On Some Paradoxes Concerning Two-Dimensional Stokes Flow Past an Obstacle

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1. Introduction. In the preceding paper [2] some problems for both the stationary and nonstationary Stokes equations are proved to be uniquely solvable within a certain class of generalized solutions. The results are the same for any number of spacial dimensions $n \geq 2$. In this paper we examine some paradoxes concerning these solutions which occur when $n = 2$; one is the Stokes paradox and the others are related to it but concern nonstationary problems. These paradoxes are all obtained as consequences of a single lemma which asserts that when $n = 2$ the function space H_0 that defines the class of solutions being considered contains elements which equal nonzero constants in a neighborhood of infinity. One may infer from the lemma that the boundary condition at infinity is not strictly imposed on solutions of this class when $n = 2$. Roughly speaking the paradoxes concern particular violations of the condition at infinity.

The Stokes paradox of hydrodynamics states that it is impossible to solve the Stokes equations in a two-dimensional exterior domain for a time independent solution which vanishes on the boundary and which tends to a nonzero limit at infinity. In Section 3 we consider a class of generalized solutions of the two-dimensional exterior stationary problem which includes the class H_0 solutions of [2]. Theorem 1 is a version of the Stokes paradox; it states that the unique generalized solution of this class vanishes identically. We also consider in Section 3 the initial boundary value problem for a two-dimensional exterior domain with an initial velocity which tends to a nonzero limit at infinity. Theorem 2 states that the class H_0 solution of this nonstationary problem converges to zero in every finite region as $t \rightarrow \infty$.

In Section 4 the initial boundary value problem for a two-dimensional exterior domain is studied with nonzero prescribed forces. Theorem 3 states that there exist square-summable forces such that the unique class H_0 solution of this problem violates the prescribed condition at infinity. This is particularly surprising because there also exists a unique finite energy solution of the problem which satisfies the condition at infinity.

2. Preliminary lemma.

Lemma. *There exists a sequence of functions $\{\omega_k\}$ mapping $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ such that (i) each ω_k is continuous, piece-wise continuously differentiable and solenoidal, (ii) $\omega_k(x) = (1, 0)$ for $|x| \leq e^k$ and $\omega_k(x) = (0, 0)$ for $|x| \geq e^{2k}$, (iii) $\|\nabla \omega_k\|^2 \leq C/k$ for some constant C , and (iv) there is a bound independent of x and k for the values of $|\omega_k(x)|$.*

Proof. We seek ω_k in the form $\omega_k(x) = (\partial(x_2\phi_k(r))/\partial x_2, -\partial(x_2\phi_k(r))/\partial x_1)$ where $r = |x|$ and where $\phi_k(r) : [0, \infty) \rightarrow [0, 1]$ will be defined below in such a way as to be continuously differentiable, piece-wise twice continuously differentiable, equal to one for $r \in [0, e^k]$, and equal to zero for $r \in [e^{2k}, \infty)$. Thus ω_k is assured of being continuous, piece-wise continuously differentiable, solenoidal, equal to $(1, 0)$ for $|x| < e^k$, and equal to $(0, 0)$ for $|x| > e^{2k}$. Further, we will define ϕ_k so that $|\phi_k''(r)| < C/kr^2$ and $|\phi_k'(r)| < C$ for $r \in [e^k, e^{2k}]$. This ensures that $|\partial\phi_k(r)/\partial x_i| < C/kr$ and that $|x_2\partial^2\phi_k(r)/\partial x_i\partial x_j| < C/kr$ for $e^k \leq |x| \leq e^{2k}$. An easy computation shows that

$$\begin{aligned} \|\nabla \omega_k\|^2 &\leq C \sum_{i=1}^2 \left\| \frac{\partial}{\partial x_i} \phi_k(r) \right\|^2 + C \sum_{i,j=1}^2 \left\| x_2 \frac{\partial^2}{\partial x_i \partial x_j} \phi_k(r) \right\|^2 \\ &\leq C \int_0^{2\pi} \int_{e^k}^{e^{2k}} \left(\frac{1}{kr} \right)^2 r \, dr d\theta = C/k. \end{aligned}$$

Furthermore the values of $|\omega_k(x)|$ are bounded:

$$\begin{aligned} |\omega_k(x)| &= \left| \left(x_2 \frac{\partial \phi_k}{\partial x_2} + \phi_k, -x_2 \frac{\partial \phi_k}{\partial x_1} \right) \right| \\ &\leq |r\phi_k'(r)| + |\phi_k(r)| \leq C/k + 1. \end{aligned}$$

