x, t)$ in $(a, 1)$ with 1 as the entrance boundary. We find that there exists a Borel measure $\tilde{c}(ds) \geq 0$ on $[0, T)$ such that

$$\begin{aligned} \int_0^t v_1(x, \theta) d\theta &= \int_0^t v_1(a, \theta) P_x(\tau_a \leq t - \theta) d\theta \\ &\quad + \int_0^t \int_0^\theta p_{a1}(\theta - u, 1, x) \tilde{c}(du) d\theta \quad (0 < a < x) \end{aligned}$$

and by (9.6) for $a \rightarrow 0$, x fixed

$$\int_0^t v_1(x, \theta) d\theta = \int_0^t \int_0^\theta p(\theta - u, 1, x) \tilde{c}(du) d\theta.$$

Hence for a.e. (x, t)

$$\begin{aligned} v_1(x, t) &= u(x, t) - \int_R p(t, x, y) F(dy) - \int_0^t p(t - s, 0, x) c(ds) \\ &= \int_0^t p(t - s, 1, x) \tilde{c}(ds). \end{aligned}$$

10. Existence. The purpose here is to prove

Theorem 10.1. *Given non-negative Borel measures $c(ds)$, $\tilde{c}(ds)$ in $[0, T)$ and $F(dy)$ on R , define $u(x, t)$ by the right-hand side of (2.1). Assume $u(x_0, t_0) < \infty$ for some $(x_0, t_0) \in R \times (0, T]$. Then $u(x, t)$ is a weak solution of $u_t = \mathcal{L}u$ in $R \times (0, t_0)$; moreover, the initial measure $F(dy)$ and boundary measures $c(ds)$, $\tilde{c}(ds)$ (where the boundaries are accessible or entrance) can be recovered as in Theorem 2.3.*

The proof will be broken up into a series of lemmas. We begin with the boundary terms.

Lemma 10.1. *For each $\epsilon > 0$, $u(x, t) = p(t, \epsilon, x)$ satisfies*

$$\begin{aligned} (10.1) \quad \int_0^t u(x, \theta) d\theta &= - \int_a^b g_{ab}(x, y) u(y, t) m(dy) \\ &\quad + \frac{s(b) - s(x)}{s(b) - s(a)} \int_0^t u(a, \theta) d\theta + \frac{s(x) - s(a)}{s(b) - s(a)} \int_0^t u(b, \theta) d\theta \end{aligned}$$

for all $a, b, \epsilon \in (\epsilon, 1)$, $a < x < b$, and all $t > 0$.

Proof. $u(x, t)$ is a "classical" solution and hence a weak solution by Theorem A1 in the Appendix. In particular equation (A1) in the Appendix holds identically in (a, x, b, t_1, t_2) ; however

$$\int_a^b g_{ab}(x, y)u(y, t_1)m(dy) = \int_a^b g_{ab}(x, y)p(t_1, \epsilon, y)m(dy) \rightarrow 0$$

as $t_1 \rightarrow 0$ since $\epsilon < a$, and (A1) reduces to (10.1).

Corollary 10.1. *Let $c(ds)$ be as in Theorem 10.1, and set*

$$(10.2) \quad u(x, t) = \int_0^t p(t-s, \epsilon, x)c(ds).$$

Then $u(x, t)$ and $\int_0^t u(x, s)ds$ are locally integrable in $R \times (0, T)$, and $u(x, t)$ satisfies (10.1) for $\epsilon < a < x < b < 1$ and $0 < t < T$.

Proof. Local integrability follows from $\int p(t, a, x)m(dx) \leq 1$; (10.1) follows by integration directly in (10.1) and the identity

$$\int_0^t u(x, s) ds = \int_0^t \int_0^{t-s} p(u, \epsilon, x) duc(ds).$$

Lemma 10.2. *Let $u(x, t) \geq 0$ be any function satisfying (10.1) in $R \times (0, T)$ such that $u(x, t)$ and $\int_0^t u(x, s)ds$ are locally integrable in $R \times (0, T)$. Then $u(x, t)$ is a weak solution of $u_t = \mathcal{L}u$ and moreover*

$$(10.3) \quad u(x, t)m(dx) \rightarrow 0 \text{ weakly on compact sets in } R \text{ as } t \rightarrow 0.$$

Proof. Subtracting (10.1) with $t = t_1$ from (10.1) with $t = t_2$ yields (A1); hence $u(x, t)$ is a weak solution. In general, for any weak solution $u(x, t) \geq 0$, $u(x, t)m(dx) \rightarrow F(dx)$ for some measure $F(dx)$ by §4. If we now let $t_1 \rightarrow 0$ in (A1) we obtain (10.1) again ($t = t_2$) plus the extra term

$$h(x) = \int_a^b g_{ab}(x, y)F(dy) = 0.$$

But

$$\frac{dh}{ds}(x_2) - \frac{dh}{ds}(x_1) = \int_{x_1}^{x_2} F(dy) = 0$$

by explicit calculation, and the measure $F(dy) = 0$.

Lemma 10.3. *Assume 0 is an entrance boundary, and set*

$$(10.4) \quad u(x, t) = \int_0^t p(t-s, 0, x)c(ds)$$

where $c(ds)$ is as in Theorem 10.1. Then $u(x, t)$ is a weak solution of $u_t = \mathcal{L}u$ in $R \times (0, T)$ and $u(x, t)m(dx) \rightarrow 0$ weakly on compact sets in R as $t \rightarrow 0$.

Proof. Define $u_\epsilon(x, t)$ by (10.2); then u_ϵ satisfies (10.1) in $(\epsilon, 1) \times (0, T)$. I claim $u(x, t)$ satisfies (10.1) for $0 < a < x < b < 1$, $0 < t < T$. To prove (10.1) is preserved as $\epsilon \rightarrow 0$, it is sufficient to show

$$(i) \quad \lim_{\epsilon \rightarrow 0} \int \varphi(x) u_\epsilon(x, t) m(dx) = \int \varphi(x) u(x, t) m(dx)$$

for all $\varphi \in C(R)$ of compact support, and

$$(ii) \quad \lim_{\epsilon \rightarrow 0} \int_0^t u_\epsilon(x, s) ds = \int_0^t u(x, s) ds, \quad 0 < x < 1.$$

Property (i) follows as in §7. If $s(1) < \infty$, the density $p^*(t, x, y)$ can be defined and 0 is accessible for $\mathfrak{L}^*$; (ii) follows as in the proof of Theorem 7.3. In particular (10.1) obtains for $u(x, t)$ given by (10.4) with $p = p_{0d}$ for $d > b$; now let $d \rightarrow 1$.

Lemma 10.4. Assume 0 is an accessible boundary, and set

$$(10.5) \quad u(x, t) = \int_0^t \frac{dp}{ds}(t-s, 0, x) c(ds)$$

where $c(ds)$ is as in Theorem 10.1. Then $u(x, t)$ is a weak solution of $u_t = \mathfrak{L}u$ in $R \times (0, T)$ and $u(x, t)m(dx) \rightarrow 0$ weakly on compact sets in R as $t \rightarrow 0$.

