Extreme Points and Finite Rank Operators in the Set of Compact Approximants

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Communicated by the Editors

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. A *compact approximant* of T is a compact operator K of minimum distance from T ; that is,

$$\|T - K\| = \inf \{\|T - C\| : C \text{ compact}\}.$$

We call this distance μ , the *essential norm* of T [8]. The set $\mathcal{K}_T$ of compact approximants is a closed convex subset of the sphere of radius μ and center T in $\mathcal{B}(\mathcal{H})$. In this note we explicitly describe the extreme points of $\mathcal{K}_T$, and exhibit the structure of $\mathcal{K}_T$ in terms of these extreme points. There may be no extreme points in $\mathcal{K}_T$; however if there is one, then every element of $\mathcal{K}_T$ is a midpoint between two extreme points. Similarly (but under unrelated conditions) there may be no finite rank operators in $\mathcal{K}_T$; but if there is one, then they are uniformly dense in $\mathcal{K}_T$.

These results contribute to the continuing study of non-commutative approximation theory initiated by P. R. Halmos [5]. Also, they focus on the Banach space structure of $\mathcal{B}(\mathcal{H})$ by examining the position of the distinguished closed subspace of compact operators. It was originally shown by I. Gohberg and M. G. Krein in [3] that a compact approximant exists for every $T \in \mathcal{B}(\mathcal{H})$. More general existence theorems are given in [1] concerning M -ideals in Banach spaces, and in [9] for closed two-sided ideals in C^* -algebras. Compact approximants and related topics have also been studied in [6], [7], and [8].

The author wishes to thank William R. Zame and Richard Bourgin for a number of very helpful conversations on this subject.

We write $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, and $\mathcal{K}$ for the closed ideal in $\mathcal{B}(\mathcal{H})$ of compact operators. The map $\nu: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})/\mathcal{K}$ is the natural homomorphism of $\mathcal{B}(\mathcal{H})$ onto the C^* -algebra quotient, the *Calkin algebra*. Let $\sigma(T)$ and $\sigma(\nu(T))$ be the spectra of $T \in \mathcal{B}(\mathcal{H})$ and of $\nu(T)$ in the Calkin algebra, respectively. The operator $(T^*T)^{1/2}$

is denoted by $|T|$. For general facts about Hilbert space operators, the reader is referred to [4].

Theorem 1. *Let $T \in \mathcal{B}(\mathcal{H})$ and let K be a compact approximant of T with $\mu = \|T - K\|$. The following statements are equivalent:*

- (i) K is an extreme point of $\mathcal{K}_T$;
- (ii) $T - K$ is an extreme point of the ball of radius μ in $\mathcal{B}(\mathcal{H})$;
- (iii) $T - K = \mu U$ where U is an isometry or a coisometry.

Proof. It is well known that (ii) and (iii) are equivalent [4, p. 69], and it is easy to see that (ii) implies (i): for, if $K = \lambda K_1 + (1 - \lambda)K_2$ with $K_i \in \mathcal{K}_T$ and $0 \leq \lambda \leq 1$, then $\|T - K_i\| = \mu$ and

$$T - K = \lambda(T - K_1) + (1 - \lambda)(T - K_2).$$

The proof is completed by showing that if $T - K \neq \mu U$, where U is an isometry or coisometry, then K is not an extreme point of $\mathcal{K}_T$. This hypothesis on $T - K$ ensures that both $\sigma(|T - K|)$ and $\sigma(|(T - K)^*|)$ must contain more than the single point μ . Hence if $E(\lambda)$ is the spectral resolution associated with $|T - K|$ and $F(\lambda)$ that of $|(T - K)^*|$, there exists an interval $[a, b]$ with $0 \leq a < b < \mu$, and with $E = E[a, b] \neq 0$ and $F = F[a, b] \neq 0$. Choose unit vectors $x \in E\mathcal{H}$ and $y \in F\mathcal{H}$ and define a rank one operator $C \in \mathcal{B}(\mathcal{H})$ by $Cx = \epsilon y$ and $Cz = 0$ if $\langle z, x \rangle = 0$. For a sufficiently small choice of $\epsilon > 0$ (so that $b + \epsilon < \mu$) we have $K + C$ and $K - C$ both compact approximants for T , and K is the mid-point between these two operators. That is, since $F(T - K) = (T - K)E$ and $\|(T - K)E\| \leq b$, we have

$$\begin{aligned} \|T - (K + C)\| &= \max(\|(T - K - C)E\|, \|(T - K - C)(1 - E)\|) \\ &= \max(b + \epsilon, \mu) \\ &= \mu, \end{aligned}$$

and the same for $K - C$. The proof is complete.

