Real Flat Chains, Cochains and Variational Problems

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1. Introduction. Both chains with real coefficients and chains with integer coefficients have been studied from the point of view of geometric measure theory, in terms of the vector spaces $\mathbf{F}_{m,K}(U)$ of real flat chains and the groups $\mathfrak{F}_{m,K}(U)$ of integral flat chains defined in [5, 4.1.12] and [5, 4.1.24]. However, the theory developed in Chapters 4 and 5 of [5] deals in much greater depth with integral flat chains, which are certainly more important. Here we present additional facts about real flat chains, showing that also they are useful for geometry and the calculus of variations.

In §2 we relate limit properties of integrals over real chains with finite mass to certain pointwise properties of parametric integrands, such as convexity and flatness.

Even though the isoperimetric inequality fails for chains with real coefficients (as was shown in [5, 4.5.13]), we derive in §3 a somewhat weaker estimate concerning real flat (relative) boundaries, which enables us to prove flat closedness and compactness properties of real homology classes, hence the existence of real homologically minimizing chains with respect to any convex positive parametric integrand.

In §4 we study locally flat cochains, which are defined as continuous real valued linear functions on the spaces of real flat chains, and which together with their coboundaries can be represented by locally bounded Lebesgue measurable differential forms. After proving some extension theorems we show that locally flat real cohomology is dual to real flat homology on the local Lipschitz category. These results generalize, and imply by regularization, the classical theorems of De Rham about the duality between smooth singular chains and smooth differential forms. Next we apply our theory to variational problems involving any convex positive integrand Ψ . We establish a duality between homologically (or absolutely) Ψ minimizing real flat chains and locally flat cocycles (or coboundaries) which are pointwise bounded by the integrand Ψ . Thus we put into a general context the previous uses of closed differential forms

in sufficient conditions for variational minima, such as geodesic fields [11] and the Kaehler form [5, 5.4.19].

In §5 we compare variational minima defined in terms of real and integral comparison chains, respectively. We prove, for instance, that if Ψ is a convex parametric integrand of degree m which has the Lipschitz chain approximation property (related to flatness in §2) with respect to a compact Lipschitz neighborhood retract A in $\mathbf{R}^n$, then for every m dimensional rectifiable current T in A the infimum of the set of numbers

$$\int_{T+R} \Psi$$

corresponding to all m dimensional *real flat* boundaries (or cycles) R with finite mass in A equals the infimum of the set of numbers

$$k^{-1} \int_{kT+S} \Psi$$

corresponding to all positive integers k and all m dimensional *rectifiable* boundaries (or cycles) S in A . Under additional hypotheses, including positivity and evenness of Ψ , we also study the question whether restriction to $k = 1$ increases the second infimum. When $m = 1$ there is no increase in the cases of cycles, but sometimes an increase in the case of boundaries. When A is an $m + 1$ dimensional manifold, there is no increase in the case of boundaries if A is orientable, but sometimes an increase for both boundaries and cycles if A is not orientable. Our general theory subsumes the proposition that for any Riemannian manifold A and any homology classes

$$\lambda \in \mathbf{H}_m(A; \mathbf{Z}), \quad \mu \in \mathbf{H}_m(A; \mathbf{R}) \quad \text{with} \quad \lambda \subset \mu,$$

it is true that

$$\inf \{ \mathbf{M}(Q) : Q \in \mu \} = \inf \{ \mathbf{M}(X)/k : 0 < k \in \mathbf{Z}, X \in k\lambda \}.$$

In §6 we establish a sufficient condition, weaker than a Lipschitz condition, for a differential form to define a flat cochain. We illustrate this result by a simplified discussion of certain m dimensional area minimizing currents with singularities in $\mathbf{R}^{m+1}$, previously treated by different methods in [2] and [9].

We use the terminology of [5], which contains glossaries of basic notations on pages 669 to 671.

2. Lowersemicontinuity and polyhedral approximation of integrals related to convexity and flatness of integrands. Suppose U is an open subset of $\mathbf{R}^n$, Ψ is a parametric integrand of degree m on U (see [5, 5.1.1]), and

$$\mathbf{M}_m(U)$$

is the subspace of $\mathcal{E}_m(U)$ consisting of those m dimensional currents with compact support in U which are representable by integration. Representing any $T \in \mathbf{M}_m(U)$ in the form

$$T = \mu \wedge \xi$$

where μ is a (nonnegative) Radon measure with compact support and $\xi \in \mathbf{L}_1(\mu, \wedge_m \mathbf{R}^n)$, we define

$$\int_T \Psi = \int \Psi[x, \xi(x)] d\mu x \in \mathbf{R};$$

since $\Psi_x = \Psi(x, \cdot)$ is positively homogeneous whenever $x \in U$, all such representations of T yield the same integral (see [5, 4.1.7, page 358]).

In case $\Psi(x, \eta + \zeta) \leq \Psi(x, \eta) + \Psi(x, \zeta)$ for $x \in U$ and $\eta, \zeta \in \wedge_m \mathbf{R}^n$ we call Ψ a **convex integrand**. In case $\Psi(x, -\eta) = \Psi(x, \eta)$ we term Ψ an **even integrand**.

When ϕ is a differential form on U , we write $\Psi \geq \phi$ if and only if $\Psi(x, \eta) \geq \langle \eta, \phi(x) \rangle$ for $x \in U$ and $\eta \in \wedge_m \mathbf{R}^n$.

2.1. *If Ψ is a convex positive integrand and $T \in \mathbf{M}_m(U)$, then*

$$\int_T \Psi = \sup \{T(\phi) : \Psi \geq \phi \in \mathfrak{D}^m(U)\}.$$

Proof. We represent $T = \vec{T} \wedge ||T||$. First we observe that

$$T(\phi) = \int \langle \vec{T}(x), \phi(x) \rangle d||T|| x \leq \int \Psi[x, \vec{T}(x)] d||T|| x = \int_T \Psi$$

whenever $\Psi \geq \phi \in \mathfrak{D}^m(U)$. Next we choose a neighborhood V of $\text{spt } T$ in U , and $1 \leq \lambda \in \mathbf{R}$, so that

$$\lambda^{-1} ||\eta|| \leq \Psi(x, \eta) \leq \lambda ||\eta|| \quad \text{for } (x, \eta) \in V \times \wedge_m \mathbf{R}^n.$$

Given $0 < \epsilon \in \mathbf{R}$ we construct compact sets $C_1, \dots, C_k$ with disjoint neighborhoods $U_1, \dots, U_k$ in V , as well as $a_i \in C_i$ and $\zeta_i \in \wedge_m \mathbf{R}^n$ with $||\zeta_i|| = 1$ corresponding to $j \in \{1, \dots, k\}$, such that

$$||T|| \left(U \sim \bigcup_{i=1}^k C_i \right) < \epsilon,$$

$$||\vec{T}(x) - \zeta_i|| < \epsilon \quad \text{for } x \in C_i,$$

$$|\Psi(x, \eta) - \Psi(a_i, \eta)| \leq \epsilon ||\eta|| \quad \text{for } (x, \eta) \in U_i \times \wedge_m \mathbf{R}^n.$$

Applying the Hahn-Banach Theorem to the positively homogeneous convex function Ψ_{a_i} on $\wedge_m \mathbf{R}^n$ we obtain $f_i \in \wedge^m \mathbf{R}^n$ with

$$\langle \zeta_i, f_i \rangle = \Psi(a_i, \zeta_i),$$

$$\langle \eta, f_i \rangle \leq \Psi(a_i, \eta) \quad \text{for } \eta \in \wedge_m \mathbf{R}^n.$$

We also select $g_i \in \mathfrak{D}^0(U)$ so that $0 \leq g_i(x) \leq 1$ for $x \in U$,

$$g_i(x) = 1 \quad \text{for } x \in C_i, \quad g_i(x) = 0 \quad \text{for } x \in U \sim U_i,$$

and consider the differential forms

$$\theta = \sum_{i=1}^k g_i f_i, \quad \phi = (1 + \epsilon\lambda)^{-1} \theta \in \mathfrak{D}^m(U).$$

We see that $\Psi \geq \phi$ because if $x \in U_i$ and $\eta \in \wedge_m \mathbf{R}^n$, then

$$\begin{aligned} \langle \eta, \theta(x) \rangle &= g_i(x) \langle \eta, f_i \rangle \leq g_i(x) \Psi(a_i, \eta) \leq \Psi(a_i, \eta) \\ &\leq \Psi(x, \eta) + \epsilon \|\eta\| \leq (1 + \epsilon\lambda) \Psi(x, \eta). \end{aligned}$$

Since $\langle \eta, f_i \rangle \leq \lambda \|\eta\|$ for $\eta \in \wedge_m \mathbf{R}^n$, $\|f_i\| \leq \lambda$, $\|\theta(x)\| \leq \lambda$ for $x \in U_i$,

$$\sum_{i=1}^k \int_{U_i \sim C_i} |\langle \vec{T}(x), \theta(x) \rangle| d\|T\| x \leq \lambda \sum_{i=1}^k \|T\| (U_i \sim C_i) \leq \lambda\epsilon.$$

Moreover

$$\int_T \Psi - \sum_{i=1}^k \int_{C_i} \Psi[x, \vec{T}(x)] d\|T\| x \leq \lambda\epsilon.$$

If $x \in C_i$, then

$$\begin{aligned} \langle \vec{T}(x), \theta(x) \rangle &= \langle \vec{T}(x), f_i \rangle \geq \langle \xi_i, f_i \rangle - \|\vec{T}(x) - \xi_i\| \cdot \|f_i\| \\ &\geq \Psi(a_i, \xi_i) - \epsilon\lambda \geq \Psi(x, \xi_i) - \epsilon - \epsilon\lambda \\ &\geq \Psi[x, \vec{T}(x)] - \Psi[x, \vec{T}(x) - \xi_i] - \epsilon - \epsilon\lambda \\ &\geq \Psi[x, \vec{T}(x)] - \lambda\epsilon - \epsilon - \epsilon\lambda \geq \Psi[x, \vec{T}(x)] - 3\lambda\epsilon. \end{aligned}$$

We conclude that

$$\begin{aligned} T(\theta) &\geq \sum_{i=1}^k \int_{C_i} (\Psi[x, \vec{T}(x)] - 3\lambda\epsilon) d\|T\| x - \lambda\epsilon \\ &\geq \int_T \Psi - \lambda\epsilon - 3\lambda\epsilon \|T\| U - \lambda\epsilon, \end{aligned}$$

hence

$$T(\phi) \geq (1 + \epsilon\lambda)^{-1} \left[\int_T \Psi - (2 + 3\|T\| U) \lambda\epsilon \right].$$

The proposition 2.1 has the following corollary:

2.2. *If Ψ is a convex positive integrand, then the function $\int \Psi$, mapping $T \in \mathbf{M}_m(U)$ onto $\int_T \Psi \in \mathbf{R}$, is positively homogeneous, convex, and lowersemicontinuous with respect to weak convergence.*

2.3. *If Ψ is a convex nonnegative integrand, then $\int \Psi$ is positively homogeneous, convex, and lowersemicontinuous with respect to massbounded weak convergence.*

Proof. Suppose $c \in \mathbf{R}$, $S_j \rightarrow T$ in $\mathcal{E}_m(U)$ as $j \rightarrow \infty$, and $\mathbf{M}(S_j) \leq c$ for all positive integers j . Given $0 < \epsilon \in \mathbf{R}$ we define

$$\Omega(x, \eta) = \Psi(x, \eta) + \epsilon \|\eta\| \quad \text{for } (x, \eta) \in U \times \wedge_m \mathbf{R}^n$$

and apply 2.2 with Ψ replaced by Ω to infer

$$\begin{aligned} \int_T \Psi &\leq \int_T \Omega \leq \liminf_{i \rightarrow \infty} \int_{S_i} \Omega \\ &= \liminf_{i \rightarrow \infty} \left[\int_{S_i} \Psi + \epsilon \mathbf{M}(S_i) \right] \leq \liminf_{i \rightarrow \infty} \int_{S_i} \Psi + \epsilon c. \end{aligned}$$

2.4. If the function $\int \Psi$ is lowersemicontinuous with respect to massbounded weak convergence, then Ψ is a convex integrand.