Proof. Let $s(x)$ be a scale with $s(0) = 0$. Then

$$u(x, t) = s(x) \int_0^t p^*(t-s, 0, x) c(ds)$$

by definition, and $u^*(x, t) = u(x, t)/s(x)$ is a weak solution of $u_t = \mathfrak{L}^*u$ by Lemma 10.3. Now use Lemma 7.3.

Lemma 10.5. (i) If 0 is accessible and $u(x, t)$ is given by (10.5), $\int_0^t u(x, s) ds \rightarrow c(t-)$ for $0 < t < T$, as $x \rightarrow 0$.

(ii) If 0 is entrance and $u(x, t)$ is given by (10.4),

$$(10.6) \quad \lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u(x, \theta) d\theta = c(t-) = \int_{0-}^{t-} c(ds), \quad 0 < t < T.$$

Proof. (i) By Lemma A2 and Theorem 7.3,

$$\int_0^t u(x, s) ds = \int_0^t P_x(\tau_0 \leq t-s, \tau_0 < \tau_1) c(ds) \rightarrow c(t-).$$

(ii) For $0 < \epsilon < a < x < b$,

$$\int_0^t p_{\epsilon b}(\theta, a, x) d\theta = g_{\epsilon b}(a, x) - \int_\epsilon^b p_{\epsilon b}(t, a, y) g_{\epsilon b}(y, x) m(dy)$$

and letting $\epsilon \rightarrow 0$ ($s(0) = -\infty$ since 0 is entrance)

$$\begin{aligned}
\int_0^t p_{0b}(\theta, a, x) d\theta &= s(b) - s(x) - \int_0^b p_{0b}(t, a, y)(s(b) - s(x \vee y))m(dy) \\
\int_0^t p_{0b}(\theta, a, x) d\theta - \int_0^t p_{0b}(\theta, a, x_0) d\theta \\
&= s(x_0) - s(x) - \int_0^b p_{0b}(t, a, y)(s(x_0 \vee y) - s(x \vee y))m(dy) \\
&= s(x_0) - s(x) - \int_0^{x_0} p_{0b}(t, a, y)(s(x_0) - s(x \vee y))m(dy)
\end{aligned}$$

where x_0 is fixed, $x < x_0 < b < 1$. Let $b \rightarrow 1$ and $a \rightarrow 0$ in any order; then

$$\begin{aligned}
\int_0^t p(\theta, 0, x) d\theta &= \int_0^t p(\theta, 0, x_0) d\theta + s(x_0) - s(x) \\
&\quad - \int_0^{x_0} p(t, a, y)(s(x_0) - s(x \vee y))m(dy)
\end{aligned}$$

and

$$\lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t p(\theta, 0, x) d\theta = 1, \quad 0 < t < \infty$$

which implies (10.6).

Lemma 10.6. Let.

$$u(x, t) = \int p(t, x, y)F(dy)$$

and assume $u(x_0, t_0) < \infty$. Then $u(x, t)$ is a weak solution of $u_t = \mathfrak{L}u$ in $R \times (0, t_0)$. Moreover

$$(10.7) \quad u(x, t)m(dx) \rightarrow F(dx) \text{ weakly on compact sets in } R,$$

$$(10.8) \quad (a) \text{ If } 0 \text{ is accessible, } \int_0^t u(x, s)ds \rightarrow 0 \text{ as } x \rightarrow 0,$$

$$(b) \text{ If } 0 \text{ is entrance,}$$

$$\lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u(x, \theta) d\theta = 0 \quad 0 < t < T.$$

Proof. In general $u(x, t) = \int p(t - s, x, y)u(y, s)m(dy)$ for $u \geq 0$, and $u(x, t)$ is locally integrable in $R \times (0, t_0)$. For fixed y , $p(t, x, y)$ satisfies (A1) identically; hence by linearity so does $u(x, t)$ for a.e. (a, x, b, t_1, t_2) , and $u(x, t)$ is a weak solution in $R \times (0, t_0)$. Thus $u(x, t)m(dx) \rightarrow G(dx)$ weakly on compact sets in R for some measure $G(dx)$ by §4. However by comparison of equations (4.7) and (4.9) or (5.9)

$$\int_a^b p_{ab}(t, x, y)G(dy) = \int_a^b p_{ab}(t, x, y)F(dy)$$

identically in (a, x, b, t) . By integration in t

$$\int_a^b g_{ab}(x, y)G(dy) = \int_a^b g_{ab}(x, y)F(dy)$$

and $G(dy) = F(dy)$ as before. Hence $u(x, t)m(dx) \rightarrow F(dx)$ as $t \rightarrow 0$.

If 0 is accessible, (10.8a) is Lemma 5.2. If 0 is entrance,

$$(10.9) \quad u(x, t) = \int_0^t p_{ob}(t-s, 0, x)c(ds) + \int_0^b p_{ob}(t, x, y)F(dy) \\ - \int_0^t \frac{dp_{ob}}{ds}(t-s, b, x)\mathfrak{c}(ds)$$

for all $b \in R$ by Theorem 9.2, where (see (9.4))

$$(10.10) \quad \lim_{x \rightarrow 0} \frac{1}{|s(x)|} \int_0^t u(x, \theta) d\theta = c(t-), \quad 0 < t < T.$$

However, letting $b \rightarrow 1$ in (10.9) implies

$$\int_0^t p(t-s, 0, x)c(ds) = 0 \quad \text{for a.e. } (x, t)$$

and $c(t) \equiv 0$ by Lemma 10.5. Equation (10.8b) now follows from (10.10).

If both boundaries are accessible or entrance, the measures $F(dy)$ which can appear in Lemma 10.6 can be explicitly characterized. Indeed

Theorem 10.2. (See Hartman-Wintner [10] for $\mathfrak{L}u = u_{xx}$.) If 0 is accessible and $(x, t) \in R \times (0, \infty)$, $a \in R$,

$$\int_0^a p(t, x, y)F(dy) < \infty \quad \text{iff} \quad \int_0^a [s(y) - s(0)]F(dy) < \infty$$

and if 0 is entrance

$$\int_0^a p(t, x, y)F(dy) < \infty \quad \text{iff} \quad \int_0^a F(dy) < \infty.$$

The proof follows from Corollary 5.2, Lemma 7.4, duality, and (10.9).

11. Theorem 2.2. As in §2, set

$$\mathfrak{L}_1 = \frac{d}{dm(x)} \left[\frac{d}{ds(x)} - \int^x \cdot k(dy) \right]$$

for $s(x)$, $m(dx)$ as in §1 and a Borel measure $k(dx) \geq 0$. Let r_t be the diffusion process in R generated by the operator

$$\mathfrak{L} = \frac{d}{dm(x)} \frac{d}{ds(x)}$$

(see §1). Then the operator $\mathfrak{L}_1$ can be associated with a process r_t' in R which is created from r_t by randomly "killing" r_t ; i.e. by interrupting r_t at a random time $\zeta(\omega)$ and sending it to a point $\Delta \notin \bar{R}$. Once at Δ , r_t' never returns to R . If for example

$$(11.1) \quad P_x[\zeta(\omega) \leq t + dt \mid r_s, 0 < s < \infty, r_t \in R] = c(r_t)dt$$

(or equivalently $P_x(\zeta > t \mid r) = \exp(-\int_0^t c(r_s)ds)$) where $k(dx) = c(x)m(dx)$, the operators

$$(11.2) \quad S_t f(x) = E_x[f(r_t)\chi_{\{\zeta > t\}}] = E_x\left[f(r_t) \exp\left(-\int_0^t c(r_s) ds\right)\right]$$

form a semi-group which has $\mathfrak{L}_1$ as its infinitesimal generator (with zero boundary conditions where appropriate). If $k(dx)$ is a general Borel measure, (11.1) is replaced by

$$P_x(\zeta > t \mid r_s, 0 < s < \infty) = \exp\left(-\int_R \ell_t(\omega, y)k(dy)\right)$$

where $\ell_t(\omega, a)$ is the occupation time process of r_t ([13]).