The next result is the observation that an isometry in the Calkin algebra lifts to an isometry or coisometry in $\mathcal{B}(\mathcal{H})$. The given equivalent conditions characterize those $T \in \mathcal{B}(\mathcal{H})$ whose set of compact approximants has extreme points.

Theorem 2. *For an operator $T \in \mathcal{B}(\mathcal{H})$ and the real number $\mu \geq 0$, the following conditions are equivalent (μ will be the essential norm of T):*

- (i) there is a compact K and an isometry or coisometry U with $T - K = \mu U$;
- (ii) $\nu(T)^*\nu(T) = \mu^2$ or $\nu(T)\nu(T)^* = \mu^2$;
- (iii) the spectrum of $(T^*T)^{1/2}$ or of $(TT^*)^{1/2}$ is at most a sequence of isolated eigenvalues of finite multiplicity converging to μ .

Proof. It is obvious that (i) implies (ii). To show (ii) implies (iii), we argue that (iii) describes $\sigma(|T|)$ whenever $\nu(|T|) = \mu I$: since $|T|$ is positive, Weyl's theorem says

$\sigma(|T|) = \bigcap_{K \in \mathcal{K}} \sigma(|T| + K) \cup$ isolated eigenvalues of finite multiplicity.

Now, for any $S \in \mathcal{B}(\mathcal{H})$, $\bigcap_{K \in \mathcal{K}} \sigma(S + K)$ is obtained by filling in some bounded components of the complement of $\sigma(\nu(S))$ [2, 2.4]; in this case $\sigma(\nu(|T|)) = \{\mu\}$, so these sets are equal. Thus the compact set $\sigma(|T|)$ satisfies (iii).

Suppose (iii) holds; without loss of generality, assume $\sigma(|T|)$ is described by (iii). Then there is an orthogonal sequence (E_n) of projections with

$$|T| = \mu E_0 + \sum_{n \geq 1} \mu_n E_n, \quad \sum_{n \geq 0} E_n = I,$$

where $\lim_n \mu_n = \mu$ and where $E_n \mathcal{H}$ is the finite-dimensional eigenspace for μ_n , $n \geq 1$. Defining a compact operator C by

$$C = \sum_{n \geq 1} (\mu_n - \mu) E_n,$$

we have $|T| - C = \mu I$. If the polar decomposition for T is $T = V |T|$, then it is always possible to extend the partial isometry V to an isometry or coisometry U . Set $K = UC$; then K is compact, and

$$T - K = U |T| - UC = \mu U,$$

so (i) holds. Thus the conditions are equivalent.

In [7] it is observed that the set $\mathcal{K}_T$ is closed, bounded, convex and infinite-dimensional. We shall now see that if T satisfies the conditions of the preceding theorem, then $\mathcal{K}_T$ reflects the structure of the ball in $\mathcal{B}(\mathcal{H})$ in terms of its extreme points.

Theorem 3. *The set $\mathcal{K}_T$ of compact approximants of an operator $T \in \mathcal{B}(\mathcal{H})$ is uniformly closed, bounded and convex. The operator T is always in the strong closure of $\mathcal{K}_T$; thus $\mathcal{K}_T$ is not compact in any of the usual operator topologies, unless T is compact. If T satisfies the conditions of the preceding theorem, then each element of $\mathcal{K}_T$ is a midpoint of two extreme points; in particular, $\mathcal{K}_T$ is the convex hull of its extreme points. If T does not satisfy these conditions, $\mathcal{K}_T$ has no extreme points.*

Proof. The first statement is obvious. For any T , choose $K \in \mathcal{K}_T$ and a sequence $(E_n)_n$ of finite rank projections of supremum I . Then $TE_n + K(I - E_n)$, $n \geq 1$, are compact approximants of T which converge strongly to T . Thus whenever T is not compact, the set $\mathcal{K}_T$ is not strongly closed; and hence $\mathcal{K}_T$ is not compact in any Hausdorff topology comparable to the strong operator topology.

It is immediate from Theorems 1 and 2 that $\mathcal{K}_T$ will have extreme points if and only if T satisfies the conditions of Theorem 2.

As for the remaining assertion, if we can establish for any isometry U that an arbitrary element of $\mathcal{K}_U$ is a midpoint between two extreme points, then simply by taking adjoints we will have this result for any coisometry. By

multiplying and translating we will have the result for any T with $T - K = \mu V$, where K is compact, and V is any isometry or coisometry.