Proof. Given $a \in U$ and $\eta, \zeta \in \wedge_m \mathbf{R}^n$, we choose $0 \neq b \in \mathbf{R}^n$ and infer that $\delta_a \wedge \eta + \delta_{a+\epsilon b} \wedge \zeta \rightarrow \delta_a \wedge (\eta + \zeta)$ in $\mathbf{M}_m(U)$ as $\epsilon \rightarrow 0$, hence

$$\begin{aligned} \Psi(a, \eta) + \Psi(a, \zeta) &= \lim_{\epsilon \rightarrow 0} [\Psi(a, \eta) + \Psi(a + \epsilon b, \zeta)] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\delta_a \wedge \eta + \delta_{a+\epsilon b} \wedge \zeta} \Psi \geq \int_{\delta_a \wedge (\eta + \zeta)} \Psi = \Psi(a, \eta + \zeta). \end{aligned}$$

2.5. If Ψ is a convex nonnegative integrand and $T \in \mathbf{M}_m(U)$ with regularizations T_ϵ (see [5, 4.1.2]), then

$$\lim_{\epsilon \downarrow 0} \int_{T_\epsilon} \Psi = \int_T \Psi.$$

Proof. Since $T_\epsilon = \mathcal{L}^n \wedge g_\epsilon$ with $g_\epsilon(z) = \int \Phi_\epsilon(x - z) \vec{T}(x) d ||T|| x$ for $z \in \mathbf{R}^n$, it follows from Jensen's inequality as well as Fubini's theorem and Lebesgue's bounded convergence theorem that

$$\begin{aligned} \int_{T_\epsilon} \Psi &= \int \Psi \left[z, \int \Phi_\epsilon(x - z) \vec{T}(x) d ||T|| x \right] d \mathcal{L}^n z \\ &\leq \iint \Phi_\epsilon(x - z) \Psi[z, \vec{T}(x)] d ||T|| x d \mathcal{L}^n z \\ &= \int \Phi_\epsilon(w) \int \Psi[x - w, \vec{T}(x)] d ||T|| x d \mathcal{L}^n w \\ &\rightarrow \int \Psi[x, \vec{T}(x)] d ||T|| x = \int_T \Psi \quad \text{as } \epsilon \downarrow 0. \end{aligned}$$

Moreover T_ϵ converges weakly to T as $\epsilon \downarrow 0$, hence 2.3 implies

$$\liminf_{\epsilon \downarrow 0} \int_{T_\epsilon} \Psi \geq \int_T \Psi.$$

2.6. If K is a compact subset of U , A is an $\mathcal{L}^n$ measurable subset of $\text{Int } K$, $0 \neq \eta \in \wedge_m \mathbf{R}^n$, η is simple and $\epsilon > 0$, then there exists $P \in \mathbf{P}_m(U)$ such that

$$\mathbf{F}_K[P - (\mathcal{L}^n \llcorner A) \wedge \eta] < \epsilon, \quad \left| \int_P \Psi - \int_{(\mathcal{L}^n \llcorner A) \wedge \eta} \Psi \right| < \epsilon$$

and P is a finite real linear combination of m dimensional cubes with orientation $\eta/|\eta|$ in $\text{Int } K$.

To prove this proposition we follow the method of [5, 4.1.23] beginning with the last four lines on page 379, and observe that in the special case treated on page 380 the integral

$$\int_B \int_{R \times [b]} \Psi \, d\mathfrak{L}^{n-m} b = \int_{(\mathfrak{L}^n \llcorner A) \wedge \eta} \Psi$$

can be approximated by the sum

$$\sum_{k=1}^p \mathfrak{L}^n(B_k) \int_{R \times [b_k]} \Psi = \int_P \Psi.$$

For each $(x, \eta) \in U \times \wedge_m \mathbf{R}^n$ we define

$$\Psi^{\flat}(x, \eta)$$

as the infimum of the set of numbers

$$\sum_{i=1}^N \Psi(x, \zeta_i)$$

corresponding to all finite sequences of simple m vectors $\zeta_1, \dots, \zeta_N$ in $\wedge_m \mathbf{R}^n$ such that

$$\eta = \sum_{i=1}^N \zeta_i.$$

Suppose Ψ is nonnegative. For each $r \in \mathbf{R}$ we infer that $\Psi^{\flat}(x, \eta) < r$ if and only if η belongs to the convex hull of

$$\{\zeta : \zeta \in \wedge_m \mathbf{R}^n, \zeta \text{ is simple}, \Psi(x, \zeta) < r\}.$$

Accordingly we can take $N \leq \dim \wedge_m \mathbf{R}^n = \binom{n}{m}$, by Fenchel's extension of Carathéodory's theorem [4, page 35]. Using these facts one readily verifies:

2.7. *If Ψ is a positive integrand, then $\Psi^{\flat}$ is a convex positive integrand and $(\Psi^{\flat})^{\flat} = \Psi$.*

We note that if Ψ is merely nonnegative, then $\Psi^{\flat}$ is uppersemicontinuous, but $\Psi^{\flat}$ need not be lowersemicontinuous; for instance when $U = \mathbf{R}^2$, $m = 1$ and

$$\Psi(x, \eta) = \text{distance}(\eta, \mathbf{R}^2 \cap \{\zeta : |\zeta_2| \leq |x| \cdot |\zeta_1|\})$$

for $(x, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2$, then $\Psi^{\flat}(x, \eta) = 0$ for $x \neq 0$, but $\Psi^{\flat}(0, \eta) = |\eta_2|$.

The construction of $\Psi^{\flat}$ from Ψ generalizes the example [5, 1.8.1] where $\Psi(x, \eta) = |\eta|$ and $\Psi^{\flat}(x, \eta) = ||\eta||$.

We call Ψ a **flat integrand** if and only if $0 \leq \Psi = \Psi^{\flat}$.

2.8. If Ψ is a flat integrand, then the following conditions (1) and (2) hold for every compact subset K of U :

(1) If $S_1, S_2, S_3, \dots$ and $T \in \mathbf{F}_{m,K}(U)$ with

$$\lim_{i \rightarrow \infty} \mathbf{F}_K(S_i - T) = 0 \quad \text{and} \quad \limsup_{i \rightarrow \infty} \mathbf{M}(S_i) < \infty,$$

then

$$\liminf_{i \rightarrow \infty} \int_{S_i} \Psi \geq \int_T \Psi.$$

(2) If $T \in \mathbf{F}_m(U)$, $\mathbf{M}(T) < \infty$ and $\text{spt } T \subset \text{Int } K$, then there exist $P_1, P_2, P_3, \dots \in \mathbf{P}_m(U)$ with $\text{spt } P_i \subset \text{Int } K$ such that

$$\lim_{i \rightarrow \infty} \mathbf{F}_K(P_i - T) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{P_i} \Psi = \int_T \Psi.$$

2.9. If Ψ is a positive integrand and the preceding conditions (1) and (2) hold for every compact subset K of U , then Ψ is a flat integrand.

Proof of 2.8. (1) follows from 2.3, while (2) follows from 2.5 and 2.6 (through generalization of the argument in [5, 4.1.23]).

Proof of 2.9. First we will show that $\Psi^b \geq \Psi$.

Suppose $0 \neq \eta \in \wedge_m \mathbf{R}^n$ and $\zeta_1, \dots, \zeta_N$ are simple m -vectors with sum η , such that ζ_i and ζ_k are linearly independent for $i \neq k$. Given any compact $K \subset U$, any $\mathcal{L}^n$ measurable $A \subset \text{Int } K$, and any $\epsilon > 0$, we use 2.6 to obtain $Q_i \in \mathbf{P}_m(U)$ such that

$$\mathbf{F}_K[Q_i - (\mathcal{L}^n \lfloor A) \wedge \zeta_i] < \epsilon, \quad \left| \int_{Q_i} \Psi - \int_{(\mathcal{L}^n \lfloor A) \wedge \zeta_i} \Psi_i \right| < \epsilon$$

and Q_i is a finite linear combination of m dimensional cubes with orientation $\zeta_i/|\zeta_i|$ in $\text{Int } K$. Then

$$S_\epsilon = \sum_{i=1}^N Q_i \in \mathbf{P}_m(U), \quad \mathbf{F}_K[S_\epsilon - (\mathcal{L}^n \lfloor A) \wedge \eta] < N\epsilon, \quad \int_{S_\epsilon} \Psi = \sum_{i=1}^N \int_{Q_i} \Psi$$

because the m dimensional vectorspaces associated with $\zeta_1, \dots, \zeta_N$ are distinct, hence

$$\left| \int_{S_\epsilon} \Psi - \sum_{i=1}^N \int_{(\mathcal{L}^n \lfloor A) \wedge \zeta_i} \Psi \right| < N\epsilon.$$

Letting ϵ approach 0 we infer from (1) that

$$\int_A \sum_{i=1}^N \Psi(x, \zeta_i) d\mathcal{L}^n x = \lim_{\epsilon \downarrow 0} \int_{S_\epsilon} \Psi \geq \int_{(\mathcal{L}^n \lfloor A) \wedge \eta} \Psi = \int_A \Psi(x, \eta) d\mathcal{L}^n x,$$

and recalling the arbitrariness of A we conclude

$$\sum_{i=1}^N \Psi(x, \zeta_i) \geq \Psi(x, \eta) \quad \text{for } x \in U.$$

Thus $\Psi^b(x, \eta) \geq \Psi(x, \eta)$, with equality in case η is simple. It follows that $\int_P \Psi^b = \int_P \Psi$ for $P \in \mathbf{P}_m(U)$.

Next we see that $\Psi \geq \Psi^b$, because for any $T \in \mathbf{M}_m(U)$ it follows from (2) that

$$\int_T \Psi = \lim_{i \rightarrow \infty} \int_{P_i} \Psi = \lim_{i \rightarrow \infty} \int_{P_i} \Psi^b \geq \int_T \Psi^b,$$

by virtue of 2.7, and 2.2 applied to Ψ^b .

According to 2.7 and 2.8 (applied to Ψ^b), replacement of Ψ by Ψ^b helps polyhedral approximation. However, according to 2.10, it impedes differentiation.

2.10. *If Ψ is a convex positive integrand, $x \in U$ and $\Psi_x \neq \Psi_x^b$, then Ψ_x^b is nondifferentiable at some simple m vector.*

Proof. Observing that $C = \{\eta : \Psi_x^b(\eta) \leq 1\}$ is the convex hull of

$$D = \{\zeta : \Psi_x(\zeta) = 1 \text{ and } \zeta \text{ is simple}\},$$

we choose $\xi \in \wedge_m \mathbf{R}^n$ with $\Psi_x^b(\xi) > 1 = \Psi_x(\xi)$, $\alpha \in \wedge^m \mathbf{R}^n$ with

$$\langle \xi, \alpha \rangle > 1 = \sup \{\langle \eta, \alpha \rangle : \eta \in C\} = \sup \{\langle \zeta, \alpha \rangle : \zeta \in D\},$$

$\theta \in D$ with $\langle \theta, \alpha \rangle = 1$, and $\beta \in \wedge^m \mathbf{R}^n$ with

$$\langle \theta, \beta \rangle = 1 = \sup \{\langle \eta, \beta \rangle : \Psi_x(\eta) \leq 1\} \geq \sup \{\langle \eta, \beta \rangle : \eta \in C\}.$$

Then $\Psi_x^b(\theta) = \Psi_x(\theta) = 1$, $\langle \theta, \alpha \rangle = 1 = \langle \theta, \beta \rangle$, $\langle \xi, \alpha \rangle > 1 \geq \langle \xi, \beta \rangle$, α and β are linearly independent,

$$C \subset \{\eta : \langle \eta, \alpha \rangle \leq 1\} \cap \{\eta : \langle \eta, \beta \rangle \leq 1\},$$

hence Ψ_x^b is not differentiable at θ .

2.11. For the purpose of extending the relation between flatness and approximation, in particular 2.8(2), to real flat chains with supports in a subset E of U , we will replace polyhedral chains by m dimensional **Lipschitz chains of E** , which are defined as currents of the type

$$f_* P \in \mathbf{N}_m(U)$$

where, for some open subset Z of some Euclidean space and some compact subset C of Z ,

$$f : Z \rightarrow U \text{ is Lipschitzian, } f(C) \subset E, P \in \mathbf{P}_{m,c}(Z);$$

in case $P \in \mathcal{P}_{m,c}(Z)$ we term $f_* P$ an **integral Lipschitz chain of E** .