Since $k(dx)$ is a Borel measure in R , $P_a'(\tau_b < \infty) > 0$ for all a, b in R , where P_x', E_x' refer to the process r_t' , and the behavior of r_t and r_t' in R is qualitatively similar. However the extra term in $\mathfrak{L}_1$ can change the behavior at the boundary. For example, 0 can be accessible for $\mathfrak{L}$ (i.e. $P_x(\tau_0 < \infty) > 0$) but inaccessible for $\mathfrak{L}_1$; i.e. $P_x'(\tau_0 < \infty) = P_x(\tau_0 < \zeta) \equiv 0$. Indeed in this case $P_x'(\tau_0 < \infty) > 0$ (any x) if and only if

$$(11.3) \quad \int_0^a (s(y) - s(0))k(dy) < \infty, \quad a \in R.$$

Similarly, 0 can be entrance for $\mathfrak{L}$ but with $\int_0^a |s(y)|k(dy) = \infty$; this implies $\lim_{x \rightarrow 0} P_x'(\tau_b < \infty) = 0$ for all $b \in R$. In these cases the boundary behavior resembles that of an I.E. or natural boundary; however boundary terms for the general positive solution of $u_t = \mathfrak{L}_1 u$ do appear (see below). Any nice probabilistic explanation of these boundary terms, in the sense of §3, completely escapes me.

For $0 < c < x < d < 1$, define

$$(11.4) \quad \begin{aligned} h_+(x) &= P_x'(\tau_d < \infty) & (=P_x(\tau_d < \zeta)). \\ h_-(x) &= P_x'(\tau_c < \infty) & (=P_x(\tau_c < \zeta)). \end{aligned}$$

If $0 < x < d < d_1 < 1$, by the strong Markov property

$$P_x'(\tau_{d_1} < \infty) = P_x'(\tau_d < \infty)P_d'(\tau_{d_1} < \infty)$$

and

$$h_+(x) = P_x'(\tau_{d_1} < \infty)/P_d'(\tau_{d_1} < \infty).$$

In this way we can extend $h_{\pm}(x)$ to all of R . By arguing as in [13, §4.6], $h_{\pm}(x) \in$

$\mathfrak{D}(\mathfrak{L}_1)$ (see the Appendix) and $\mathfrak{L}_1 h_{\pm}(x) = 0$. Also, by Lemma A1 and the remarks following (A6) in the Appendix

$$(11.5) \quad K = \frac{dh_+}{ds}(x)h_-(x) - \frac{dh_-}{ds}(x)h_+(x) = \text{const.} > 0.$$

The parabolic equation $u_t = \mathfrak{L}_1 u$ of (1.1) or (2.3) can be converted into an equation of type $u_t = \mathfrak{L}u$ by a transformation similar to the one described in §7. Specifically

Lemma 11.1. *Define*

$$(11.6) \quad s^*(x) = \frac{h_+(x)}{h_-(x)}, \quad m^*(dx) = K^{-1}h_-(x)^2 m(dx)$$

for K as in (11.5). Then

$$\mathfrak{L}^* = \frac{d}{dm^*(x)} \frac{d}{ds^*(x)} = \frac{1}{h_-(x)} \mathfrak{L}_1 h_-(x)$$

in values and in domain; i.e. in the sense of Theorem 7.1.

Proof. If $f(x)$ is such that $f(x)h_-(x) \in \mathfrak{D}(\mathfrak{L}_1)$, then by Lemma A1

$$\begin{aligned} \frac{df}{ds^*}(x) &= \frac{h_-(x)^2}{K} \frac{d}{ds(x)} \frac{fh_-}{h_-} = K^{-1}[h_-(d/ds)fh_- - fh_-(d/ds)h_-], \\ \frac{df}{ds^*}(b) - \frac{df}{ds^*}(a) &= K^{-1} \int_a^b [h_- \mathfrak{L}_1 fh_- - fh_- \mathfrak{L}_1 h_-] m(dx) \\ &= \int_a^b \frac{1}{h_-(x)} \mathfrak{L}_1 h_-(x) m^*(dx) \end{aligned}$$

and $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$, etc. by (A3), (A4). Similarly, if $f(x) \in \mathfrak{D}(\mathfrak{L}^*)$,

$$\begin{aligned} \frac{d}{ds}(fh_-)(b) - \frac{d}{ds}(fh_-)(a) &= \left[h_- \frac{df}{ds} + f \frac{dh_-}{ds} \right]_a^b \\ &= \left[\frac{K}{h_-(y)} \frac{df}{ds^*}(y) \right]_a^b + \left[f(y) \frac{dh_-}{ds}(y) \right]_a^b. \end{aligned}$$

If we now use equation (A8) in the Appendix to expand the last two expressions (with $\mathfrak{L}^*$ in place of $\mathfrak{L}_1$ in the first term), we conclude $fh_- \in \mathfrak{D}(\mathfrak{L}_1)$ (i.e. (A5)) etc.

Corollary 11.1. $f(x) \in \mathfrak{D}(\mathfrak{L}_1)$ in R if and only if

$$f(x) = - \int_a^b g_{ab}(x, y) \ell(y) m(dy) + C_1 h_+(x) + C_2 h_-(x)$$

for all $a, b \in R$, where $\mathfrak{L}_1 f(x) = \ell(x) \in C(R)$ and C_1, C_2 vary with a, b . Here

$$(11.7) \quad g_{ab}(x, y) = \frac{(h_+(x \wedge y) - \alpha h_-(x \wedge y))(h_-(x \vee y) - \beta h_+(x \vee y))}{K(1 - \alpha\beta)}$$

where α, β are chosen so that $g_{ab}(a, y) = g_{ab}(b, y) \equiv 0$.