Let U be an isometry and let $K \in \mathcal{K}_U$. Write $U - K = VP$, where $P = |U - K|$ and V is an isometry: extend the partial isometry in the polar decomposition for $U - K$ to an isometric or coisometric V . We can assume without loss of generality that V is an isometry (in fact, index considerations ensure that V must be an isometry).

Continuing to argue as P. R. Halmos does in [4, p. 265], we see that $P = \frac{1}{2}(W + W^*)$ where $W = P + i(I - P^2)^{1/2}$ is unitary. Thus

$$U - K = \frac{1}{2}(VW + VW^*),$$

where VW, VW^* are isometries. Furthermore,

$$\begin{aligned} U - VW &= U - V[P + i(I - P^2)^{1/2}] \\ &= U - VP - iV[I - (U - K)^*(U - K)]^{1/2} \\ &= K - iV[K^*U + U^*K - K^*K]^{1/2} \end{aligned}$$

which is compact, and similarly $U - VW^*$ is compact. Thus if $K_1 = U - VW$ and $K_2 = U - VW^*$, then K_1, K_2 are extreme points of $\mathcal{K}_U$ and $K = \frac{1}{2}(K_1 + K_2)$. The theorem is proved.

Turning now to finite rank approximants, we remark that for certain $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{K}_T$ contains no finite rank operators: for example, if T is a diagonal operator with diagonal sequence $(1 + 1/n)_n$. In fact if T is an operator with essential norm μ , and if $\|Tx\| > \mu \|x\|$ for all non-zero x in some infinite-dimensional closed subspace $\mathcal{E} \subset \mathcal{H}$, then there is no finite rank element of $\mathcal{K}_T$. To see this, let E be the projection with range $\mathcal{E}$, and let K be any finite rank operator. Then the rank of KE is finite, and the null space of KE is strictly greater than $\mathcal{H} \ominus \mathcal{E}$. Thus there is a vector $x \in \mathcal{E}$ with $KEx = Kx = 0$, so

$$\|(T - K)x\| = \|Tx\| > \mu \|x\|,$$

and so $K \notin \mathcal{K}_T$.

Conversely, if there is no such infinite-dimensional closed subspace $\mathcal{E}$, then there is at least one finite rank approximant for T . In particular, if $E(\lambda)$ is the spectral resolution for $|T|$, then $E = E((\mu, \|T\|])$ must be a finite rank projection since

$$\|Tx\| = \| |T|x \| > \mu \|x\|,$$

for any $x \in \text{range } E$. Furthermore, $TE \in \mathcal{K}_T$, since

$$\|T - TE\| = \| |T| (I - E) \| = \mu.$$

We have established the following:

Theorem 4. *Let T be an operator in $\mathcal{B}(\mathcal{H})$, with essential norm μ . Then $\mathcal{K}_T$ contains a finite rank approximant if and only if there is no infinite-dimensional closed subspace $\mathcal{E} \subset \mathcal{H}$ with $\|Tx\| > \mu \|x\|$, for all non-zero $x \in \mathcal{E}$.*

Recall that an operator $U \in \mathfrak{B}(\mathfrak{H})$ is a partial isometry if U^*U is a projection.

Theorem 5. *The finite rank elements of $\mathfrak{K}_U$ are uniformly dense in $\mathfrak{K}_U$ whenever U is a partial isometry.*

Proof. Suppose K is a compact approximant of a partial isometry U , so that $\|U - K\| = 1$. Let E and F be the projections defined by $E = U^*U$, $F = UU^*$. We define two sequences of finite rank projections $(E_n)_n$ and $(F_n)_n$ each converging strongly to I : choose three non-decreasing sequences of finite rank projections, $(P_{0n})_n$ converging strongly to $I - E$, $(P_n)_n$ converging strongly to E , and $(Q_{0n})_n$ converging strongly to $I - F$. Now specify a similar sequence $(Q_n)_n$ by $Q_n = UP_nU^*$, so $(Q_n)_n$ converges strongly to F . Set

$$E_n = P_{0n} + P_n, \quad F_n = Q_{0n} + Q_n, \quad n = 1, 2, \dots$$

We claim our construction ensures that $F_nU = UE_n$, for all n . Note first that $U^*U = E$ and $P_{0n} \leq I - E$ imply that $U^*UP_{0n} = 0$. Since U^* is isometric on the range of U , $UP_{0n} = 0$. Similarly, $Q_{0n}UU^* = 0$, so $Q_{0n}U = 0$. Furthermore,