We note that the m dimensional Lipschitz chains of E form a vectorsubspace of $\mathbf{N}_m(U) \cap \{S : \text{spt } S \subset E\}$.

(For the purpose of this paper we could alternately have used the vectorspace generated by $\mathbf{I}_m(U) \cap \{S : \text{spt } S \subset E\}$.)

We say that Ψ has the **Lipschitz chain approximation property** with respect to a compact subset K of U if and only if for each $T \in \mathbf{F}_{m,K}(U)$ with $\mathbf{M}(T) < \infty$ there exists a sequence of m dimensional Lipschitz chains S_i of K such that

$$\lim_{i \rightarrow \infty} \mathbf{F}_K(S_i - T) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{S_i} \Psi = \int_T \Psi.$$

In 2.12 and 2.13 we will find sufficient conditions for this approximation property, often applicable to sets which are somewhat more general than the stratified objects of classical geometry.

2.12. Suppose H is a submanifold of class 1 of U . By saying that Ψ is H flat we mean that if $x \in H$ and $\eta \in \wedge_m \text{Tan}(H, x)$, then $\Psi(x, \eta)$ is nonnegative and equals the infimum of the set of numbers

$$\sum_{i=1}^N \Psi(x, \zeta_i)$$

corresponding to all finite sequences of simple m vectors $\zeta_1, \dots, \zeta_N$ in $\wedge_m \text{Tan}(H, x)$ such that

$$\eta = \sum_{i=1}^N \zeta_i.$$

If Ψ is H flat, K is a compact subset of U and

$$T \in \mathbf{F}_{m,K}(U) \text{ with } \mathbf{M}(T) < \infty, \text{ spt } T \subset H \sim \text{Clos}(H \sim K),$$

then there exists a sequence of m dimensional Lipschitz chains S_i of $H \sim \text{Clos}(H \sim K)$ such that

$$\lim_{i \rightarrow \infty} \mathbf{F}_K(S_i - T) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{S_i} \Psi = \int_T \Psi.$$

Proof. Letting $\kappa = \dim H$ we choose a finite sequence of open subsets $U_1, \dots, U_\nu$ of U so that

$$\text{spt } T \subset \bigcup_{i=1}^{\nu} U_i$$

and for each $i \in \{1, \dots, \nu\}$ there exists an open subset V_i of $\mathbf{R}^k$ as well as Lipschitzian maps of class 1

$$f_i : U_i \rightarrow V_i, g_i : V_i \rightarrow U_i \text{ with } \text{im } g_i = U_i \cap H, f_i \circ g_i = \mathbf{1}_{V_i}.$$

Given any $\epsilon > 0$, we select disjoint compact sets

$$C_i \subset (U_i \cap H) \sim \text{Clos}(H \sim K)$$

so that

$$\mathbf{M}\left(T - \sum_{i=1}^{\nu} T \llcorner C_i\right) < \epsilon \quad \text{and} \quad \left| \int_T \Psi - \sum_{i=1}^{\nu} \int_{T \llcorner C_i} \Psi \right| < \epsilon,$$

and also select compact subsets K_i of V_i so that

$$f_i(C_i) \subset \text{Int } K_i, \quad g_i(K_i) \subset (U_i \cap H) \sim \text{Clos } (H \sim K),$$

and $g_1(K_1), \dots, g_\nu(K_\nu)$ are disjoint. Observing that $g_i^* \Psi$ is a flat integrand on V_i we apply 2.8(2) to

$$T_i = f_{i*}(T \lfloor C_i) \in \mathbf{F}_m(V_i) \quad \text{with} \quad \mathbf{M}(T_i) < \infty, \quad \text{spt } T_i \subset \text{Int } K_i,$$

and obtain $P_i \in \mathbf{P}_m(V_i)$ with $\text{spt } P_i \subset \text{Int } K_i$,

$$\mathbf{F}_{K_i}(P_i - T_i) < \epsilon [\nu \sup \{ \text{Lip } (g_i)^m, \text{Lip } (g_i)^{m+1} \}]^{-1}, \quad \left| \int_{P_i} g_i^* \Psi - \int_{T_i} g_i^* \Psi \right| \leq \epsilon \nu^{-1}.$$

Accordingly $S = \sum_{i=1}^\nu g_{i*} P_i$ is an m dimensional Lipschitz chain of $H \sim \text{Clos } (H \sim K)$ with

$$\mathbf{F}_K(S - T) < 2\epsilon \quad \text{and} \quad \left| \int_S \Psi - \int_T \Psi \right| < 2\epsilon.$$

For example, if Ψ is the mass integrand,

$$\Psi(x, \eta) = \|\eta\| \quad \text{whenever } (x, \eta) \in U \times \wedge_m \mathbf{R}^n,$$

then Ψ is H flat for every submanifold H of class 1 of U , because for every vector-subspace X of $\mathbf{R}^n$ and any simple m vectors $\zeta_1, \dots, \zeta_N$ in $\wedge_m \mathbf{R}^n$ such that

$$\sum_{i=1}^N \zeta_i = \eta \in \wedge_m X$$

the orthogonal projection p retracting $\mathbf{R}^n$ onto X yields simple m vectors $\zeta'_i = \langle \zeta_i, \wedge_m p \rangle$ in $\wedge_m X$ such that

$$\eta = \langle \eta, \wedge_m p \rangle = \sum_{i=1}^N \zeta'_i, \quad |\zeta'_i| \leq |\zeta_i| \quad \text{for } i \in \{1, \dots, N\}.$$

If $m = 1$ or $\dim H = m + 1$, then every convex nonnegative integrand is H flat, because in these cases all m vectors of all tangent spaces of H are simple.

2.13. If $K = \bigcup_{i=1}^\infty H_i$ is a compact subset of U and each H_i is a submanifold of class 1 of U such that Ψ is H_i flat, then Ψ has the Lipschitz chain approximation property with respect to K .

Proof. Suppose $T \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U)$ and $\epsilon > 0$. Since the manifolds $H_1, H_2, H_3, \dots$ are Borel sets whose union contains the support of T , we can choose a positive integer ν and disjoint compact subsets C_i of H_i corresponding to $i = 1, 2, \dots, \nu$ so that

$$\mathbf{M}\left(T - \sum_{i=1}^\nu T \lfloor C_i\right) < \epsilon \quad \text{and} \quad \left| \int_T \Psi - \sum_{i=1}^\nu \int_{T \lfloor C_i} \Psi \right| < \epsilon.$$

Inasmuch as $\text{spt}(T \llcorner C_i) \subset C_i \subset H_i$ we can apply 2.12 to find m dimensional Lipschitz chains S_i of H_i such that

$$\mathbf{F}_K(S_i - T \llcorner C_i) < \epsilon/\nu, \left| \int_{S_i} \Psi - \int_{T \llcorner C_i} \Psi \right| < \epsilon/\nu$$

for $i = 1, 2, \dots, \nu$ and the supports of $S_1, S_2, \dots, S_\nu$ are disjoint. Thus we construct the m dimensional Lipschitz chain

$$S = \sum_{i=1}^{\nu} S_i$$

of K with $\mathbf{F}_K(T - S) < 2\epsilon$ and $|\int_T \Psi - \int_S \Psi| < 2\epsilon$.

For example the mass integrand has the Lipschitz chain approximation property with respect to any compact class 1 submanifold with boundary.

3. Flat closedness of real homology classes. *We assume that*

$$\mathbf{R}^n \supset A \supset B, B \text{ is relatively closed in } A,$$

U and V are neighborhoods of A and B in $\mathbf{R}^n$, f and g are locally Lipschitzian retractions of U and V onto A and B , hence

$$W = V \cap \{x : (1-t)g(x) + tx \in U \text{ for } 0 \leq t \leq 1\}$$

is a neighborhood of B in $U \cap V$. We define

$$\mathbf{F}_m(A)$$

as the union of the Banach spaces $\mathbf{F}_{m,K}(U)$ corresponding to all compact subsets K of A , with the topology induced by their inclusion maps. This topology consists of those subsets of $\mathbf{F}_m(A)$ whose intersection with $\mathbf{F}_{m,K}(U)$ is $\mathbf{F}_K$ open for every compact subset K of A . Recalling [5, 4.1.14, 4.1.15] we see that

$$\mathbf{F}_m(A) = \mathbf{F}_m(U) \cap \{T : \text{spt } T \subset A\},$$

and that f_ retracts $\mathbf{F}_m(U)$ continuously onto $\mathbf{F}_m(A)$.*

From the definitions in [5, 4.4.6, page 473] we infer immediately that $\mathbf{Z}_m(A, B)$ is closed in $\mathbf{F}_m(A)$. We will deduce the closedness of $\mathbf{B}_m(A, B)$ from the following basic estimate:

3.1. *For each compact subset K of A there exist $1 \leq \tau_K \in \mathbf{R}$ and a compact set E_K such that $K \subset E_K \subset A$ and the following statement is true:*

If $X \in \mathbf{M}_m(U) \cap \mathbf{Z}_m(A, B) \cap \mathbf{B}_m(U, W)$ with $\text{spt } X \subset K$, then there exists $Y \in \mathbf{N}_{m+1, E_K}(U)$ with

$$\text{spt}(X - \partial Y) \subset B, \quad \mathbf{M}(Y) + \mathbf{M}(X - \partial Y) \leq \tau_K \mathbf{M}(X).$$

To prove this proposition we adapt certain parts of the argument in [5, 4.4.2, pages 466 to 469]. We choose $\alpha, \beta, L, \lambda, M, \mu, \gamma, \delta$ with the properties stated on lines 10 to 13 of [5, page 466], let $\epsilon = \delta$, define

$$\begin{aligned}\Omega &= \mathbf{N}_m(U) \cap \{R : \text{spt } R \subset L \cap \mathbf{u}_\epsilon(\mathbf{W}_m'), \quad \text{spt } \partial R \subset \mathbf{u}_\epsilon(\mathbf{W}_{m-1}')\}, \\ \Phi &= \Omega \cap \mathbf{B}_m(U, W),\end{aligned}$$

and let h be the affine homotopy from g to $\mathbf{1}_U$.

From [5, 4.1.12, 4.1.24] we see that for each $R \in \Phi$ there exist

$$\begin{aligned}F &\in \mathbf{F}_{m+1}(U) \quad \text{with} \quad \text{spt } (R - \partial F) \subset W, \quad \mathbf{M}(F) < \infty, \\ G &\in \mathbf{F}_m(W), \quad H \in \mathbf{F}_{m+1}(W) \quad \text{with} \quad \mathbf{M}(G) < \infty, \quad \mathbf{M}(H) < \infty, \\ R - \partial F &= G + \partial H,\end{aligned}$$

hence $F + H \in \mathbf{N}_{m+1}(U)$ and $\text{spt } [R - \partial(F + H)] \subset W$. Since Φ is a finite dimensional vectorspace over $\mathbf{R}$, normed by $\mathbf{M}$, there exist an $\mathbf{R}$ linear map Γ of Φ into $\mathbf{N}_{m+1}(U)$ and a finite positive number ξ such that

$$\begin{aligned}\text{spt } [R - \partial\Gamma(R)] &\subset W, \\ \mathbf{M}f_*[\Gamma(R) + h_*([0, 1] \times [R - \partial\Gamma(R)])] &\leq \xi \mathbf{M}(R), \\ \mathbf{M}g_*[R - \partial\Gamma(R)] &\leq \xi \mathbf{M}(R)\end{aligned}$$

whenever $R \in \Phi$. Moreover

$$C = \cup \{\text{spt } \Gamma(R) : R \in \Phi\}, \quad D = \cup \{\text{spt } [R - \partial\Gamma(R)] : R \in \Phi\}$$

are compact subsets of U, W respectively.