It is easy to check that $\phi_k(r)$ will have all the properties mentioned above if we define it by setting $\phi_k(r) = 2 - k^{-1} \log \rho_k(r)$ where $\rho_k : [0, \infty) \rightarrow [e^k, e^{2k}]$ is a continuously differentiable, piece-wise twice continuously differentiable function satisfying (i) $\rho_k(r) = e^k$ for $r \in [0, e^k]$, (ii) $|r\rho_k''(r)| \leq 2$, $|\rho_k'(r)| \leq 4$, and $|r^{-1}\rho_k(r)| \geq 2 \log(3/2)$ for $r \in [e^k, e^{2k}]$, and (iii) $\rho_k(r) = e^{2k}$ for $r \in [e^{2k}, \infty)$.

We now construct, at least for $k \geq 4$, functions $\rho_k(r)$ having all the properties mentioned. Let $f_k(r)$ be the solution of $f_k''(r) = 2/r$ on $[e^k, e^{k+2}]$ which satisfies $f_k(e^k) = e^k$ and $f_k'(e^k) = 0$. Let $g_k(r)$ be the solution of $g_k''(r) = -2/r$ on $[e^{2k-2}, e^{2k}]$ which satisfies $g_k(e^{2k}) = e^{2k}$ and $g_k'(e^{2k}) = 0$. It is easily checked that $f_k'(e^{k+2}) = g_k'(e^{2k-2}) = 4$ and that $f_k(e^{k+2}) > g_k(e^{2k-2})$. It follows that the graphs of f_k and g_k have a common tangent line ℓ_k meeting f_k at a point with r coordinate $a_k \in [e^k, e^{k+2}]$ and g_k at a point with r coordinate $b_k \in [e^{2k-2}, e^{2k}]$; see figure. The slope of ℓ_k is less than 4. We define $\rho_k(r)$ by setting $\rho_k(r) = e^k$ for $r \in [0, e^k]$, $\rho_k(r) = f_k(r)$ for $r \in [e^k, a_k]$, $\rho_k(r) = \ell_k(r)$ for $r \in [a_k, b_k]$, $\rho_k(r) = g_k(r)$ for $r \in [b_k, e^{2k}]$, and $\rho_k(r) = e^{2k}$ for $r \in [e^{2k}, \infty)$. The functions ρ_k , $k \geq 4$, have all the required properties; in particular $|r^{-1}\rho_k(r)| \geq 2 \log(3/2)$ for $r \in [e^k, e^{2k}]$ because $f'(r) = r^{-1}f(r)$ only when $r = 3e^k/2$ and $f'(3e^k/2) = 2 \log(3/2) < 1$.

Except where otherwise qualified Ω will always denote a domain in $\mathbf{R}^2$ which contains a complete neighborhood of infinity and whose complement Ω^c contains an arc. To see why it is important that Ω^c should contain an arc (or at least have nonzero capacity) consider the principal function spaces

$$\mathbf{D}(\Omega) = \{\phi : \phi \in \mathbf{C}_0^\infty(\Omega) \text{ and } \nabla \cdot \phi = 0\},$$

$$\mathbf{J}_0(\Omega) = \text{Completion of } \mathbf{D}(\Omega) \text{ in the norm } \|\nabla \cdot\|$$

used in the preceding paper [2], and earlier in [3], to study the exterior stationary problem. The boldface type here indicates that functions are $\mathbf{R}^2$ -valued and $\|\cdot\|$ denotes an L^2 norm over Ω ; we use the same notation as in [2]. Suppose now we allow $\Omega = \mathbf{R}^2$ so that spaces $\mathbf{D}(\mathbf{R}^2)$ and $\mathbf{J}_0(\mathbf{R}^2)$ come under consideration. By using averaging convolutions it may be shown that the functions ω_k constructed in the lemma belong to $\mathbf{J}_0(\mathbf{R}^2)$. Now as elements of the Hilbert space $\mathbf{J}_0(\mathbf{R}^2)$ the sequence $\{\omega_k\}$ converges to zero, but as a sequence of functions $\{\omega_k\}$ converges to the constant function $(1, 0)$. Evidently elements of $\mathbf{J}_0(\mathbf{R}^2)$ are equivalence classes of functions which differ by constants. By assuming Ω^c contains an arc we ensure that elements of $\mathbf{J}_0(\Omega)$ can be identified with functions because elements of $\mathbf{D}(\Omega)$ then satisfy an inequality $\|\phi\|_{L^2(\Omega')} \leq C_{\Omega'} \|\nabla \phi\|$ for every bounded subset Ω' of Ω with a constant $C_{\Omega'}$ depending only on Ω' ; see [2].