Proof. $g(x) \in \mathfrak{D}(\mathfrak{L}^*)$ iff $g(x) = \int_b^a g_{ab}^*(x, y) \ell(y) m^*(dy) + C_1 s^*(x) + C_2$, where (see Appendix)

$$g_{ab}^*(x, y) = \frac{[s^*(b) - s^*(x \vee y)][s^*(x \wedge y) - s^*(a)]}{s^*(b) - s^*(a)}.$$

Expanding the right-hand side in terms of $h_{\pm}(x)$ (see (11.6))

$$g_{ab}^*(x, y) = K \frac{g_{ab}(x, y)}{h_-(x)h_-(y)}, \text{ etc.}$$

In particular Theorem A1 (that “classical” solutions are weak solutions) generalizes, and by extending Lemmas 7.2 and 7.3:

Lemma 11.2. *Let $p^*(t, x, y)$ be the fundamental solution of $u_t = \mathfrak{L}^*u$ in R , and $p(t, x, y)$ the fundamental solution of $u_t = \mathfrak{L}_1u$ in R . Then*

$$p^*(t, x, y) = K \frac{p(t, x, y)}{h_-(x)h_-(y)}, \text{ all } t > 0, x, y \in R.$$

Lemma 11.3. *Let $u(x, t)$ be a locally integrable function in $R \times (0, T)$. Then $u(x, t)$ is a weak solution of $u_t = \mathfrak{L}_1u$ in $R \times (0, T)$ iff*

$$u^*(x, t) = \frac{u(x, t)}{h_-(x)}$$

is a weak solution of $u_t = \mathfrak{L}^*u$ in $R \times (0, T)$.

Proof of Theorems 2.2, 2.3. By (11.4), $h_+(0+) = 0$ whenever $s(0)$ is finite, and similarly it can be shown that $h_-(0+) = \lim_{a \rightarrow 0} 1/P'_c(\tau_a < \infty) = \infty$ whenever $s(0) = -\infty$ and $k(R) > 0$. Hence

$$s^*(0) = 0, \quad s^*(1) = \infty.$$

Hence by Theorem 2.1 and Lemma 11.3

$$(11.8) \quad u^*(x, t) = \int_0^t \frac{dp^*}{ds^*}(t - \theta, 0, x) \alpha(d\theta) + \int p^*(t, x, y) F^*(dy) \\ + \int_0^t p^*(t - \theta, 1, x) \beta(d\theta)$$

for Borel measures $\alpha(ds), \beta(ds) \geq 0$ on $[0, T]$ and $F^*(dy) \geq 0$ on R . By Theorem 2.1, the first integral in (11.8) vanishes unless

$$(11.9) \quad \int_0^a s^*(x) m^*(dx) = K^{-1} \int_0^a h_+(x) h_-(x) m(dx) < \infty,$$

in which case 0 is accessible for $\mathfrak{L}^*$, and by Lemma 11.2 (and §7)

$$\int_0^t \frac{dp^*}{ds^*}(u, 0, x) du = \lim_{a \rightarrow 0} \int_0^t \frac{p^*(u, a, x)}{s^*(a)} du \\ = \frac{K}{h_-(x)} \lim_{a \rightarrow 0} \int_0^t \frac{p(u, a, x)}{h_+(a)} du.$$

similarly by §5

$$\lim_{x \rightarrow 0} \int_0^t u^*(x, s) ds = \lim_{x \rightarrow 0} \int_0^t \frac{u(x, s)}{h_-(x)} ds = \alpha(t-), \quad 0 < t < T,$$

and, if $c(ds) = K\alpha(ds)$, the first term in (11.8) reduces to the first term of (2.1) predicted by Theorem 2.2, except for a factor of $h_-(x)$. Similarly, by §4, $u^*(x, t)m^*(dx) \rightarrow F^*(dx)$ weakly on compact sets in R ; thus $u(x, t)m(dx) \rightarrow F(dx) = (K/h_-(x))F^*(dx)$ by (11.6) and the definition of $u^*(x, t)$. Then $F^*(dx) = (h_-(x)/K)F(dx)$ and by Lemma 11.2

$$\int p^*(t, x, y)F^*(dy) = \int p(t, x, y)F(dy)/h_-(x).$$

The third term in (11.8) can be handled similarly. In particular, we obtain the same criterion (11.9) that 1 be entrance for $\mathfrak{L}^*$.

Remark. Thus a boundary term can appear for $\mathfrak{L}_1$ iff the corresponding boundary is accessible or entrance for $\mathfrak{L}^*$. However it turns out that $\mathfrak{L}^*$ cannot have fewer accessible or entrance boundaries than $\mathfrak{L}$:

Theorem 11.1. *If 0 is accessible or entrance for $\mathfrak{L}$, then 0 is accessible for $\mathfrak{L}^*$ (i.e., a boundary term at 0 appears in general for positive solutions of $u_t = \mathfrak{L}_1 u$).*

Proof. Given a, b in R , define $g_{ab}(x, y, k)$ for $\mathfrak{L}_1$ by (11.7) and $g_{ab}(x, y, 0)$ for $\mathfrak{L}$ by (A2) in the Appendix. Since in general

$$g_{ab}(x, y) = \int_0^\infty p_{ab}(t, x, y) dt$$

we have

$$(11.10) \quad \int_a^b g_{ab}(x, y, k)f(y)m(dy) = E_x' \left(\int_0^{\tau_a \wedge \tau_b} f(r_s') ds \right) = E_x \left(\int_0^{\tau_a \wedge \tau_b \wedge \zeta} f(r_s) ds \right)$$

with a similar identity for $g_{ab}(x, y, 0)$. Assume $f(y) \geq 0$. Since $\zeta \leq \infty$,

$$\int_a^b g_{ab}(x, y, k)f(y)m(dy) \leq \int_a^b g_{ab}(x, y, 0)f(y)m(dy)$$

for all non-negative $f(y)$, and by continuity

$$g_{ab}(x, y, k) \leq g_{ab}(x, y, 0), \quad 0 < a < x, y < b < 1.$$

For $y = x$, this reduces to

$$\frac{[h_+(x) - s^*(a)h_-(x)](h_-(x) - \beta h_+(x))}{K(1 - \beta s^*(a))} \leq \frac{(s(b) - s(x))(s(x) - s(a))}{s(b) - s(a)}.$$

Let $a \rightarrow 0$. Since $s^*(0) = 0$,

$$h_+(x)[h_-(x) - \beta h_+(x)] \leq K(s(x) - s(0)), \quad 0 < x < b,$$

if $s(0)$ is finite. If 0 is entrance,

$$h_+(x)[h_-(x) - \beta h_+(x)] \leq K(s(b) - s(x)).$$

In either event $\int_0^a h_+(x)h_-(x)m(dx) < \infty$, and the theorem follows from (11.9).

As Example 2 in §2 indicates, it is possible to have a boundary which is natural for $\mathfrak{L}$ but entrance for $\mathfrak{L}^*$. Hence it is of interest to find conditions on $k(dx)$ which make Theorem 11.1 necessary and sufficient; i.e., conditions on $k(dx)$ such that if 0 is I.E. or natural for $\mathfrak{L}$, no boundary term can appear for positive solutions of $u_t = \mathfrak{L}_t u$. As the remarks around equation (11.3) suggest,

$$(11.11) \quad \int_0^a (s(x) - s(0))k(dx) < \infty \quad \left(\text{resp. } \int_0^a |s(x)| k(dx) < \infty \right)$$

is a sufficient condition on $k(dx)$. A slightly better result, whose proof we shall give, is

Theorem 11.2. *Assume*

$$(11.12) \quad k(dx) = c(x)m(dx), \quad c(x) \leq \lambda < \infty.$$

Then, 0 is accessible or entrance for $\mathfrak{L}$ iff it is accessible for $\mathfrak{L}^$.*

Remark. As Example 2 of §2 indicates, (11.12) is far from sharp. It would be better to have sharper conditions. As the proof below shows, it is only necessary to have (11.12) satisfied in a neighborhood of 0 .