Given $X \in \mathbf{M}_m(U) \cap \mathbf{Z}_m(A, B) \cap \mathbf{B}_m(U, W)$ with $\text{spt } X \subset K$ we follow [5, page 467] to construct first $T \in \mathbf{N}_m(U)$ as in lines 5 to 11, next $P \in \Omega, Q \in \mathbf{N}_m(U), S \in \mathbf{N}_{m+1}(U)$ and $Z \in \mathbf{N}_{m+1}(U)$ as in lines 18 to 30. Then we obtain Y by proceeding from line 28 of [5, page 468] to line 5 of [5, page 469], and verify the stated conclusions with

$$\tau_K = (\mu^m + \xi + \eta)2\gamma, \quad E_K = f[L \cup C \cup h(\{t : 0 \leq t \leq 1\} \times D)].$$

3.2. *If K is a compact subset of A and*

$$R \in \mathbf{F}_{m,K}(U) \cap \mathbf{Z}_m(A, B) \cap \mathbf{B}_m(U, W),$$

then there exists $S \in \mathbf{F}_{m+1,E_K}(U)$ with

$$\text{spt } (R - \partial S) \subset B, \quad \mathbf{M}(S) + \mathbf{M}(R - \partial S) \leq \tau_K \mathbf{F}_K(R).$$

Proof. Using [5, 4.1.24] we represent $R = X + \partial Z$ with

$$X \in \mathbf{F}_{m,K}(U), \quad Z \in \mathbf{F}_{m+1,K}(U), \quad \mathbf{F}_K(R) = \mathbf{M}(X) + \mathbf{M}(Z)$$

and apply 3.1 to obtain $S = Z + Y \in \mathbf{F}_{m+1,E_K}(U)$ with

$$\begin{aligned}
R - \partial S &= X - \partial Y, \quad \text{spt}(R - \partial S) \subset B, \\
\mathbf{M}(S) + \mathbf{M}(R - \partial S) &\leq \mathbf{M}(Z) + \mathbf{M}(Y) + \mathbf{M}(X - \partial Y) \\
&\leq \mathbf{M}(Z) + \tau_K \mathbf{M}(X) \leq \tau_K \mathbf{F}_K(R).
\end{aligned}$$

Corollary. $\mathbf{Z}_m(A, B) \cap \mathbf{B}_m(U, W) = \mathbf{B}_m(A, B)$.

3.3. $\mathbf{B}_m(A, B)$ is closed in $\mathbf{F}_m(A)$.

Proof. For any compact subset K of A and any sequence of real flat chains $R_i \in \mathbf{F}_{m,K}(U) \cap \mathbf{B}_m(A, B)$ with

$$\sum_{i=1}^{\infty} \mathbf{F}_K(R_i) < \infty, \quad \text{hence} \quad Q = \sum_{i=1}^{\infty} R_i \in \mathbf{F}_{m,K}(U),$$

we apply 3.2 to obtain $S_i \in \mathbf{F}_{m+1,E_K}(U)$ with

$$\text{spt}(R_i - \partial S_i) \subset B \quad \text{and} \quad \mathbf{M}(S_i) \leq \tau_K \mathbf{F}_K(R_i),$$

hence $T = \sum_{i=1}^{\infty} S_i \in \mathbf{F}_{m+1,E_K}(U)$, $\text{spt}(Q - \partial T) \subset B$, $Q \in \mathbf{B}_m(A, B)$. We conclude that $\mathbf{F}_{m,K}(U) \cap \mathbf{B}_m(A, B)$ is $\mathbf{F}_K$ complete.

3.4. For each compact subset K of A we define

$$\begin{aligned}
\mathbf{Z}_{m,K}(A, B) &= \mathbf{F}_{m,K}(U) \cap \mathbf{Z}_m(A, B), \\
\mathbf{B}_{m,K}(A, B) &= \mathbf{F}_{m,K}(U) \cap \mathbf{B}_m(A, B), \\
\mathbf{H}_{m,K}(A, B; \mathbf{R}) &= \mathbf{Z}_{m,K}(A, B) / \mathbf{B}_{m,K}(A, B).
\end{aligned}$$

Combining 3.3 and the real analogue of [5, 4.4.2(4)] stated in [5, 4.4.6] we see that $\mathbf{H}_{m,K}(A, B; \mathbf{R})$ is a finite dimensional Banach space, with the norm

$$\inf \{ \mathbf{F}_K(S) : S \in \chi \} \quad \text{for} \quad \chi \in \mathbf{H}_{m,K}(A, B; \mathbf{R}).$$

Moreover the inclusion map of $\mathbf{Z}_{m,K}(A, B)$ into $\mathbf{Z}_m(A, B)$ induces a continuous monomorphism of $\mathbf{H}_{m,K}(A, B; \mathbf{R})$ into $\mathbf{H}_m(A, B; \mathbf{R})$.

The collection of such monomorphisms corresponding to all compact subsets K of A induces the quotient topology on $\mathbf{H}_m(A, B; \mathbf{R})$. Any linear map of $\mathbf{H}_m(A, B; \mathbf{R})$ into any topological vectorspace is continuous.

It happens often that C is a compact subset of A and $\mathbf{F}_{m,C}(U)$ is neither homeomorphically embedded in $\mathbf{F}_m(A)$ nor closed in $\mathbf{F}_m(A)$. For example, if $U = \mathbf{R}$, $A = \{x : -1 \leq x \leq 1\}$, $C = \{0\} \cup \{1/j : j \in \mathbf{Z}\}$, then $\{\mathfrak{d}_a : a \in C\}$ is $\mathbf{F}_C$ discrete, but $\mathfrak{d}_a \rightarrow \mathfrak{d}_0$ in $\mathbf{F}_0(A)$ as $a \rightarrow 0$ in C , and

$$\sum_{i=1}^{\infty} (-1)^i \mathfrak{d}_{1/i}$$

is a limit point of $\mathbf{F}_{0,C}(\mathbf{R})$ in $\mathbf{F}_0(A)$ which does not belong to $\mathbf{F}_{0,C}(\mathbf{R})$. There is, however, the following useful fact:

3.5. If C is a compact subset of A and C is a Lipschitz neighborhood retract in U , then $\mathbf{F}_{m,c}(U)$ is homeomorphically embedded in $\mathbf{F}_m(A)$ and closed in $\mathbf{F}_m(A)$.

Proof. We choose a Lipschitzian retraction $r : D \rightarrow C$, where D is a compact subset of U such that $C \subset \text{Int } D$, and a function $\gamma \in \mathfrak{D}^0(U)$ with $\text{spt } \gamma \subset D$, $\text{spt } (1 - \gamma) \subset U \sim C$.

Suppose K is any compact subset of A . Using [5, 4.1.12, 4.1.14, 4.1.15] we see that if $T \in \mathbf{F}_{m,K}(U) \cap \mathbf{F}_{m,c}(U)$, then $T = T \llcorner \gamma = r_*(T \llcorner \gamma)$ and

$$\mathbf{F}_c(T) \leq \alpha \mathbf{F}_K(T \llcorner \gamma) \leq \alpha \beta \mathbf{F}_{K \cap \text{spt } \gamma}(T) \leq \alpha \beta \mathbf{F}_K(T)$$

where $\alpha = \sup \{(\text{Lip } r)^m, (\text{Lip } r)^{m+1}\}$, $\beta = \mathbf{M}(\gamma) + \mathbf{M}(d\gamma)$. It follows that the intersection of $\mathbf{F}_{m,K}(U)$ with any $\mathbf{F}_c$ closed subset of $\mathbf{F}_{m,c}(U)$ is $\mathbf{F}_K$ closed.

3.6. If $c \in \mathbf{R}$ and K, K' are compact subsets of A with

$$K \cap \text{Clos } (A \sim K') = \emptyset$$

then $\mathbf{N}_{m,K}(U) \cap \{R : \mathbf{N}(R) \leq c\}$ is $F_{K'}$ compact.

Proof. Noting that $f^{-1}[A \sim \text{Clos } (A \sim K')]$ is a neighborhood of K in U , we obtain from [5, 4.1.29] a compact Lipschitz neighborhood retract C with

$$K \subset C \subset f^{-1}[A \sim \text{Clos } (A \sim K')] \subset f^{-1}(K').$$

We infer that the set

$$\mathbf{N}_{m,K}(U) \cap \{R : \mathbf{N}(R) \leq c\} = \mathbf{N}_{m,c}(U) \cap \{R : \text{spt } R \subset K, \mathbf{N}(R) \leq c\}$$

is $\mathbf{F}_c$ compact, by [5, 4.2.17], and invariant under f_* , hence $\mathbf{F}_{f(C)}$ compact and $\mathbf{F}_{K'}$ compact.

Corollary. The sets

$$\mathbf{Z}_{m,K}(A, \emptyset) \cap \{R : \mathbf{M}(R) \leq c\} \quad \text{and} \quad \mathbf{B}_{m,K}(A, \emptyset) \cap \{R : \mathbf{M}(R) \leq c\}$$

are $F_{K'}$ compact.

In the preceding corollary one cannot replace $\emptyset$ by $B \neq \emptyset$; for instance when $A = \mathbf{R}^2$ and $B = \mathbf{R}^2 \cap \{x : x_2 = 0\}$ the mass-bounded sequence of real flat (A, B) boundaries

$$k[(0, 0), (0, k^{-2})] + k[(0, k^{-2}), (k^{-1}, k^{-2})] + k[(k^{-1}, k^{-2}), (k^{-1}, 0)]$$

corresponding to $k = 1, 2, 3, \dots$ converges in the topology of $\mathcal{E}_1(\mathbf{R}^2)$ to the current $\mathfrak{d}_0 \wedge e_1 \in \mathcal{E}_1(\mathbf{R}^2) \sim \mathbf{F}_1(\mathbf{R}^2)$. To circumvent this obstacle we define the seminorm

$$\mathbf{F}_{K',B}(Q) = \int_0^{*\infty} \mathbf{F}_{K'}[Q \llcorner H'(r)] \, d\mathcal{L}^1 r$$

for $Q \in \mathbf{F}_{m,K'}(U) \cap \mathbf{M}_m(U)$, where $H'(r) = \{x : \text{dist } (x, K' \cap B) > r\}$.

3.7. If $c \in \mathbf{R}$ and K, K' are compact subsets of A with

$$K \cap \text{Clos } (A \sim K') = \emptyset,$$

then the sets

$$\Xi = \mathbf{Z}_{m,K}(A, B) \cap \{R : \mathbf{M}(R) \leq c\},$$

$$\Upsilon = \mathbf{B}_{m,K}(A, B) \cap \{R : \mathbf{M}(R) \leq c\}$$

are $\mathbf{F}_{K',B}$ compact.

Proof. Given $R_i \in \Xi$ for $i = 1, 2, 3, \dots$ we adapt an argument from [5, 4.4.4] with K replaced by K' , starting on line 6 of page 471, to construct first

$$R_{i,j} = R_i \sqcup H'(r_{i,j}) \quad \text{with} \quad \mathbf{N}(R_{i,j}) \leq (1+j)c,$$

next (using 3.6)

$$G_i \in \mathbf{N}_{m,K}(U) \quad \text{with} \quad \lim_{i \rightarrow \infty} \mathbf{F}_{K'}(G_i - R_{i,j}) = 0,$$

and finally, proceeding through line 5 of page 472,

$$G \in \Xi \quad \text{with} \quad \lim_{i \rightarrow \infty} \mathbf{F}_{K',B}(G - R_i) = 0.$$

In case $R_i \in \Upsilon$ for $i = 1, 2, 3, \dots$ we recall 3.2 and use [5, 4.1.29] to obtain compact Lipschitz neighborhood retracts A', B' in U with

$$E_K \subset \text{Int } A' \subset A' \subset U, \quad B \cap E_K \subset \text{Int } B' \subset B' \subset W \cap \text{Int } A',$$

hence $R_i \in \mathbf{B}_m(A', B')$. When j is large, then

$$\text{dist } (B \cap K', U \sim B') > 1/j,$$

$$\text{spt } (R_i - R_{i,j}) \subset \{x : \text{dist } (x, B \cap K') \leq 1/j\} \subset B',$$

$R_i - R_{i,j} \in \mathbf{B}_m(A', B')$ and $R_{i,j} \in \mathbf{B}_m(A', B')$ for $i = 1, 2, 3, \dots$, hence $G_i \in \mathbf{B}_m(A', B')$ by 3.3. Inasmuch as

$$\text{spt } (G - G_i) \subset \{x : \text{dist } (x, B \cap K') \leq 1/j\} \subset B' \subset W,$$

we conclude with the help of the corollary of 3.2 that

$$G \in \mathbf{B}_m(U, W) \cap \mathbf{Z}_m(A, B) = \mathbf{B}_m(A, B),$$

hence $G \in \Upsilon$.