We will now show, in contrast to the case of $\Omega \subset \mathbf{R}^n$ with $n > 2$, that functions in $\mathbf{J}_0(\Omega)$ need not tend to zero at infinity. In this paper as in [2] $\mathbf{b}$ will always denote a function having the properties listed in the following corollary.

Corollary 1. $J_0(\Omega)$ contains every function $\mathbf{b}$ which is smooth, solenoidal, equal to zero in a neighborhood of $\partial\Omega$, and equal to a constant vector in a neighborhood of infinity.

Proof. Without loss of generality we may assume $\mathbf{b} = (1, 0)$ in a neighborhood of infinity. Let $\mathfrak{g}_k(x) = \mathbf{b}(x) + \omega_k(x) - (1, 0)$ where $\{\omega_k\}$ is the sequence of functions constructed in the lemma. Using averaging convolutions it may be seen that $\mathfrak{g}_k$ belongs to $J_0(\Omega)$ if k is sufficiently large that the support of $\mathfrak{g}_k$ lies in Ω . Thus we obtain a Cauchy sequence in $J_0(\Omega)$ which converges to, and may be identified with, the function $\mathbf{b}$.

We prove the following corollary in order to further indicate the possible behavior at infinity of functions in $J_0(\Omega)$.

Corollary 2. There exist functions ψ in $J_0(\Omega)$ such that $|\psi(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$.

Proof. Let $\gamma_k(x) = (1, 0) - \omega_k(x)$ where $\{\omega_k\}$ is the sequence constructed in the lemma. Let p be a nonnegative integer which is large enough so that $\{x : |x| \geq e^{2^p}\} \subset \Omega$ and set $\psi = \sum_{k=p}^{\infty} \gamma_{2^k}$. By using averaging convolutions and appealing to Corollary 1 it can be seen that each term γ_{2^k} of the series belongs to $J_0(\Omega)$. The series converges in $J_0(\Omega)$ because $\sum_{k=p}^{\infty} \|\nabla \gamma_{2^k}\| = \sum_{k=p}^{\infty} \|\nabla \omega_{2^k}\| \leq C \sum_{k=p}^{\infty} 2^{-k/2} < \infty$.

It remains to be shown that $|\psi(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$. To this end we define an integer valued function $m(x)$ on $\{x : |x| \geq e^{2^p}\}$ by the condition $e^{2^{m(x)}} \leq |x| < e^{2^{m(x)+1}}$. To evaluate $|\psi(x)|$ at a given point x note that $\gamma_{2^k}(x) = (1, 0)$ for k satisfying $p \leq k \leq m(x) - 1$, and that $\gamma_{2^k}(x) = (0, 0)$ for $k \geq m(x) + 1$. Thus $\psi(x) = (m(x) - p)(1, 0) + \gamma_{2^{m(x)}}(x)$. Because the values of $|\omega_k(x)|$ are bounded we have $|\gamma_{2^{m(x)}}(x)| < C$ and therefore $|\psi(x)| \geq m(x) - p - C$ for some constant C . Obviously $m(x) \rightarrow \infty$ as $|x| \rightarrow \infty$; in fact it is easily checked that $m(x)$ and hence $|\psi(x)|$ grow like $\log(\log |x|)$.

We have shown that elements of $J_0(\Omega)$ can grow at infinity. In [2] we pointed out that such functions must satisfy $|\phi(x)| = o(\log |x|)$ at infinity in a generalized sense.

3. Stokes paradox. The exterior stationary problem for the Stokes equations is that of finding a solution $\mathbf{w}(x)$, $p(x)$ of

$$(1) \quad \Delta \mathbf{w} = \nabla p \quad \text{in } \Omega$$

$$(2) \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega$$

$$(3) \quad \mathbf{w} = 0 \quad \text{on } \partial\Omega$$

$$(4) \quad \mathbf{w}(x) \rightarrow \mathbf{w}_{\infty} \quad \text{as } |x| \rightarrow \infty$$

where $\mathbf{w}_{\infty}$ is a prescribed constant vector. Chang and Finn [1] have proved a very strong form of the Stokes paradox: *The only classical solution of (1)–(3) which satisfies $|\mathbf{w}(x)| = o(\log |x|)$ at infinity is $\mathbf{w} \equiv 0$.* The considerations

which follow are much simpler than those in [1] but yield nearly the same result in a form conveniently applied to certain generalized solutions.