Proof of Theorem 11.2. If $f(y) \geq 0$, then by (11.2), (11.10)

$$\begin{aligned} \int_a^b g_{ab}(x, y, k)f(y)m(dy) &= \int_0^\infty E_x[f(r_t)\chi_{\{\tau_a \wedge \tau_b > t\}}\chi_{\{t > t_1\}}] dt \\ &= \int_0^\infty E_x\left[f(r_t)\chi_{\{\tau_a \wedge \tau_b > t\}} \exp\left(-\int_0^t c(r_s) ds\right)\right] dt \\ &\geq \int_0^\infty e^{-\lambda t} \int_R p_{ab}(t, x, y)f(y)m(dy) dt \end{aligned}$$

where p_{ab} above is the fundamental solution of $u_t = \mathfrak{L}u$ in (a, b) . Hence if

$$g_{ab}(x, y, \lambda) = \int_0^\infty e^{-\lambda t} p_{ab}(t, x, y) dt,$$

$$g_{ab}(x, y, \lambda) \leq g_{ab}(x, y, k) \leq g_{ab}(x, y, 0).$$

Now $g_{ab}(x, y, \lambda)$ is also given by (11.7) where $h_\pm(x)$ are replaced by

$$(11.13) \quad \begin{aligned} \varphi_+(x, \lambda) &= E_x(e^{-\lambda \tau_d} \chi_{\{\tau_d < \infty\}}), \\ \varphi_-(x, \lambda) &= E_x(e^{-\lambda \tau_c} \chi_{\{\tau_c < \infty\}}) \end{aligned}$$

for $0 < c < x < d < 1$, with the $\varphi_\pm$ extended to R the same way the $h_\pm(x)$ were extended. (See [13, Ch. 4]. Note that since $\mathfrak{L}\varphi_\pm = \lambda\varphi_\pm$, the functions $h_\pm(x)$ of

(11.4) for $\mathfrak{L}_1 = \mathfrak{L} - \lambda$ are the $\varphi_{\pm}$.) Hence as in the proof of Theorem 11.1, if $s(0)$ is finite,

$$(11.14) \quad \int_0^a \varphi_+(x, \lambda) \varphi_-(x, \lambda) m(dx) \leq C_1 \int_0^a h_+(x) h_-(x) m(dx) \\ \leq C_1 C_2 \int_0^a (s(x) - s(0)) m(dx) \quad (\text{fixed } a \in R)$$

with a similar inequality if $s(0) = -\infty$. Hence the theorem follows from (11.9) and

Lemma 11.4. *0 is accessible or entrance for $\mathfrak{L}$ iff*

$$(11.15) \quad \int_0^a \varphi_+(x, \lambda) \varphi_-(x, \lambda) m(dx) < \infty, \quad a \in R.$$

Proof. If 0 is accessible or entrance, the integral converges by (11.14); now assume (11.15) converges. As the definition (11.13) indicates, both conditions in Lemma 11.4 depend on $s(x)$ and $m(dx)$ only in $(0, d)$. Hence without loss of generality we can assume 1 is a natural boundary.

As mentioned before, the $\varphi_{\pm}$ are the functions $h_{\pm}(x)$ for $k(dx) = \lambda m(dx)$. Hence by (11.9), Lemma 11.1 and §10, there exists a non-negative weak solution $v(x, t)$ of $v_t = (\mathfrak{L} - \lambda)v$ such that

$$(11.16) \quad v(x, t) \not\equiv 0 \quad \text{and} \quad v(x, t) m(dx) \rightarrow 0 \text{ loc. weakly in } R.$$

However, an integration by parts in the definition (A1) shows that if $v(x, t)$ is a weak solution of $v_t = (\mathfrak{L} - \lambda)v$, then $\tilde{u}(x, t) = e^{\lambda t} v(x, t)$ is a weak solution of $u_t = \mathfrak{L}u$ in $R \times (0, T)$. Since 1 is a natural boundary and (11.16) holds for $u(x, t)$, it follows from Theorem 2.1 that 0 must be accessible or entrance, qed.

12. Remarks about when $u(x, t) \in L^1$. Let $u(x, t)$ be a non-negative solution of $u_t = \mathfrak{L}u$ in $R \times (0, T)$. If both endpoints of R are regular or entrance, it turns out that $\int u(x, t) m(dx) < \infty$ for a.e. $t \in (0, T)$ (see below). As the solution $u(x, t) = e^x e^t$ for the heat equation suggests, this is not true in general. We do have however

Theorem 12.1. *For $u(x, t)$ as above, assume both endpoints of R are inaccessible. Then, if $\int u(x, c) m(dx) < \infty$ for a single value of $c \in (0, T)$,*

$$(12.1) \quad \int u(x, t) m(dx) < \infty \quad \text{for all } t \in (0, T).$$

Proof. Assume for definiteness both endpoints are entrance. Then by Theorem 2.1

$$(12.2) \quad u(x, t) = \int_0^t p(t-s, 0, x) \alpha(ds) + \int p(t, x, y) F(dy) \\ + \int_0^t p(t-s, 1, x) \beta(ds)$$

for non-negative Borel measures $\alpha(ds)$, $\beta(ds)$, and $F(dy)$. Since the boundaries are inaccessible

$$\int p(t, x, y)m(dy) = 1, \quad \text{all } (x, t),$$

and by the symmetry of $p(t, x, y)$

$$\iint p(t, x, y)F(dy)m(dx) = \int_R F(dy).$$

Hence by (12.2)

$$\int_R u(x, t)m(dx) = \alpha(t-) + \int_R F(dy) + \beta(t-), \quad 0 < t < T.$$

Thus if $\int u(x, c)m(dx) < \infty$ for any c , $\int_R F(dy) < \infty$ and (12.1) follows.

Remark. By Theorem 10.2, $\int_R F(dy) < \infty$ automatically when both boundaries are entrance, and (12.1) holds in general for positive solutions in this case.

When regular boundaries are included, the situation changes slightly.

Theorem 12.2. Assume $u(x, t)$ is as before, and both endpoints are regular or inaccessible. Then, if $\int u(x, c)m(dx) < \infty$ for at least one value of $c \in (0, T)$,

$$(12.3) \quad \int u(x, t)m(dx) < \infty \quad \text{for a.e. } t \in (0, T).$$

Remarks. (1) If both endpoints are regular, (12.3) holds in general for positive solutions of $u_t = \mathcal{L}u$.

(2) The restriction “a.e. t ” in (12.3) cannot be removed. Indeed it is possible to have $\int u(x, c)m(dx) < \infty$ for a.e. $c \in (0, T)$ but $\int u(x, t)m(dx) = \infty$ for a dense null set in $(0, T)$.