3.8. Whenever Ψ is an integrand on U , $\Gamma \subset \mathbf{F}_m(U)$, $T \in \mathbf{F}_m(U)$ we define

$$\Psi_\Gamma(T) = \inf \left\{ \int_Q \Psi : Q \in \mathbf{M}_m(U), Q - T \in \Gamma \right\}.$$

Recalling 2.3 and the end of [5, 4.1.24] we infer:

If Ψ is a convex nonnegative integrand and Γ is a convex cone, then $\Psi_\Gamma : \mathbf{F}_m(U) \rightarrow \{y : 0 \leq y \leq \infty\}$ is convex and positively homogeneous.

If $\partial \mathbf{F}_{m+1,K}(U) \subset \Gamma$, $c = \sup \{ \Psi(x, \xi) : x \in K, \xi \in \wedge_m \mathbf{R}^n, ||\xi|| \leq 1 \}$ and $T \in \mathbf{F}_{m,K}(U)$, then $\Psi_\Gamma(T) \leq c \mathbf{F}_K(T)$.

If $\mathbf{B}_m(A, \phi) \subset \Gamma$ and $T \in \mathbf{F}_m(A)$, then $\Psi_\Gamma(T) < \infty$.

The following proposition 3.9 asserts the existence of real flat chains which are **absolutely**, or **homologically**, Ψ **minimizing in K with respect to (A, B)** . It follows readily from 3.7 and 2.2 by the method of [5, page 522], but is also a by-product of our proof of the subsequent continuity theorem 3.10.

3.9. If K is a compact subset of A and Ψ is a convex nonnegative integrand on U which satisfies for some $\epsilon > 0$ the condition

$$\int_S \Psi \geq \epsilon \mathbf{M}(S) \quad \text{whenever} \quad S \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U),$$

then for every $T \in \mathbf{F}_{m,K}(U)$ there exist $Q, Q' \in \mathbf{M}_m(U)$ such that

$$\Psi_{\mathbf{Z}_{m,K}(A,B)}(T) = \int_Q \Psi \quad \text{and} \quad Q - T \in \mathbf{Z}_{m,K}(U),$$

$$\Psi_{\mathbf{B}_{m,K}(A,B)}(T) = \int_{Q'} \Psi \quad \text{and} \quad Q' - T \in \mathbf{B}_{m,K}(A, B),$$

hence

$$\Psi_{\mathbf{Z}_{m,K}(A,B)}(T) = 0 \quad \text{if and only if} \quad T \in \mathbf{Z}_{m,K}(A, B),$$

$$\Psi_{\mathbf{B}_{m,K}(A,B)}(T) = 0 \quad \text{if and only if} \quad T \in \mathbf{B}_{m,K}(A, B).$$

Clearly the above condition holds in case Ψ is a positive integrand.

For 3.9 the restriction to a compact subset K of A is essential. There are simple examples where $\Gamma = \mathbf{Z}_m(A, B)$ or $\Gamma = \mathbf{B}_m(A, B)$, $T \in \mathbf{F}_m(A) \sim \Gamma$, $\Psi_\Gamma(T) = 0$.

3.10. If Ψ is a positive convex integrand on U , K is a compact subset of A , $T \in \mathbf{F}_{m,K}(A)$,

$$C_1 \supset C_2 \supset C_3 \supset \cdots \text{ are compact subsets of } U,$$

$$D_1 \supset D_2 \supset D_3 \supset \cdots \text{ are neighborhoods of } B \text{ in } W,$$

$$\bigcap_{i=1}^{\infty} C_i = K, \quad \bigcap_{i=1}^{\infty} \text{Clos}(C_i \cap D_i) = K \cap B,$$

then

$$\Psi_{\mathbf{Z}_{m,C_i(U,D_i)}}(T) \uparrow \Psi_{\mathbf{Z}_{m,K}(A,B)}(T) \quad \text{as} \quad i \uparrow \infty,$$

$$\Psi_{\mathbf{B}_{m,C_i(U,D_i)}}(T) \uparrow \Psi_{\mathbf{B}_{m,K}(A,B)}(T) \quad \text{as} \quad i \uparrow \infty.$$

Proof. Assuming $\mathbf{M}(T) < \infty$ we choose

$$R_i \in \mathbf{Z}_{m,C_i(U,D_i)} \cap \mathbf{M}_m(U) \quad \text{with} \quad \int_{T+R_i} \Psi \leq \Psi_{\mathbf{Z}_{m,C_i(U,D_i)}}(T) + i^{-1}$$

for $0 < i \in \mathbf{Z}$, and suppose $s = \lim_{i \rightarrow \infty} \int_{T+R_i} \Psi < \infty$, whence the positivity of Ψ implies

$$c = \sup \{ \mathbf{M}(R_i) : 0 < i \in \mathbf{Z} \} < \infty.$$

Assuming also that $K \cap B \neq \emptyset$ we define $u(x) = \text{dist}(x, K \cap B)$ for $x \in \mathbf{R}^n$, and observe that

$$\delta_i = \sup \{ u(x) : x \in C_i \cap D_i \} \downarrow 0 \quad \text{as } i \uparrow \infty.$$

Since $\text{spt}(\partial R_i) \subset C_i \cap D_i \subset \{x : u(x) \leq \delta_i\}$, the corollary in [5, 4.2.1] yields r_i between δ_i and $2\delta_i$ such that $\delta_i \mathbf{M}(\partial R'_i) \leq \mathbf{M}(R_i) \leq c$ with

$$R'_i = R_i \llcorner \{x : u(x) > r_i\}, \quad \text{spt}(\partial R'_i) \subset \{x : u(x) = r_i\}.$$

Recalling [5, page 467, lines 14 to 17] we pick a positive integer j so that $\{x : u(x) \leq (\mu + 2)2\delta_j\} \subset W$, and for $j \leq i \in \mathbf{Z}$ we construct the normal currents

$$R'_i = h_*([0, 1] \times \partial R'_i) \quad \text{with} \quad \text{spt} R'_i \subset \{x : u(x) \leq (\mu + 2)r_i\} \subset W,$$

$$\mathbf{M}(R'_i) \leq (\mu + 1)r_i \mathbf{M}(\partial R'_i) \leq (\mu + 1)2c, \quad \partial R'_i = \partial R'_i - g_* \partial R'_i,$$

$$R'_i - R'_i \in \mathbf{Z}_m(U, B), \quad \mathbf{M}(R'_i - R'_i) \leq (2\mu + 3)c,$$

$$\text{spt}(R'_i - R'_i) \subset K_i = C_i \cup \{x : u(x) \leq (\mu + 2)2\delta_i\}.$$

Next we choose a compact subset K' of U so that $K_j \subset \text{Int } K'$, and apply 3.7 with A, K, c replaced by $U, K', (2\mu + 3)c$ to obtain

$$S \in \mathbf{Z}_{m, K_j}(U, B) \quad \text{with} \quad \mathbf{M}(S) \leq (2\mu + 3)c$$

and

$$\liminf_{i \rightarrow \infty} \mathbf{F}_{K', B}(R'_i - R'_i - S) = 0.$$

Inasmuch as $K_j \supset K_{j+1} \supset K_{j+2} \supset \dots$ and $\bigcap \{K_i : j \leq i \in \mathbf{Z}\} = K$, we find that $\text{spt } S \subset K \subset A$, hence $S \in \mathbf{Z}_m(A, B)$. Combining the definition of $\mathbf{F}_{K', B}$ with [5, 2.4.6] we see that, for $\mathfrak{L}^1$ almost all positive numbers r ,

$$\liminf_{i \rightarrow \infty} \mathbf{F}_{K'}[(R'_i - R'_i - S) \llcorner H'(r)] = 0;$$

moreover, if i is sufficiently large, then $(\mu + 2)r_i < r$,

$$\{x : u(x) \leq (\mu + 2)r_i\} \subset U \sim H'(r),$$

$$R'_i \llcorner H'(r) = R_i \llcorner H'(r), \quad R'_i \llcorner H'(r) = 0;$$

consequently

$$\liminf_{i \rightarrow \infty} \mathbf{F}_{K'}[(R_i - S) \llcorner H'(r)] = 0,$$

and since $(R_i - S) \llcorner H'(r) = (T + R_i) \llcorner H'(r) - (T + S) \llcorner H'(r)$ we infer from 2.2 that

$$\int_{(T+S) \llcorner H'(r)} \Psi \leq \limsup_{i \rightarrow \infty} \int_{(T+R_i) \llcorner H'(r)} \Psi \leq s.$$

Inasmuch as $(T + S) \perp H'(r) \in \mathbf{N}_{m,K}(U)$ for $\mathcal{L}^1$ almost all r , and

$$\mathbf{M}[(T + S) \perp H'(r) - (T + S) \perp (U \sim B)] \downarrow 0 \quad \text{as } r \downarrow 0,$$

we find that $(T + S) \perp (U \sim B) \in \mathbf{F}_{m,K}(U)$,

$$(T + S) \perp (U \sim B) - T = S - (S + T) \perp B \in \mathbf{Z}_{m,K}(A, B),$$

$$\Psi_{\mathbf{Z}_{m,K}(A,B)}(T) \leq \int_{(T+S) \perp (U \sim B)} \Psi \leq s,$$

which confirms the first conclusion of 3.10.

To verify the second conclusion of 3.10 we choose

$$R_i \in \mathbf{B}_{m,c_i}(U, D_i) \cap \mathbf{M}_m(U) \quad \text{with} \quad \int_{T+R_i} \Psi \leq \Psi_{\mathbf{B}_{m,c_i}(U,D_i)}(T) + i^{-1}$$

for $0 < i \in \mathbf{Z}$. Proceeding as before we construct, for $j \leq i \in \mathbf{Z}$,

$$R'_i \in \mathbf{B}_m(U, W), R'_i' \in \mathbf{F}_m(W),$$

$$R'_i - R'_i' \in \mathbf{Z}_m(U, B) \cap \mathbf{B}_m(U, W) = \mathbf{B}_m(U, B).$$

The last equation results from the corollary of 3.2 through replacement of A by U . Then 3.7 yields $S \in \mathbf{B}_{m,K_i}(U, B)$, and we find that

$$S \in \mathbf{Z}_m(A, B) \cap \mathbf{B}_m(U, B) = \mathbf{B}_m(A, B),$$

$$(T + S) \perp (U \sim B) - T = S - (S + T) \perp B \in \mathbf{B}_{m,K}(A, B),$$

$$\Psi_{\mathbf{B}_{m,K}(A,B)}(T) \leq \int_{(T+S) \perp (U \sim B)} \Psi \leq s.$$

4. Flat duality. Whenever A and B are local Lipschitz neighborhood retracts in $\mathbf{R}^n$, with $B \subset A$, the space

$$\mathbf{F}^m(A)$$

of **locally flat cochains of degree m of A** consists of all continuous real valued linear functions on $\mathbf{F}_m(A)$, and the spaces of **relative cochains**, **cocycles**, **coboundaries**, **cohomology classes of (A, B)** are defined as

$$\mathbf{F}^m(A, B) = \mathbf{F}^m(A) \cap \{\alpha : \mathbf{F}_m(B) \subset \ker \alpha\},$$

$$\mathbf{Z}^m(A, B) = \mathbf{F}^m(A) \cap \{\alpha : \mathbf{B}_m(A, B) \subset \ker \alpha\},$$

$$\mathbf{B}^m(A, B) = \mathbf{F}^m(A) \cap \{\alpha : \mathbf{Z}_m(A, B) \subset \ker \alpha\},$$

$$\mathbf{H}^m(A, B; \mathbf{R}) = \mathbf{Z}^m(A, B) / \mathbf{B}^m(A, B).$$

The restriction to $\mathbf{Z}_m(A, B)$ of any cocycle in $\mathbf{Z}^m(A, B)$ induces a linear function on $\mathbf{H}_m(A, B; \mathbf{R})$; thus arises a monomorphism

$$\mathbf{H}^m(A, B; \mathbf{R}) \rightarrow \text{Hom} [\mathbf{H}_m(A, B; \mathbf{R}), \mathbf{R}],$$

which will be shown in 4.4 to be an isomorphism provided B is relatively closed

in A . We note that

$$\mathbf{Z}^0(A, B) = \mathbf{F}^0(A), \quad \mathbf{B}^0(A, B) = \{0\}.$$

The dual of the boundary operator $\partial : \mathbf{F}_{m+1}(A) \rightarrow \mathbf{F}_m(A)$ is the **coboundary operator** $d : \mathbf{F}^m(A) \rightarrow \mathbf{F}^{m+1}(A)$,

$$(d\alpha)(T) = \alpha(\partial T) \quad \text{whenever} \quad \alpha \in \mathbf{F}^m(A), \quad T \in \mathbf{F}_{m+1}(A).$$

Clearly $d\mathbf{F}^m(A, B) \subset \mathbf{B}^{m+1}(A, B)$; the opposite inclusion will be proved in 4.3 provided B is relatively closed in A .