$$Q_nU = UP_nU^*U = UP_nE = UP_n,$$

so we conclude that

$$F_nU = Q_{0n}U + Q_nU = UP_n = UP_{0n} + UP_n = UE_n.$$

The sequence of finite rank operators $(F_nKE_n)_n$ converges uniformly to K . Furthermore, this sequence is a subset of $\mathfrak{K}_U$ since

$$\begin{aligned} \|U - F_nKE_n\| &= \|F_n(U - K)E_n + (I - F_n)U(I - E_n)\| \\ &= \max(\|F_n(U - K)E_n\|, \|(I - F_n)U(I - E_n)\|) \\ &\leq \max(\|U - K\|, \|U\|) \\ &= 1. \end{aligned}$$

Thus the proof is complete.

Theorem 6. *Let T be an operator in $\mathfrak{B}(\mathfrak{H})$ with essential norm μ . If there is no closed infinite-dimensional subspace $\mathfrak{E}$ of $\mathfrak{H}$ with $\|Tx\| > \mu\|x\|$ for all $x \in \mathfrak{E}$, then the finite rank operators in $\mathfrak{K}_T$ are uniformly dense in $\mathfrak{K}_T$.*

Proof. Multiplying by a suitable scalar, we can assume $\mu = 1$. If there is no such subspace $\mathfrak{E}$, then $\mathfrak{K}_T$ contains some finite rank operator F by Theorem 4. Observe that $\mathfrak{K}_T$ is equal to the set of operators $\mathfrak{K}_{T-F} + F$; thus it suffices to establish the desired conclusion for $T - F$. In particular, we can assume that $\|T\| = \mu = 1$. Let K be a compact approximant of T , and let $\epsilon > 0$ be given. We will find a finite rank operator in $\mathfrak{K}_T$ which is within 2ϵ distance of K .

Let $T = U|T|$ be the polar decomposition for T , and let $G(\lambda)$ be the spectral resolution associated with $|T|$. Let (P_n) and (Q_n) be nondecreasing sequences

of finite rank projections converging strongly to the projections E_0 and F_0 onto the null spaces of $|T|$ and $|T^*|$ respectively. For simplicity, denote

$$G_n = G((1/n, 1]),$$

and define sequences of projections (E_n) and (F_n) by

$$E_n = P_n + G_n$$

$$F_n = Q_n + UG_nU^*, \quad n = 2, 3, \dots$$

Then (E_n) and (F_n) each converge strongly to I : G_n converges strongly to $G((0, 1]) = I - E_0$; whereas $U^*U = I - E_0$ and $UU^* = I - F_0$, so that UG_nU^* converges strongly to $I - F_0$. Thus the sequence $(F_nKE_n)_n$ converges uniformly to K . Furthermore, $F_nT = TE_n$ all n , since

$$F_nT = UG_nU^*T = UG_nU^*U|T| = UG_n|T| = U|T|G_n = TG_n = TE_n.$$

Fix n so that $\|K - F_nKE_n\| < \epsilon$. For convenience, set $S = F_nTE_n$ and $C = F_nKE_n$. To prove the theorem it will be enough to find a finite rank operator C_0 such that

$$F_nC_0E_n = C_0, \quad \|C - C_0\| < \epsilon \quad \text{and} \quad \|S - C_0\| = 1.$$

For then,

$$\|K - C_0\| = \|(K - C) + (C - C_0)\| < 2\epsilon$$

and

$$\|T - C_0\| = \|F_n(S - C_0)E_n + (I - F_n)T(I - E_n)\| = 1.$$

Adapting Halmos' idea in [4, p. 265], we will view S and $S - C$ as midpoints of partial isometries. Write the polar decompositions $S = V|S|$ and $S - C = W|S - C|$. Set

$$Y = |S| + i[E_n - S^*S]^{1/2}$$

$$X = |S - C| + i[E_n - (S - C)^*(S - C)]^{1/2},$$

where Y and X are well-defined, since $SE_n = S$ implies $S^*SE_n = E_nS^*S = S^*S$, and since $\|S^*S\| \leq 1$ (similarly for $S - C$). Then we have

$$S = \frac{1}{2}(VY + VY^*)$$

$$S - C = \frac{1}{2}(WX + WX^*).$$

Since S^*S commutes with E_n , and since $|S| = (S^*S)^{1/2}$ is a strong limit of polynomials in S^*S , we also have that $|S|$ commutes with E_n . Similarly, we deduce that $[E_n - S^*S]^{1/2}$ commutes with $|S|$. Then a straightforward computation immediately shows $Y^*Y = E_n$. In the same fashion $E_n = YY^* = X^*X = XX^*$. This implies that VY is a partial isometry: V^*V is a projection dominated by E_n , so $(VY)^*(VY) = Y^*V^*VY$ is a projection. Analogously, VY^* , WX , and WX^* are partial isometries.