Proof of Theorem 12.2. Assume for definiteness 0 is regular and 1 is a natural or I.E. boundary. Then by Theorem 2.1

$$\begin{aligned} u(x, t) &= \int_0^t \frac{dp}{ds}(t - \theta, 0, x)c(d\theta) + \int p(t, x, y)F(dy) \\ &= v(x, t) + w(x, t) \end{aligned}$$

In particular,

$$\begin{aligned} \int v(x, t)m(dx) &= \int_0^t \int \frac{dp}{ds}(t - \theta, 0, x)m(dx)c(d\theta) \\ \int_0^t \int v(x, s)m(dx) ds &= \int_0^t \int_R P_x(\tau_0 \leq t - s)m(dx)c(ds). \end{aligned}$$

Now by Theorem 7.3 and (7.1) (with $s(0) = 0$)

$$\begin{aligned}
\int_a^1 P_x(\tau_0 \leq t) m(dx) &= \int_a^1 s(x) \int_0^t p^*(u, 0, x) du m(dx) \\
&= \int_0^t \int_a^1 p^*(u, 0, x) |s^*(x)| m^*(dx) du \\
&\leq |s^*(a)| \int_0^t \int_a^1 p^*(u, 0, x) m^*(dx) du \leq t/s(a) < \infty.
\end{aligned}$$

Since 0 is regular, $\int_0^a P_x(\tau_0 \leq t) m(dx) \leq m[(0, a)] < \infty$ in general, and $\int v(x, t) m(dx) < \infty$ for a.e. $t \in (0, T)$. Similarly, since 0 is an absorbing boundary,

$$\begin{aligned}
\int w(x, t) m(dx) &= \iint p(t, x, y) F(dy) m(dx) \\
&= \int P_v(\tau_0 > t) F(dy).
\end{aligned}$$

Since $P_v(\tau_0 > t) \uparrow$ in y by the strong Markov property,

$$(12.4) \quad \int_a^1 F(dy) < \infty, \quad a \in R,$$

if $\int w(x, c) m(dx) < \infty$ for any value of c . Conversely, if $0 < y < a$,

$$\begin{aligned}
P_v(\tau_0 > t) &\leq P_v(\tau_0 \wedge \tau_a > t) + P_v(\tau_a < \tau_0) \\
&= \int_0^a p_{0a}(t, y, z) m(dz) + \frac{s(y) - s(0)}{s(a) - s(0)} \leq C(t)[s(y) - s(0)], \quad 0 < t < \infty,
\end{aligned}$$

since both 0 and a are regular endpoints of $(0, a)$ (see the remarks after equation (A10) in the Appendix). However, by Theorem 10.2, $\int_0^a (s(y) - s(0)) F(dy) < \infty$ whenever 0 is accessible. Hence, given (12.4), $\int w(x, t) m(dx) < \infty$ for $0 < t < \infty$.

Remark. (3) If 0 is accessible but not regular (i.e., if $m[(0, a)] = \infty$), the situation is less clear. For example, there exists a positive solution $u(x, t)$ of $u_t = \mathcal{L}_1 u = u_{xx} - (1/x)u_x$ for $(x, t) \in (0, \infty) \times (0, \infty)$ such that

$$\begin{aligned}
(12.5) \quad \int_0^\infty u(x, t) m(dx) &= \infty, \quad 2n < t \leq 2n + 1, \\
\int_0^\infty u(x, t) m(dx) &< \infty, \quad 2n + 1 < t \leq 2n + 2, \quad n = 0, 1, 2, \dots
\end{aligned}$$

Appendix. Let L , $s(x)$, $m(dx)$ etc. be as in §1. Then, a function $u(x, t)$ will be said to be a *weak solution* of $u_t = \mathcal{L}u$ in $R \times (0, T)$ if $u(x, t)$ is locally integrable with respect to $m(dx)dt$ in $R \times (0, T)$ and

$$\begin{aligned}
(A1) \quad \int_{t_1}^{t_2} u(x, \theta) d\theta &= - \int_a^b g_{ab}(x, y) (u(y, t_2) - u(y, t_1)) m(dy) \\
&\quad + \frac{s(b) - s(x)}{s(b) - s(a)} \int_{t_1}^{t_2} u(a, \theta) d\theta + \frac{s(x) - s(a)}{s(b) - s(a)} \int_{t_1}^{t_2} u(b, \theta) d\theta
\end{aligned}$$

for $m(dx)$ -a.e. (a, x, b) with $0 < a < x < b < 1$ and a.e. (t_1, t_2) with $0 < t_1 < t_2 < T$. Here

$$(A2) \quad g_{ab}(x, y) = \frac{(s(b) - s(x \vee y))(s(x \wedge y) - s(a))}{s(b) - s(a)}$$

where $x \vee y = \max \{x, y\}$, $x \wedge y = \min \{x, y\}$.

We now give a precise definition of $\mathfrak{L}$ and its domain.

Definition. $f(x) \in \mathfrak{D}(\mathfrak{L})$ and $\mathfrak{L}f(x) = \ell(x)$ in an interval $(a, b) \subseteq R$ (if no interval is specified, $(a, b) = R$) if $\ell(x) \in C[(a, b)]$ and

$$(A3) \quad \begin{aligned} & \text{(i) } f(x) \in C[(a, b)], \\ & \text{(ii) The one-sided derivatives } (df/ds)(x_{\pm}) = \\ & \lim_{y \rightarrow x} [f(y) - f(x)]/[s(y) - s(x)] \text{ (} y < x, y > x \text{) exist for } a < x_{\pm} < b, \\ & \text{(iii) For } a < x_1 < x_2 < b, \end{aligned}$$

$$(A4) \quad \frac{df}{ds}(x_2 \pm) - \frac{df}{ds}(x_1 \pm) = \int_{x_1 \pm}^{x_2 \pm} \ell(y) m(dy).$$

We define $f(x) \in \mathfrak{D}(\mathfrak{L}, L^1)$ if (A3), (A4) hold where $\ell(x)$ is locally $m(dx)$ -integrable in (a, b) . Condition (iii) is less restrictive than it may appear, since if $\ell(x) \in C[(a, b)]$ it is equivalent to

$$(iii)' \quad \frac{df}{ds}(x+) - \frac{df}{ds}(x-) = \ell(x)m(\{x\}), \quad a < x < b, \quad \text{and}$$

$$(iii)'' \quad \text{If } m(\{x\}) = 0, \lim_{y \rightarrow x} \left[\frac{df}{ds}(y \pm) - \frac{df}{ds}(x) \right] / (m(y) - m(x))$$

(two-sided limit) $= \ell(x)$, where $m(y) = m(x_1, y]$, some $x_1 \in (0, x)$.

Given some $h(x) \in L^1(a, b)$, set

$$f(x) = - \int_a^b g_{ab}(x, y) h(y) m(dy).$$

Then by explicit calculation, $f(x) \in \mathfrak{D}(\mathfrak{L}, L^1)$ and $\mathfrak{L}f(x) = h(x)$ in (a, b) ; also, if $f(x) \in \mathfrak{D}(\mathfrak{L}, L^1)$ and $\mathfrak{L}f(x) = 0$, then $f(x) = C_1 s(x) + C_2$ by (A3), (A4). These remarks imply that any "classical" solution of (1.3) is also a weak solution:

Theorem A1. Suppose $u(x, t) \in C(R \times (0, T))$ is continuously differentiable in t ; assume also $u(x, t) \in \mathfrak{D}(\mathfrak{L})$ for each t , $0 < t < T$, and $u_t(x, t) \equiv \mathfrak{L}u(x, t)$. Then, $u(x, t)$ is also a weak solution of this equation in $R \times (0, T)$.