Every locally Lipschitzian map $\theta : A \rightarrow A'$ induces the continuous linear map $\theta_* : \mathbf{F}_m(A) \rightarrow \mathbf{F}_m(A')$ and its dual $\theta^* : \mathbf{F}^m(A') \rightarrow \mathbf{F}^m(A)$,

$$(\theta^*\alpha)(R) = \alpha(\theta_*R) \quad \text{for} \quad \alpha \in \mathbf{F}^m(A') \quad \text{and} \quad R \in \mathbf{F}_m(A),$$

which commute with ∂ and d , respectively.

For each locally Lipschitzian homotopy $k : \{t : 0 \leq t \leq 1\} \times A \rightarrow A'$ we infer from the homotopy formulae for currents proved in [5, 4.1.9, 4.1.14] that

$$k_1^*\alpha - k_0^*\alpha = [k^*(d\alpha)]_{[0,1]} + d[(k^*\alpha)_{[0,1]}]$$

whenever $\alpha \in \mathbf{F}^m(A')$ with $m > 0$, and that

$$k_1^*\alpha - k_0^*\alpha = [k^*(d\alpha)]_{[0,1]} \quad \text{whenever} \quad \alpha \in \mathbf{F}^0(A').$$

We define $\gamma_{[0,1]} \in \mathbf{F}^m(A)$ for $\gamma \in \mathbf{F}^{m+1}(\mathbf{R} \times A)$,

$$\gamma_{[0,1]}(T) = \gamma([0, 1] \times T) \quad \text{whenever} \quad T \in \mathbf{F}_m(A).$$

These operations on locally flat cochains generalize standard operations on differential forms described in [5, 4.1.6, 4.1.8, 4.1.9]. Letting U be a neighborhood of A in $\mathbf{R}^n$ we associate with each form $\phi \in \mathcal{E}^m(U)$ the cochain $\tilde{\phi} \in \mathbf{F}^m(A)$ such that

$$\tilde{\phi}(R) = R(\phi) \quad \text{whenever} \quad R \in \mathbf{F}_m(A).$$

Then $d\tilde{\phi} = (d\phi)^\sim$, $\theta^*\tilde{\phi} = (\theta^*\phi)^\sim$, $\tilde{\psi}_{[0,1]} = (\psi_{[0,1]})^\sim$. Further relations between locally flat cochains and differential forms will be discussed in 4.6 to 4.11, and in 6.2.

For each compact subset K of A we define

$$\mathbf{F}_K^m(U)$$

as the space of all $\mathbf{F}_K$ continuous real valued linear functions on $\mathbf{F}_{m,K}(U)$, with the dual norm

$$\mathbf{F}_K(\lambda) = \sup \{ \lambda(S) : S \in \mathbf{F}_{m,K}(U), \mathbf{F}_K(S) \leq 1 \}$$

whenever $\lambda \in \mathbf{F}_K^m(U)$, and we let

$$\mathbf{F}_K^m(A, B) = \mathbf{F}_K^m(U) \cap \{ \lambda : \mathbf{F}_m(B) \cap \mathbf{F}_{m,K}(U) \subset \ker \lambda \},$$

$$\mathbf{Z}_K^m(A, B) = \mathbf{F}_K^m(U) \cap \{ \lambda : \mathbf{B}_{m,K}(A, B) \subset \ker \lambda \},$$

$$\mathbf{B}_K^m(A, B) = \mathbf{F}_K^m(U) \cap \{ \lambda : \mathbf{Z}_{m,K}(A, B) \subset \ker \lambda \},$$

$$\mathbf{H}_K^m(A, B; \mathbf{R}) = \mathbf{Z}_K^m(A, B) / \mathbf{B}_K^m(A, B).$$

If $\lambda \in \mathbf{F}_K^m(U)$, then $d\lambda = \lambda \circ \partial \in \mathbf{F}_K^{m+1}(U)$. For $\alpha \in \mathbf{F}^m(A)$ we abbreviate $\alpha|_{\mathbf{F}_{m,K}(U)} = \alpha|_K$; accordingly $d(\alpha|_K) = (d\alpha)|_K$.

For the sequel we readopt the assumptions concerning A, B, U, V, f, g, W stated at the beginning of §3.

4.1. Suppose $0 \leq m \in \mathbf{Z}$, C is a compact Lipschitz neighborhood retract in U , and D is a neighborhood of B in U .

(1) If $\alpha \in \mathbf{B}^m(U, D)$, $\mu \in \mathbf{F}_C^{m-1}(U)$ and

$$\alpha(S) = \mu(\partial S + T)$$

whenever $S \in \mathbf{F}_m(U)$, $T \in \mathbf{F}_{m-1}(D)$ with $\text{spt}(\partial S + T) \subset C$, then there exists $\beta \in \mathbf{F}^{m-1}(U, B)$ such that $\alpha = d\beta$ and $\mu = \beta|_C$.

(2) $\mathbf{B}_C^m(U, D) \subset \{(d\beta) \mid C : \beta \in \mathbf{F}^{m-1}(U, B)\}$ in case $m > 0$.

(3) If $\lambda \in \mathbf{Z}_C^m(U, D)$, then there exists $\eta \in \text{Hom}[\mathbf{Z}_m(U, D), \mathbf{R}]$ with $\mathbf{B}_m(U, D) \subset \ker \eta$ and $\lambda|_{\mathbf{Z}_{m,C}(U, D)} \subset \eta$, and for each such η there exists

$$\gamma \in \mathbf{Z}^m(U, B) \quad \text{with} \quad \lambda \cup [\eta|_{\mathbf{Z}_m(U, B)}] \subset \gamma.$$

(4) $\mathbf{Z}_C^m(U, D) \subset \{\gamma \mid C : \gamma \in \mathbf{Z}^m(U, B)\}$.

(5) $\mathbf{F}_C^m(U, D) \subset \{\gamma \mid C : \gamma \in \mathbf{F}^m(U, B)\}$.

Proof. Proceeding by induction with respect to m , we assume that either $m = 0$ or the assertions (1), (2), (3), (4), (5) with m replaced by $m - 1$ are true.

Since B is relatively closed in A and $A = U \cap \{x : f(x) = x\}$ is relatively closed in U , B is relatively closed in U and [5, 3.1.13] permits the construction of a function

$$\psi \in \mathcal{E}^0(U) \quad \text{with} \quad \text{spt } \psi \subset D, \quad \text{spt}(1 - \psi) \subset U \sim B.$$

Then the method of [5, 4.1.29] yields an infinite sequence $0 < t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < 1$ such that each set

$$B_i = \{x : \psi_i(x) \geq t_i\}$$

is a local Lipschitz neighborhood retract in U . Clearly

$$B \subset B_i \subset \text{Int } B_{i-1} \subset B_{i-1} \subset D \quad \text{for} \quad 0 < i \in \mathbf{Z}.$$

We also choose compact Lipschitz neighborhood retracts C_i in U so that

$$C_0 = C, \quad C_{i-1} \subset \text{Int } C_i \quad \text{for} \quad 0 < i \in \mathbf{Z}, \quad U = \bigcup_{i=0}^{\infty} C_i.$$

To prove (1) we observe first that in case $m > 0$ the relation

$$\xi = \{(\partial S + T, \alpha(S)) : S \in \mathbf{F}_m(U), T \in \mathbf{F}_{m-1}(B_0)\}$$

with domain $\mathbf{B}_{m-1}(U, B_0)$ is a function, because if

$$S, S' \in \mathbf{F}_m(U) \quad \text{and} \quad T, T' \in \mathbf{F}_{m-1}(B_0) \quad \text{with} \quad \partial S + T = \partial S' + T',$$

then $\partial(S - S') = T' - T \in \mathbf{F}_{m-1}(B_0)$, hence

$$S - S' \in \mathbf{Z}_m(U, B_0) \subset \mathbf{Z}_m(U, D) \subset \ker \alpha.$$

Moreover ξ is linear and $\xi|_{\mathbf{B}_{m-1,C}(U, B_0)} = \mu|_{\mathbf{B}_{m-1,C}(U, B_0)}$.

Next we verify that ξ is continuous. For each compact subset K of U we apply 3.2 with A, B, m replaced by $U, B_0, m-1$ to obtain $\tau_K \in \mathbf{R}$ and a compact set E_K with $K \subset E_K \subset U$ and the following property:

If $R \in \mathbf{B}_{m-1,K}(U, B_0)$, then there exists $S \in \mathbf{F}_{m,E_K}(U)$ with

$$R - \partial S \in \mathbf{F}_{m-1}(B_0) \quad \text{and} \quad \mathbf{M}(S) + \mathbf{M}(R - \partial S) \leq \tau_K \mathbf{F}_K(R),$$

hence $\xi(R) = \alpha(S) \leq \mathbf{F}_{E_K}(\alpha|_{E_K})\mathbf{F}_{E_K}(S) \leq \mathbf{F}_{E_K}(\alpha|_{E_K})\tau_K \mathbf{F}_K(R)$.

For $0 < i \in \mathbf{Z}$ the Hahn-Banach theorem yields

$$\mu_i \in \mathbf{F}_{C_i}^{m-1}(U) \quad \text{with} \quad \xi|_{\mathbf{B}_{m-1,C_i}(U, B_0)} = \mu_i|_{\mathbf{B}_{m-1,C_i}(U, B_0)}.$$

Finally we will inductively construct $\beta_i \in \mathbf{F}_{C_i}^{m-1}(U)$ so that

$$\xi|_{\mathbf{B}_{m-1,C_i}(U, B_i)} = \beta_i|_{\mathbf{B}_{m-1,C_i}(U, B_i)} \quad \text{for} \quad 0 \leq i \in \mathbf{Z},$$

$$\beta_0 = \mu, \quad \beta_{i-1} = \beta_i|_{C_{i-1}} \quad \text{for} \quad 0 < i \in \mathbf{Z}.$$

To obtain β_i from β_{i-1} we observe that the restrictions of β_{i-1} , ξ and μ_i to $\mathbf{B}_{m-1,C_{i-1}}(U, B_{i-1})$ are equal, hence

$$\beta_{i-1} - (\mu_i|_{C_{i-1}}) \in \mathbf{Z}_{C_{i-1}}^{m-1}(U, B_{i-1}),$$

apply (4) with m, B, C, D replaced by $m-1, B_i, C_{i-1}, \text{Int } B_{i-1}$ to find

$$\gamma_i \in \mathbf{Z}^{m-1}(U, B_i) \quad \text{with} \quad \beta_{i-1} - (\mu_i|_{C_{i-1}}) = \gamma_i|_{C_{i-1}}$$

and take $\beta_i = \mu_i + (\gamma_i|_{C_i})$. We conclude that

$$\beta = \bigcup_{i=1}^{\infty} \beta_i$$

is a continuous linear function on $\mathbf{F}_m(U)$, which extends both μ and $\xi|_{\mathbf{B}_{m-1}(U, B)}$.