If $\Omega \subset \mathbf{R}^n$ with $n > 2$, a solution of (1)–(4) can be found in the form $\mathbf{w} = \bar{\mathbf{w}} + \mathbf{b}$ where $\bar{\mathbf{w}} \in \mathbf{J}_0(\Omega)$ and where $\mathbf{b}$ is a function which has the properties listed in Corollary 1 and which equals $\mathbf{w}_\infty$ in a neighborhood of infinity. When $\Omega \subset \mathbf{R}^2$ the same method of solution can be followed [2] and it still leads to a unique “generalized solution” of this form. The following theorem shows that the condition at infinity is lost on this “generalized solution”.

Theorem 1. *The only solution of (1)–(3) having the form $\mathbf{w} = \bar{\mathbf{w}} + \mathbf{b}$ with $\bar{\mathbf{w}} \in \mathbf{J}_0(\Omega)$ and $\mathbf{b}$ as described above is $\mathbf{w} \equiv 0$.*

Proof. We need only assume (1) is satisfied in the weak sense: $\int_\Omega \nabla \mathbf{w} : \nabla \phi \, dx = 0$ for all $\phi \in \mathbf{D}(\Omega)$. Because $\mathbf{b}$ belongs to $\mathbf{J}_0(\Omega)$, by Corollary 1, so does $\mathbf{w}$. Since $\mathbf{D}(\Omega)$ is dense in $\mathbf{J}_0(\Omega)$ we get $\int_\Omega (\nabla \mathbf{w})^2 \, dx = 0$ and therefore $\mathbf{w} \equiv 0$.

Actually when Ω is a two-dimensional domain it is reasonable to seek the solution of (1)–(4) in the space $\mathbf{J}_0(\Omega)$, because then functions in $\mathbf{J}_0(\Omega)$ not only satisfy conditions (2) and (3) but also have sufficient freedom at infinity to satisfy (4). Functions in $\mathbf{J}_0(\Omega)$ necessarily have a finite Dirichlet integral, but this can probably be expected of $\mathbf{w}$ anyway since (1) is the Euler equation for the functional $\int_\Omega (\nabla \mathbf{w})^2 \, dx$ defined on solenoidal functions. In view of these considerations the following assertion is a form of the Stokes paradox. *The only function $\mathbf{w} \in \mathbf{J}_0(\Omega)$ with second derivatives in $\mathbf{L}^2_{1,0}(\Omega)$ satisfying $\Delta \mathbf{w} = \nabla p$ is $\mathbf{w} \equiv 0$.* This may be verified by noting that $\mathbf{w}$ satisfies (1) in the weak sense, and hence that $\int_\Omega (\nabla \mathbf{w})^2 \, dx = 0$ as in the proof of Theorem 1.

It is interesting to study the nonstationary solution $\mathbf{v}(x, t)$, $p(x, t)$ of

$$(5) \quad \mathbf{v}_t - \Delta \mathbf{v} = -\nabla p \quad \text{in } Q_\infty$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } Q_\infty$$

$$\mathbf{v}(x, 0) = \mathbf{b}(x) \quad \text{in } \Omega$$

$$(6) \quad \mathbf{v}(x, t) = 0 \quad \text{on } \partial\Omega \times [0, \infty)$$

$$\mathbf{v}(x, t) \rightarrow \mathbf{w}_\infty \quad \text{as } |x| \rightarrow \infty$$

as time $t \rightarrow \infty$. Here $Q_\infty = \Omega \times [0, \infty)$ and $\mathbf{b}$ again has the properties listed in Corollary 1 and equals $\mathbf{w}_\infty$ in a neighborhood of infinity. For a domain Ω which is the exterior of a circle Rusanov [4] has obtained an explicit series solution of (5), (6) and shown that it satisfies $|\mathbf{v}(x, t)| < C \log R / \log t$ for $|x| < R$ as $t \rightarrow \infty$. For an arbitrary two dimensional exterior domain and for the class of solutions studied in [2] we have:

Theorem 2. *The unique class $\mathbf{H}_0$ solution of (5), (6) converges to zero in $\mathbf{L}^2_{1,0}(\Omega)$ as $t \rightarrow \infty$.*

In [2] we proved that the class $\mathbf{H}_0$ solution of (5), (6) converges to the class

$\mathbf{H}_0$ solution of (1)–(4) in $L^2_{1,0,0}(\Omega)$ as $t \rightarrow \infty$. Because Theorem 1 applies to class $\mathbf{H}_0$ solutions, Theorem 2 follows from it. Alternatively we may apply Theorems 1, 2, and 4 of [2] to get the same result because we have shown that $\mathbf{b} \in \mathbf{J}_0(\Omega)$ and therefore that $\mathbf{b}$ satisfies the hypothesis made of the initial data in those theorems.