The proof proceeds by demonstrating that C is a midpoint between certain compact operators in $\mathcal{K}_{VY}$ and $\mathcal{K}_{VY^*}$. Since VY and VY^* are partial isometries, Theorem 5 allows these compact operators to be approximated in norm by finite rank elements of $\mathcal{K}_{VY}$ and $\mathcal{K}_{VY^*}$. The midpoint of these elements then turns out to be the desired finite rank element of $\mathcal{K}_S$ which approximates C in norm.

We now wish to show that $VY - WX$ is compact. Observe first that

$$Y - X = |S| - |S - C| + i[E_n - S^*S]^{1/2} - i[E_n - (S - C)^*(S - C)]^{1/2},$$

which is compact. That is, this expression is in the kernel of the homomorphism ν , since ν maps the unique positive square root of an operator to the unique positive square root of its image in the Calkin algebra. We claim that $(V - W)Y$ is also compact: note first that

$$\begin{aligned} C &= V|S| - W|S - C| \\ &= (V - W)|S| + W(|S| - |S - C|). \end{aligned}$$

Thus $(V - W)|S|$ must be compact, since $\nu(|S| - |S - C|) = 0$, as above. Now, $|T|$ is bounded away from zero by $1/n$ on the range of $G_n = E((1/n, 1])$. Hence T , S and $|S|$ are similarly bounded away from zero on this subspace. Therefore, there is a $Z \in \mathcal{B}(\mathcal{H})$ such that $|S|Z = G_n$. In particular, since

$$\begin{aligned} (V - W)E_n &= (V - W)P_n + (V - W)G_n \\ &= (V - W)P_n + (V - W)|S|Z, \end{aligned}$$

where P_n is finite rank, it follows that $(V - W)E_n$ is compact. Thus also $(V - W)Y = (V - W)E_n Y$ is compact, as claimed. Finally, by writing

$$VY - WX = (V - W)Y + W(Y - X)$$

we conclude that $VY - WX$ is compact.

Set $A = VY - WX$; then $A \in \mathcal{K}_{VY}$ since $\|VY - A\| = \|WX\| = 1$. Since VY is a partial isometry, Theorem 5 implies the existence of a finite rank operator $A_0 \in \mathcal{K}_{VY}$ with $\|A - A_0\| < \epsilon$. We can assume $F_n A_0 E_n = A_0$ for,

$$F_n V = V, F_n W = W, X E_n = X, Y E_n = Y,$$

so that

$$F_n V Y E_n = VY, F_n W X E_n = WX, F_n A E_n = A.$$

Hence

$$\begin{aligned} \|A - F_n A_0 E_n\| &\leq \|A - A_0\| < \epsilon \\ \|VY - F_n A_0 E_n\| &\leq \|VY - A_0\| = 1. \end{aligned}$$

The same arguments as above show that $B = VY^* - WX^*$ is a compact operator in $\mathcal{K}_{VY^*}$, and yield a finite rank $B_0 \in \mathcal{K}_{VY^*}$ with $\|B - B_0\| < \epsilon$ and

with $F_n B_0 E_n = B_0$. Set $C_0 = \frac{1}{2}(A_0 + B_0)$, so C_0 is finite rank and $F_n C_0 E_n = C_0$. Furthermore,

$$\begin{aligned} \|S - C_0\| &= \|\tfrac{1}{2}(VY + VY^*) - \tfrac{1}{2}(A_0 + B_0)\| \\ &\leq \tfrac{1}{2} \|VY - A_0\| + \tfrac{1}{2} \|VY^* - B_0\| \\ &= 1. \end{aligned}$$

Write

$C = S - (S - C) = \tfrac{1}{2}(VY + VY^*) - \tfrac{1}{2}(WX + WX^*) = \tfrac{1}{2}(A + B)$,
to see that

$$\|C - C_0\| = \|\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A_0 + B_0)\| < \epsilon.$$

Thus the proof is complete.

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Date communicated: AUGUST 10, 1973