To prove (2) we suppose $\lambda \in \mathbf{B}_C^m(U, D)$, let r be a Lipschitzian retraction of a neighborhood U' of C in U onto C , choose a compact Lipschitz neighborhood retract B' in U' with

$$C \cap B_0 \subset B' \subset r^{-1}(D),$$

observe that $r^*\lambda \in \mathbf{B}^m[U', r^{-1}(D)]$, apply (1) with U replaced by U' , B by B' , C by $\emptyset$, D by $r^{-1}(D)$, α by $r^*\lambda$, μ by 0 , to obtain

$$\beta \in \mathbf{F}^{m-1}(U', B') \quad \text{with} \quad r^*\lambda = d\beta,$$

infer that $\beta|_C \in \mathbf{F}_C^{m-1}(U, B_0)$ and use (5) with m replaced by $m-1$, D by $\text{Int } B_0$, to find

$$\gamma \in \mathbf{F}^{m-1}(U, B) \quad \text{with} \quad \beta|_C = \gamma|_C,$$

hence $(d\gamma)|_C = d(\gamma|_C) = d(\beta|_C) = (r^*\lambda)|_C = \lambda$.

To deduce the first conclusion of (3) we consider the commutative diagram of linear maps

$$\begin{array}{ccc} \mathbf{Z}_{m,c}(U, D) & \xrightarrow{\rho} & \mathbf{H}_{m,c}(U, D; \mathbf{R}) \\ \downarrow \cap & & \downarrow \kappa \\ \mathbf{Z}_m(U, D) & \xrightarrow{\sigma} & \mathbf{H}_m(U, D; \mathbf{R}) \end{array}$$

where ρ , σ are the usual coset maps, and κ is a monomorphism because $\ker \rho = \mathbf{B}_{m,c}(U, D) = \mathbf{Z}_{m,c}(U, D) \cap \ker \sigma$. Since

$$\ker \lambda \supset \ker \rho = \ker (\kappa \circ \rho),$$

there exists a linear function

$$\zeta : \mathbf{H}_m(U, D; \mathbf{R}) \rightarrow \mathbf{R} \quad \text{with} \quad \lambda \mid \mathbf{Z}_{m,c}(U, D) = \zeta \circ \kappa \circ \rho,$$

and it follows that $\eta = \zeta \circ \sigma$ has the required properties. (No continuity of ζ or η is asserted here!)

To prove the second conclusion of (3) we will inductively construct $\gamma_i \in \mathbf{Z}_{C_i}^m(U, B_i)$ for $0 \leq i \in \mathbf{Z}$ so that

$$\begin{aligned} \gamma_i \mid \mathbf{Z}_{m,C_i}(U, B_i) &= \gamma_i \mid \mathbf{Z}_{m,C_i}(U, B_i), \\ \lambda &= \gamma_0, \quad \gamma_{i-1} = \gamma_i \mid C_{i-1} \quad \text{for} \quad 0 < i \in \mathbf{Z}. \end{aligned}$$

To obtain γ_i from γ_{i-1} we observe first that

$$\eta_i = \eta \mid \mathbf{Z}_{m,C_i}(U, B_{i-1}) \quad \text{is} \quad \mathbf{F}_{C_i} \quad \text{continuous,}$$

because $\mathbf{B}_{m,C_i}(U, B_{i-1}) \subset \mathbf{B}_{m,C_i}(U, D) \subset \ker \eta_i$, hence η_i is representable as composition

$$\mathbf{Z}_{m,C_i}(U, B_{i-1}) \xrightarrow{\rho_i} \mathbf{H}_{m,C_i}(U, B_{i-1}; \mathbf{R}) \xrightarrow{\phi_i} \mathbf{R}$$

where the usual coset map ρ_i is a continuous epimorphism with finite dimensional image, by 3.4, and consequently the linear function ϕ_i is also continuous. Using the Hahn-Banach theorem to procure

$$\zeta_i \in \mathbf{Z}_{C_i}^m(U, B_{i-1}) \quad \text{with} \quad \eta_i = \zeta_i \mid \mathbf{Z}_{m,C_i}(U, B_{i-1}),$$

we find that

$$\gamma_{i-1} - (\zeta_i \mid C_{i-1}) \in \mathbf{B}_{C_{i-1}}^m(U, B_{i-1})$$

because the restrictions of γ_{i-1} , η , η_i , ζ_i to $\mathbf{Z}_{m,C_{i-1}}(U, B_{i-1})$ are equal. In case $m = 0$ we infer $\gamma_{i-1} = \zeta_i \mid C_{i-1}$ and take $\gamma_i = \zeta_i$. In case $m > 0$ we apply (2) with B, C, D replaced by $B_i, C_{i-1}, \text{Int } B_{i-1}$ to obtain

$$\beta_i \in \mathbf{F}^{m-1}(U, B_i) \quad \text{with} \quad \gamma_{i-1} - (\zeta_i \mid C_{i-1}) = (d\beta_i) \mid C_{i-1},$$

and take $\gamma_i = \zeta_i + (d\beta) \mid C_i$. We conclude that

$$\gamma = \bigcup_{i=0}^{\infty} \gamma_i \in \mathbf{Z}^m(U, B) \quad \text{and} \quad \lambda \cup [\eta \mid \mathbf{Z}_m(U, B)] \subset \gamma.$$

We note that (4) merely reasserts part of (3).

To prove (5) we suppose $\mu_0 \in \mathbf{F}_C^m(U, D)$, construct successive extensions

$$\mu_i \in \mathbf{F}_{C_i}^m(U) \quad \text{with} \quad \mu_{i-1} = \mu_i \mid \mathbf{F}_{m, C_{i-1}}(U)$$

for $0 < i \in \mathbf{Z}$ by means of the Hahn–Banach theorem, which is applicable in view of 3.5, define

$$\mu = \bigcup_{i=0}^{\infty} \mu_i \in \mathbf{F}^m(U),$$

$$\gamma \in \mathbf{F}^m(U, B), \quad \gamma(T) = \mu(T - T \mid \psi) \quad \text{for} \quad T \in \mathbf{F}_m(U),$$

and infer $\mu_0 = \gamma \mid C$, because if $T \in \mathbf{F}_{m, C}(U)$, then $T \mid \psi \in \mathbf{F}_m(D)$ and $\mu_0(T) = \mu_0(T - T \mid \psi) = \mu(T - T \mid \psi) = \gamma(T)$.

4.2. Replacement of D by B in 4.1 (2), (3), (4), (5) is impossible, as shown by the example where

$$U = \mathbf{R}^2, \quad B = \mathbf{R}^2 \cap \mathbf{B}[(-1, 0), 1], \quad C = \mathbf{R}^2 \cap \mathbf{B}[(1, 0), 1],$$

$$\lambda \in \mathbf{B}_C^1(\mathbf{R}^2, B), \quad \lambda(R) = R(dX_2) \quad \text{for} \quad R \in \mathbf{F}_{1, C}(\mathbf{R}^2),$$

$$\lambda \notin \{\gamma \mid C : \gamma \in \mathbf{F}^1(\mathbf{R}^2, B)\}.$$

In fact for $A = \mathbf{R}^2 \cap \mathbf{B}[(0, 0), 2]$ there exists no $\nu \in \mathbf{F}_A^1(A, B)$ such that $\lambda = \nu \mid \mathbf{F}_{1, C}(\mathbf{R}^2)$, because this would imply

$$S_\theta = [(0, 0), (1 - \cos(\theta), \sin(\theta))] \in \mathbf{F}_1(C), \quad \lambda(S_\theta) = \sin(\theta),$$

$$T_\theta = [(0, 0), (\cos(\theta) - 1, \sin(\theta))] \in \mathbf{F}_1(B), \quad \nu(T_\theta) = 0,$$

$$\mathbf{F}_A(S_\theta - T_\theta) \leq 2[1 - \cos(\theta)] + [1 - \cos(\theta)] \cdot |\sin(\theta)|,$$

$$|\sin(\theta)| = |\lambda(S_\theta)| = |\nu(S_\theta - T_\theta)| \leq \mathbf{F}_A(\nu) \cdot \mathbf{F}_A(S_\theta - T_\theta)$$

for $\theta \in \mathbf{R}$, which is absurd since $\mathbf{F}_A(S_\theta - T_\theta)/|\sin(\theta)| \rightarrow 0$ as $\theta \rightarrow 0$. We observe that $\mathbf{F}_1(B)$ and $\mathbf{F}_1(C)$ are $\mathbf{F}_A$ closed linear subspaces of $\mathbf{F}_1(A)$ whose vector sum $\mathbf{F}_1(B) + \mathbf{F}_1(C)$ fails to be $\mathbf{F}_A$ closed.

The relation between α and μ hypothesized in 4.1(1) cannot be weakened to $d\alpha \mid C = \mu$, as seen from the example where

$$U = \mathbf{R}, \quad B = D = \emptyset, \quad C = \mathbf{R} \cap \{x : 1 \leq |x| \leq 2\},$$

$$\alpha \in \mathbf{B}^1(\mathbf{R}), \quad \alpha(S) = S(dX_1) \quad \text{for} \quad S \in \mathbf{F}_1(\mathbf{R}),$$

$$q \in \mathcal{E}^0(\mathbf{R}), \quad q(x) = x + x/|x| \quad \text{for} \quad x \in C,$$

$$\mu \in \mathbf{F}_C^0(\mathbf{R}), \quad \mu(T) = T(q) \quad \text{for} \quad T \in \mathbf{F}_{1, C}(\mathbf{R}).$$

4.3. B has a neighborhood G in U such that

$$\mathbf{Z}^m(U, B) = \{\alpha + d\beta : \alpha \in \mathbf{Z}^m(U, G), \quad \beta \in \mathbf{F}^{m-1}(U, B)\},$$

$$\mathbf{B}^m(U, B) = \{d\beta : \beta \in \mathbf{F}^{m-1}(U, B)\}$$

in case $m > 0$, and $\mathbf{Z}^0(U, B) = \mathbf{Z}^0(U, G)$.

Proof. We construct ψ as in the proof of 4.1 taking $D = W$, let $G = U \sim \text{spt}(1 - \psi)$, and consider the homotopy

$$k : \{t : 0 \leq t \leq 1\} \times U \rightarrow U,$$

$$k(t, x) = x \quad \text{if } 0 \leq t \leq 1 \quad \text{and } x \in U \sim W,$$

$$k(t, x) = [1 - t\psi(x)]x + t\psi(x)g(x) \quad \text{if } 0 \leq t \leq 1 \quad \text{and } x \in W.$$

If $\gamma \in \mathbf{Z}^m(U, B)$, then $k_1^*\gamma \in \mathbf{Z}^m(U, G)$ because $k_1(G) \subset B$, $k_0^*\gamma = \gamma$ because $k_0(x) = x$ for $x \in U$, and $d\gamma = 0$.

In case $m = 0$, $\gamma = k_1^*\gamma$ by the homotopy formula.

In case $m > 0$, $\gamma = k_1^*\gamma - d[(k^*\gamma)_{|0,1}]$ by the homotopy formula, and $(k^*\gamma)_{|0,1} \in \mathbf{F}^m(U, B)$ because

$$k(t, x) = x \quad \text{if } 0 \leq t \leq 1 \quad \text{and } x \in B,$$

$$k_*([0, 1] \times T) = 0 \quad \text{for } T \in \mathbf{F}_m(B).$$

In case $m > 0$ and $\gamma \in \mathbf{B}^m(U, B)$, then $k_1^*\gamma \in \mathbf{B}^m(U, G)$ because $k_1(G) \subset B$, and 4.1(1) applied with C, D, α, μ replaced by $\emptyset, G, k_1^*\gamma, 0$ yields $\beta \in \mathbf{F}^m(U, B)$ such that

$$k_1^*\gamma = d\beta, \text{ hence } \gamma = d[\beta - (k^*\gamma)_{|0,1}].$$

4.4. $\mathbf{B}^m(A, B) = \{d\beta : \beta \in \mathbf{F}^{m-1}(A, B)\}$ in case $m > 0$.

Proof. If $\gamma \in \mathbf{B}^m(A, B)$, then $f^*\gamma \in \mathbf{B}^m(U, B)$ because $f(B) = B$, hence 4.3 yields $\beta \in \mathbf{F}^{m-1}(U, B)$ with $f^*\gamma = d\beta$, and $\theta : A \subsetneq U$ gives $\theta^*\beta \in \mathbf{F}^{m-1}(A, B)$ with $d\theta^*\beta = \theta^*f^*\gamma = \gamma$.