Proof. Assume $a, b, x \in R$, $a < x < b$. Then by the above remarks

$$u(x, t) = - \int_a^b g_{ab}(x, y) \mathfrak{L}u(y, t) m(dy) + C_1(t)s(x) + C_2(t)$$

where $C_1(t)$, $C_2(t)$ can be evaluated by setting $x = a$, $x = b$. If we substitute $\mathfrak{L}u(y, t) = u_t(y, t)$ and integrate in t from t_1 at t_2 , we obtain (A1) identically in a, x, b, t_1, t_2 .

We define $\mathfrak{D}(\mathfrak{L}_1)$ and $\mathfrak{D}(\mathfrak{L}_1, L^1)$ as before (see (2.3)), with (A4) replaced by

$$(A5) \quad \frac{df}{ds}(x_2) - \frac{df}{ds}(x_1) - \int_{x_1}^{x_2} f(y)k(dy) = \int_{x_1}^{x_2} \mathfrak{L}_1 f(y)m(dy).$$

A weak solution of $u_t = \mathfrak{L}_1 u$ is defined as in (A1) with the coefficients of the last integrals replaced by the appropriate linear combinations of $h_{\pm}(x)$ (see (2.4), (11.4); i.e. so that (A1) is formally continuous in x) and

$$g_{ab}(x, y) = \frac{[h_+(x \wedge y) - \alpha h_-(x \wedge y)][h_-(x \vee y) - \beta h_+(x \vee y)]}{K(1 - \alpha\beta)}.$$

Here α, β are chosen so that $g_{ab}(a, y) = g_{ab}(b, y) \equiv 0$, and K is the Wronskian

$$(A6) \quad K = \frac{dh_+}{ds}(x)h_-(x) - \frac{dh_-}{ds}(x)h_+(x) = \text{const.}$$

($K = \text{const.}$ by Lemma A1 below, since $\mathfrak{L}_1 h_{\pm}(x) = 0$. If $K = 0$, $(d/ds)(h_+(x)/h_-(x)) = 0$ and $h_+(x) = Ch_-(x) = \text{const.}$, since $h_+ \uparrow, h_- \downarrow, h_+(x) > 0, h_-(x) > 0$. But then (A5) ($\mathfrak{L}_1 h_{\pm}(x) = 0$) implies $k(R) = 0$, and $K > 0$ whenever $k(R) > 0$.)

A useful identity for $\mathfrak{L}$ or $\mathfrak{L}_1$ is

Lemma A1. For any two functions $f(x), g(x) \in \mathfrak{D}(\mathfrak{L}_1, L^1)$,

$$(A7) \quad \int_{a^{\pm}}^{b^{\pm}} [f(y)\mathfrak{L}_1 g(y) - g(y)\mathfrak{L}_1 f(y)]m(dy) = \left[f(y) \frac{dg}{ds}(y) - g(y) \frac{df}{ds}(y) \right]_{a^{\pm}}^{b^{\pm}}$$

for all $0 < a < b < 1$, where $[h(y)]_a^b = h(b) - h(a)$.

Proof. By (A5) and two integrations by parts,

$$(A8) \quad \begin{aligned} \int_a^b f(y)\mathfrak{L}_1 g(y)m(dy) &= \left[f(y) \left(\frac{dg}{ds}(y) - \int_a^y g(z)k(dz) \right) \right]_a^b \\ &\quad - \int_a^b \frac{df}{ds}(y) \left[\frac{dg}{ds}(y) - \int_a^y g(z)k(dz) \right] ds(y) \\ &= \left[f(y) \frac{dg}{ds}(y) \right]_a^b - \int_a^b \frac{df}{ds}(y) \frac{dg}{ds}(y) ds(y) \\ &\quad - \int_a^b f(y)g(y)k(dy) \end{aligned}$$

from which (A7) follows.

We will also need the following lemma.

Lemma A2. Let $g(t)$ be a bounded Borel function and $\mu(ds)$ a Borel measure on $[0, T)$. Set $f(t) = \int_0^{t-} g(t-s)\mu(ds)$. Then

$$\int_0^t \int_0^{t-s} g(u) ds \mu(ds) = \int_0^t f(s) ds = \int_0^t g(t-s)\mu(s) ds$$

where $\mu(t) = \int_{0-}^t \mu(ds)$.

Proof.

$$\begin{aligned} \int_0^t f(s) ds &= \int_0^t \int_0^s g(s-u) \mu(du) ds = \int_0^t \int_u^t g(s-u) ds \mu(du) \\ &= \int_0^t \int_0^{t-u} g(s) ds \mu(du) = \int_0^t \int_0^{t-s} \mu(du) g(s) ds \end{aligned}$$

by several applications of Fubini's theorem.

In the following, we restrict ourselves to $\mathfrak{L}$, although the arguments hold equally well for $\mathfrak{L}_1$. Assume $u(x, t)$, $g(x, t)$ are both locally $m(dx) \times dt$ -integrable and locally absolutely continuous in t for $(x, t) \in (a, b) \times (0, T)$, with u_t , g_t also locally integrable; assume also $u(x, t)$, $g(x, t) \in \mathfrak{D}(\mathfrak{L}, L^1)$ in (a, b) (or can be so modified on a null set in x) for a.e. $t \in (0, T)$. Finally, assume that for a.e. t , u , $g \in C([a, b])$ (or can be so modified on a null set) with $\mathfrak{L}u$, $\mathfrak{L}g \in L^1(a, b)$. Then by Lemma A1, for $0 < c < t < T$,

$$\begin{aligned} \text{(A9)} \quad \int_c^t \int_a^b [g(y, s)(\mathfrak{L}u - u_s) - u(y, s)(\mathfrak{L}g + g_s)] m(dy) ds \\ = \int_c^t \left[\frac{du}{ds}(y, \theta) g(y, \theta) - u(y, \theta) \frac{dg}{ds}(y, \theta) \right]_a^b d\theta \\ - \int_a^b [u(y, t)g(y, t) - u(y, c)g(y, c)] m(dy). \end{aligned}$$

Given $0 < a < b < 1$, let $p_{ab}(t, x, y)$ be the fundamental solution of the equation $u_t = \mathfrak{L}u$ in (a, b) with zero boundary conditions. Then

$$\text{(A10)} \quad p_{ab}(t, x, y) = \sum_1^\infty e^{-\lambda_n t} \alpha_n(x) \alpha_n(y)$$

where $\{\alpha_n(x)\}$ are the normalized eigenfunctions of the Fredholm equation

$$\alpha_n(x) = \lambda_n \int_a^b g_{ab}(x, y) \alpha_n(y) m(dy).$$

One has $\lambda_n > 0$ and $\Sigma(1/\lambda_n^2) < \infty$; also, $|\alpha_n(x)| \leq C_{ab} \lambda_n$. Hence for fixed x , $g(y, s) = p_{ab}(t + \epsilon - s, y, x)$ satisfies the above hypotheses in $[a, b]$ for any $\epsilon > 0$, and by letting $\epsilon \rightarrow 0$ in (A9)

$$\begin{aligned} \text{(A11)} \quad u(x, t) &= \int_c^t u(a, \theta) \frac{dp_{ab}}{ds}(t - \theta, a, x) d\theta \\ &\quad - \int_c^t u(b, \theta) \frac{dp_{ab}}{ds}(t - \theta, b, x) d\theta \\ &\quad + \int_a^b p_{ab}(t - c, x, y) u(y, c) m(dy) \\ &\quad + \int_c^t \int_a^b p_{ab}(t - \theta, y, x) (u_t - \mathfrak{L}u)(y, \theta) m(dy) d\theta, \text{ a.e. } x. \end{aligned}$$