4. A disparity between solution classes. In formulating a notion of generalized solution for the exterior problem

$$(7) \quad \mathbf{u}_t - \Delta \mathbf{u} = -\nabla p + \mathbf{f} \quad \text{in } Q_T = \Omega \times (0, T)$$

$$(8) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } Q_T$$

$$(9) \quad \mathbf{u}(x, 0) = 0 \quad \text{in } \Omega$$

$$(10) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T)$$

$$(11) \quad \mathbf{u}(x, t) \rightarrow 0 \quad \text{on } |x| \rightarrow \infty,$$

and for similar problems, it is convenient to impose the condition at infinity (11), as well as conditions (8) and (10), by requiring $\mathbf{u}$ to belong to appropriate function spaces. We cite two possibilities.

More or less following [3] let $\mathbf{J}_1(Q_T)$ be the completion of $\mathbf{D}(Q_T) = \{\phi : \phi \in C_0^\infty(Q_T) \text{ and } \nabla \cdot \phi = 0\}$ in the norm $(\|\phi\|_{Q_T^2} + \|\nabla \phi\|_{Q_T^2})^{1/2}$, where $\|\cdot\|_{Q_T}$ denotes an L^2 norm over Q_T . Also, let $\mathbf{J}(Q_T)$ be the closure of $\mathbf{D}(Q_T)$ in $L^2(Q_T)$. Elements of $\mathbf{J}_1(Q_T)$ satisfy (8), (10), and (11) in a reasonable generalized sense; condition (11) is satisfied at least in the sense that elements belong to $L^2(Q_T)$. Therefore $\mathbf{u}$ is called a *finite energy solution* of (7)–(11) if, for some scalar p , $\mathbf{u}$ is a solution (or weak solution) of (7) such that $\mathbf{u} \in \mathbf{J}_1(Q_T)$, $\mathbf{u}_t \in \mathbf{J}(Q_T)$, and $\|\mathbf{u}(t)\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow 0^+$. It is easy to show that problem (7)–(11) possesses a unique finite energy solution if $\mathbf{f} \in L^2(Q_T)$.

Another procedure was used in the preceding paper [2] to study certain problems with $\mathbf{f} \notin L^2(Q_T)$. Letting $\mathbf{J}_0(Q_T)$ denote the completion of $\mathbf{D}(Q_T)$ in norm $\|\nabla \phi\|_{Q_T}$ we defined $\mathbf{K}_0(Q_T)$ to be the set of all $\mathbf{u} \in \mathbf{J}_0(Q_T)$ such that $\int_{Q_T} \nabla \mathbf{u} : \nabla \phi = -\int_{Q_T} \mathbf{f} \cdot \phi$ for some $\mathbf{f} \in \mathbf{D}(Q_T)$ and all $\phi \in \mathbf{J}_0(Q_T)$. Noting that $\mathbf{f}$ is uniquely determined by $\mathbf{u}$ we set $\tilde{\Delta} \mathbf{u} = -\mathbf{f}$ and defined $\mathbf{H}_0(Q_T)$ to be the completion of $\mathbf{K}_0(Q_T)$ in norm $(\|\nabla \mathbf{u}\|_{Q_T^2} + \|\tilde{\Delta} \mathbf{u}\|_{Q_T^2})^{1/2}$. At least when $\Omega \subset \mathbb{R}^n$ with $n > 2$ elements of $\mathbf{H}_0(Q_T)$ satisfy (8), (10), and (11) in a reasonable generalized sense. Therefore we called $\mathbf{u}$ a class $\mathbf{H}_0$ solution of problem (7)–(11) if, for some scalar p , $\mathbf{u}$ is a solution of (7) such that $\mathbf{u} \in \mathbf{H}_0(Q_T)$, $\mathbf{u}_t \in \mathbf{J}_0(Q_T)$, and $\|\nabla \mathbf{u}(t)\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow 0^+$.