The first two integrals in this familiar Green's identity can be interpreted probabilistically. Let r_t be the diffusion process in R generated by $\mathcal{L}$, and set $\tau_a = \inf \{t : r_t = a\}$ ($\tau_a = \infty$ if $r_t \neq a$ for all t) for $a \in \bar{R}$. If $\lambda > 0$, it is standard that

$$\varphi(x) = \varphi_-(x, \lambda) = E_x(e^{-\lambda \tau_a} \chi_{\{\tau_a < \tau_b\}}),$$

where " E_x " means probabilistic expectation given $r_0 = x$ and χ is the characteristic function, $\varphi \in \mathcal{D}(\mathcal{L})$ and is a solution of

$$\mathcal{L}\varphi(x) = \lambda\varphi(x), \quad \varphi(a) = 1, \quad \varphi(b) = 0$$

(see Breiman [3], Ito-McKean [13, Ch. 4]). Hence $u(x, t) = e^{\lambda t} \varphi(x)$ is a solution of $u_t = \mathcal{L}u$ and by (A11)

$$e^{\lambda t} \varphi(x) = \int_0^t e^{\lambda \theta} \frac{dp_{ab}}{ds}(t - \theta, a, x) d\theta + \int_a^b p_{ab}(t, x, y) \varphi(y) m(dy).$$

Multiplying by $e^{-\lambda t}$ and letting $t \rightarrow \infty$

$$\varphi(x) = \varphi_-(x, \lambda) \equiv \int_0^\infty e^{-\lambda t} \frac{dp_{ab}}{ds}(t, a, x) dt.$$

Hence by the uniqueness of the Laplace transformation,

$$P_x(\tau_a \leq t, \tau_a < \tau_b) \equiv \int_0^t \frac{dp_{ab}}{ds}(\theta, a, x) d\theta$$

where " P_x " means "probability given $r_0 = x$ ". Hence, if $u(x, t)$ is a solution of $u_t = \mathcal{L}u$ in the above sense, (A11) can be rewritten

$$\begin{aligned} \text{(A12)} \quad u(x, t) &= \int_0^{t-c} u(a, t - \theta) P_x(\tau_a \in d\theta, \tau_a < \tau_b) \\ &\quad + \int_0^{t-c} u(b, t - \theta) P_x(\tau_b \in d\theta, \tau_b < \tau_a) \\ &\quad + \int_a^b p_{ab}(t - c, x, y) u(y, c) m(dy). \end{aligned}$$

We can now state the main result of the Appendix:

Theorem A2. *Let $u(x, t)$ be a locally integrable function in $R \times (0, T)$. Then, $u(x, t)$ is a weak solution of (1.3) in $R \times (0, T)$ if and only if $u(x, t)$ satisfies (A12) for $m(dx)$ -a.e. (a, x, b) with $0 < a < x < b < 1$ and a.e. (c, t) with $0 < c < t < T$. (Here local integrability is with respect to $m(dx)dt$.)*

Proof. ($\Rightarrow$) Let $v(x, t) = \int_c^t u(x, \theta) d\theta$. Then, assuming (A1), for a.e. (c, t) , $v(x, t)$ can be changed on a null set so that $v(x, t) \in \mathcal{D}(\mathcal{L}, L^1)$ with $\mathcal{L}v(x, t) = u(x, t) - u(x, c)$. Clearly $v(x, t)$ is locally absolutely continuous in t for a.e. x , and v satisfies $(v_t - \mathcal{L}v)(x, t) = u(x, c)$. Hence by (A11)

$$\begin{aligned}
 (A13) \quad v(x, t) = & \int_0^{t-c} v(a, t - \theta) P_x(\tau_a \leq \theta) d\theta, \tau_a < \tau_b \\
 & + \int_0^{t-c} v(b, t - \theta) P_x(\tau_b \leq \theta) d\theta, \tau_b < \tau_a \\
 & + \int_0^{t-c} \int_a^b p_{ab}(\theta, x, y) u(y, c) m(dy) d\theta.
 \end{aligned}$$

Since it is sufficient to assume $c = 0$ in Lemma A2, (A13) is the indefinite integral in t of equation (A12). Hence (A12) follows for *a.e.* (a, x, b, c, t) by taking the derivative.

($\Leftarrow$). (A12) is equivalent to (A11) without the final term. However, by results in §10 above, the right-hand side of (A11) is a weak solution of $u_t = \mathcal{L}u$ in $(a, b) \times (c, T)$. Now choose (a_n, b_n, c_n) such that (A12) holds for *a.e.* $(x, t) \in (a_n, b_n) \times (c_n, T)$ and $c_n \downarrow 0$, $(a_n, b_n) \uparrow R$. Thus $u(x, t)$ is *a.e.* equal to a weak solution in $R \times (0, T)$.

Remarks. (1) Thus, modulo some smoothness, $u(x, t)$ is a weak solution of $u_t = \mathcal{L}u$ iff it is a “parabolic function” in the sense of Doob [6]. (See also Remark (3) below and Lai [14].)

(2) By results of Lai [14], the right-hand side of (A12) is a continuous function of $(x, t) \in (a, b) \times (c, T)$. Hence a weak solution can always be modified on a null set in $R \times (0, T)$ to be a continuous function in $R \times (0, T)$. An inspection of the proofs of Theorem A2 and Theorem 10.1 above then shows

Corollary A1. *A weak solution can always be modified on a null set to be continuous in $R \times (0, T)$; once modified, (A1) and (A12) hold identically in (a, x, b, c, t) and (a, x, b, t_1, t_2) .*

Remark. (3) Since r_t is Markovian, (A12) is essentially the property that $y_t = u[r(t \wedge \tau_a \wedge \tau_b), T - t \wedge \tau_a \wedge \tau_b]$ ($0 < t < T$) is a martingale. Hence by optimal stopping, so is $z_t = u[r(t \wedge \beta), T - t \wedge \beta]$ for any stopping time β such that for some a, b with $0 < a < r_0 < b < 1$, $\beta \leq \tau_a \wedge \tau_b$. In this sense $u(r_t, T - t)$ is a local martingale.

FOOTNOTES

1. The condition $c(x) \leq M$ can be removed if there exists some $g(x) \geq 0$ in R such that $\mathcal{L}g = \lambda g$ for some λ . (Feller [7].)
2. Either side of (2.1) can be modified on a null set in $R \times (0, T)$ to be continuous in $R \times (0, T)$; see the Appendix and §10.

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