When $\Omega \subset \mathbb{R}^2$ it may be seen from Corollary 1 that $\mathbf{H}_0(Q_T)$ contains functions which tend continuously to nonzero limits at infinity and that consequently condition (11) is not directly imposed on class $\mathbf{H}_0$ solutions. It is therefore surprising that uniqueness holds within this class of solutions when $\Omega \subset \mathbb{R}^2$ just as when $\Omega \subset \mathbb{R}^n$ with $n > 2$; see [2]. When $\Omega \subset \mathbb{R}^2$ the solution seems to be determined without regard to condition (11); it is apparently determined

through the relation between $\mathbf{H}_0(Q_T)$ and solutions of the exterior stationary problem. In any case this presents a uniqueness paradox similar to the Stokes paradox for stationary solutions.

The question arises of whether, when $\Omega \subset \mathbf{R}^2$, class $\mathbf{H}_0$ solutions of (7)–(11) do in fact satisfy (11). There is also the question of whether, when $\mathbf{f}$ satisfies hypotheses which insure existence in both solution classes, the solutions from each class are necessarily identical. The following theorem answers both questions negatively; there is a paradoxical disparity between the two solution classes.

Theorem 3. *There are forces $\mathbf{f} \in \mathbf{L}^2(Q_T)$ for which the unique class $\mathbf{H}_0$ solution of (7)–(11) tends continuously to a nonzero limit at infinity and therefore differs from the unique finite energy solution.*

Proof. In either the finite energy or $\mathbf{H}_0$ theories two forces are equivalent if they differ by a term $\nabla q \in \mathbf{L}^2_{1,0}(Q_T)$ where q is a scalar. That is, $\mathbf{u}$ is a generalized solution for $\mathbf{f} = \bar{\mathbf{f}} + \nabla q$ if and only if $\mathbf{u}$ is a generalized solution for $\bar{\mathbf{f}}$.

Now suppose $\mathbf{b}(x)$ is a smooth, solenoidal function such that $\mathbf{b} = \mathbf{0}$ in a neighborhood of $\partial\Omega$ and $\mathbf{b} = (1, 0)$ in a neighborhood of infinity. Using Corollary 1 we see that $\mathbf{b}(x)$ and $t\mathbf{b}(x)$ both belong to $\mathbf{H}_0(Q_T)$. Also $\|\nabla(t\mathbf{b})\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow 0^+$. Therefore $\mathbf{u}(x, t) = t\mathbf{b}(x)$ is the class $\mathbf{H}_0$ solution determined by $\bar{\mathbf{f}} = \mathbf{b}(x) - t\Delta\mathbf{b}$. Adding $-\nabla q$ to $\bar{\mathbf{f}}$, where $q = x_1$, gives an equivalent force $\mathbf{f} = [\mathbf{b}(x) - (1, 0)] - t\Delta\mathbf{b}(x)$. The unique class $\mathbf{H}_0$ solution determined by $\mathbf{f}$ is still $\mathbf{u}(x) = t\mathbf{b}(x)$. But $\mathbf{f} \in \mathbf{L}^2(Q_T)$ and so there is also a finite energy solution.

Suppose that instead of (11) we set the inhomogeneous condition at infinity

$$(12) \quad \mathbf{u}(x, t) \rightarrow \alpha(t)\mathbf{w}_\infty \quad \text{as } x \rightarrow \infty$$

where $\mathbf{w}_\infty$ is a constant vector, and $\alpha(t)$ is a smooth scalar function satisfying $\alpha(0) = 0$. Let $\mathbf{b}$ be a function as in Corollary 1 which equals $\mathbf{w}_\infty$ in a neighborhood of infinity. Condition (12) can be imposed by requiring $\mathbf{u}$ to be of the form $\mathbf{u} = \bar{\mathbf{u}} + \alpha(t)\mathbf{b}$, with $\bar{\mathbf{u}} \in \mathbf{J}_1(Q_T)$ in the finite energy theory, or with $\bar{\mathbf{u}} \in \mathbf{H}_0(Q_T)$ in the $\mathbf{H}_0$ theory if $n > 2$. The condition at infinity is completely lost on class $\mathbf{H}_0$ solutions if $n = 2$. For example, if $\mathbf{f} \equiv \mathbf{0}$ one will get $\bar{\mathbf{u}} = -\alpha(t)\mathbf{b}$ and thus $\mathbf{u} \equiv \mathbf{0}$. Again the class $\mathbf{H}_0$ solution is determined without regard to the condition at infinity.

A theory of the heat equation can be developed along lines similar to the $\mathbf{H}_0$ theory of the Stokes equations. In this theory there are analogues of the Stokes paradox. However the disparity between solution classes which is encountered with the Stokes equations does not occur.

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