4.5. $\mathbf{H}^m(A, B; \mathbf{R}) \simeq \text{Hom} [\mathbf{H}_m(A, B; \mathbf{R}), \mathbf{R}]$.

Proof. Given $v \in \text{Hom} [\mathbf{H}_m(A, B; \mathbf{R}), \mathbf{R}]$ we complete the commutative diagram of linear maps

$$\begin{array}{ccccc} \mathbf{Z}_m(A, B) & \longrightarrow & \mathbf{H}_m(A, B; \mathbf{R}) & & \\ \downarrow & & \downarrow & \searrow v & \\ \mathbf{Z}_m(U, W) & \xrightarrow{\quad \rho \quad} & \mathbf{H}_m(U, W; \mathbf{R}) & \longrightarrow & \mathbf{R} \\ \downarrow & & \nearrow \gamma & \nearrow \omega & \\ \mathbf{F}_m(U) & & & & \end{array}$$

where ρ is the usual coset map, $\theta : A \subset U$, ω exists because $\mathbf{H}_m(\theta; \mathbf{R})$ is a monomorphism according to 3.2, and γ is obtained from 4.1(3) with C, D, λ, η replaced by $\emptyset, W, 0, \omega \circ \rho$. Then $\gamma \mid \mathbf{F}_m(A) = \theta^* \gamma \in \mathbf{Z}^m(A, B)$ induces ν by restriction to $\mathbf{Z}_m(A, B)$.

4.6. We observe that a real valued linear function α on $\mathbf{F}_m(U)$ is a **locally flat cochain** of the open set U if and only if

$$\mathbf{F}_K(\alpha \mid K) < \infty \text{ for each compact subset } K \text{ of } U,$$

but in order that α be a **flat cochain** of U in the sense of [5, 4.1.19] it must meet the stronger requirement

$$\mathbf{F}(\alpha) = \sup \{ \mathbf{F}_K(\alpha \mid K) : K \text{ is a compact subset of } U \} < \infty.$$

However, if α is locally flat, then the restriction of α to the interior of each compact subset of U is flat, hence the results of [5, 4.1.19] yield $\mathcal{L}^n \lfloor U$ measurable differential forms ϕ and ψ of degrees m and $m+1$ on U satisfying the four conditions:

- (1) $\alpha(\mathcal{L}^n \wedge \xi) = \int_U \langle \xi, \phi \rangle d\mathcal{L}^n$ for $\xi \in \mathbf{L}_1(\mathcal{L}^n, \wedge_m \mathbf{R}^n)$ with compact support in U .
- (2) $\alpha[\partial(\mathcal{L}^n \wedge \eta)] = \int_U \langle \eta, \psi \rangle d\mathcal{L}^n$ for $\eta \in \mathbf{L}_1(\mathcal{L}^n, \wedge_{m+1} \mathbf{R}^n)$ with compact support in U .
- (3) $\sup \{ \|\phi(x)\|, \|\psi(x)\| \} \leq \mathbf{F}_K(\alpha \mid K)$ for $\mathcal{L}^n$ almost all x in the interior of each compact subset K of U .
- (4) $\int_U (\langle \operatorname{div} \eta, \phi \rangle + \langle \eta, \psi \rangle) d\mathcal{L}^n = 0$ for $\eta \in \mathfrak{D}(U, \wedge_{m+1} \mathbf{R}^n)$.

Moreover ϕ and ψ are $\mathcal{L}^n \lfloor U$ almost unique. In view of [5, 2.3.6] we may require ϕ and ψ to be Baire forms.

Conversely, any two locally bounded $\mathcal{L}^n \lfloor U$ measurable differential forms ϕ and ψ of degrees m and $m+1$ on U satisfying (4) are associated with a unique $\alpha \in \mathbf{F}_m(U)$ such that (1), (2), (3) hold.

Representing $\eta = \mathbf{D}^{n-m-1} \omega$ we see from [5, 4.1.7] that (4) is equivalent to the condition

$$(4)' \quad \mathbf{E}^n[\psi \wedge \omega + (-1)^m \phi \wedge d\omega] = 0 \text{ for } \omega \in \mathfrak{D}^{n-m-1}(U).$$

In particular, if $\phi \in \mathcal{E}^m(U)$, then ϕ and $d\phi$ are associated with $\tilde{\phi} \in \mathbf{F}^m(U)$, where $\tilde{\phi}(T) = T(\phi)$ for $T \in \mathbf{F}_m(U)$. Since $\mathbf{F}_m(U)$ is dense in $\mathcal{E}_m(U)$, $\tilde{\phi} = 0$ if and only if $\phi = 0$.

4.7. In order to regularize locally flat cochains of the open set U we generalize the procedures of [5, 4.1.2, 4.1.18] as follows: Choosing Φ, Φ_* as in [5, 4.1.2] we also select

$$v \in \mathcal{E}^0(U) \text{ with } 0 < v(x) < \operatorname{dist}(x, \mathbf{R}^n \sim U) \text{ for } x \in U.$$

(Such a function v is readily constructed, for instance through use of [5, 3.1.13]. In case $U = \mathbf{R}^n$ one may take $v(x) = 1$ for $x \in \mathbf{R}^n$, and recover standard smoothing.) We note that

$$\Delta^z = (\mathbf{R} \times U) \cap \{(t, x) : x + tv(x)z \in U\} \supset \{t : 0 \leq t \leq 1\} \times U$$

whenever $z \in \mathbf{R}^n$ with $|z| \leq 1$, define

$$H^z : \Delta^z \rightarrow U, \quad H^z(t, x) = x + tv(x)z \quad \text{for } (t, x) \in \Delta^z,$$

$$H_t^z : U \rightarrow U \quad \text{for } 0 \leq t \leq 1, \quad H_t^z(x) = H(t, x) \quad \text{for } x \in U,$$

and compute $\dot{H}_t^z(x) = v(x)z$, $DH_t^z = \mathbf{1}_{\mathbf{R}^n} + tDv(x)z$.

Suppose $0 < \epsilon \leq 1$. Whenever K is a compact subset of U , so is

$$K_\epsilon = \cup \{\mathbf{B}[x, v(x)\epsilon] : x \in K\};$$

if $0 \leq t \leq 1$, $x \in K$ and $\{z, w\} \subset \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$, then

$$H^z(t, x) \in K_\epsilon, \quad |H^z(t, x) - H^w(t, x)| \leq v(x) |z - w|,$$

$$\|DH^z(t, x)\| \leq \sup \{v(x)\epsilon, 1 + \|Dv(x)\|\epsilon\} \leq 1 + \mathbf{F}_K(v)\epsilon;$$

also $\text{Lip } [H^z | (\{t : 0 \leq t \leq 1\} \times K)] \leq 1 + \text{Lip } (v | K)\epsilon$.

We define the regularization $\phi_\epsilon \in \mathcal{E}^m(U)$ of any locally $\mathcal{L}^n$ summable differential form ϕ of degree m on U by the formula

$$\begin{aligned} \phi_\epsilon(x) &= \int \Phi_\epsilon(-z) [H_1^{z*} \phi](x) d\mathcal{L}^n z \\ &= \int \langle \phi[x + v(x)z], \Phi_\epsilon(-z) \wedge^m [\mathbf{1}_{\mathbf{R}^n} + Dv(x)z] \rangle d\mathcal{L}^n z \\ &= \int \langle \phi(y), \theta_\epsilon^m(x, y) \rangle d\mathcal{L}^n y \end{aligned}$$

for $x \in U$, where $\theta_\epsilon^m : U \times U \rightarrow \text{Hom} [\wedge^m \mathbf{R}^n, \wedge^m \mathbf{R}^n]$,

$$\theta_\epsilon^m(x, y) = v(x)^{-n} \Phi_\epsilon \left[\frac{x - y}{v(x)} \right] \wedge^m \left[\mathbf{1}_{\mathbf{R}^n} + Dv(x) \frac{y - x}{v(x)} \right]$$

for $(x, y) \in U \times U$. If ϕ is of class 1, then $(d\phi)_\epsilon = d(\phi_\epsilon)$, because d and H_1^{z*} commute. If ϕ is of class ∞ , then $\phi_\epsilon \rightarrow \phi$ in $\mathcal{E}^m(U)$ as $\epsilon \downarrow 0$.

Whenever $T \in \mathcal{E}_m(U)$ we define the regularization $T_\epsilon \in \mathcal{E}_m(U)$ and the smoothing deformation chain $H_\epsilon T \in \mathcal{E}_{m+1}(U)$ by the formulae

$$T_\epsilon(\phi) = T(\phi_\epsilon) = \int \Phi_\epsilon(-z) [H_1^{z*} T](\phi) d\mathcal{L}^n z \quad \text{for } \phi \in \mathcal{E}^m(U),$$

$$[H_\epsilon T](\psi) = \int \Phi_\epsilon(-z) H_\epsilon^z([0, 1] \times T)(\psi) d\mathcal{L}^n z \quad \text{for } \psi \in \mathcal{E}^{m+1}(U),$$

and infer the smoothing homotopy formulae for currents:

$$T_\epsilon - T = \partial(H_\epsilon T) + H_\epsilon(\partial T) \text{ if } m > 0, \quad T_\epsilon - T = \partial(H_\epsilon T) \text{ if } m = 0.$$

Since $T(\phi_\epsilon) = T_x \int \langle \phi(y), \theta_\epsilon^m(x, y) \rangle d\mathcal{L}^n y = \int T_x \langle \phi(y), \theta_\epsilon^m(x, y) \rangle d\mathcal{L}^n y$, we obtain the representation

$$T_\epsilon = \mathcal{L}^n \wedge \xi_{T, \epsilon} \quad \text{with} \quad \xi_{T, \epsilon} \in \mathcal{D}(U, \wedge_m \mathbf{R}^n),$$

$$\langle \xi_{T, \epsilon}(y), \kappa \rangle = T_x \langle \kappa, \theta_\epsilon^m(x, y) \rangle \quad \text{for} \quad y \in U, \kappa \in \wedge^m \mathbf{R}^n.$$

Moreover $T_\epsilon \rightarrow T$ in $\mathcal{E}_m(U)$ as $\epsilon \downarrow 0$.

In regard to the **smoothing of normal currents and flat chains** we gain additional information from [5, 4.1.13, 4.1.14]:

If $T \in \mathbf{N}_{m, K}(U)$ and $\{z, w\} \subset \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$, then

$$\mathbf{F}_{K, \epsilon}(H_{1\#}^z T - H_{1\#}^w T) \leq \mathbf{N}(T) \mathbf{F}_K(v) \cdot |z - w| \cdot [1 + \mathbf{F}_K(v)\epsilon]^m,$$

$$\mathbf{F}_{K, \epsilon}[H_\#^z([0, 1] \times T) - H_\#^w([0, 1] \times T)] \leq 3\mathbf{N}(T) \mathbf{F}_K(v) \cdot |z - w| \cdot [1 + \mathbf{F}_K(v)]^{m+1}.$$

If $T \in \mathbf{F}_{m, K}(U)$ and $z \in \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$, then

$$\mathbf{F}_{K, \epsilon}(H_{1\#}^z T) \leq [1 + \text{Lip}(v | K)\epsilon]^{m+1} \mathbf{F}_K(T),$$

$$\mathbf{F}_{K, \epsilon}[H_\#^z([0, 1] \times T)] \leq [1 + \text{Lip}(v | K)\epsilon]^{m+2} 3\mathbf{F}_K(T).$$

If $T \in \mathbf{F}_{m, K}(U)$, then the functions mapping $z \in \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$ onto $H_{1\#}^z T \in \mathbf{F}_{m, K, \epsilon}(U)$ and $H_\#^z([0, 1] \times T) \in \mathbf{F}_{m+1, K, \epsilon}(U)$ are continuous, hence

$$T_\epsilon = \int \Phi_\epsilon(-z) H_{1\#}^z T d\mathcal{L}^n z \in \mathbf{F}_{m, K, \epsilon}(U),$$

$$H_\epsilon T = \int \Phi_\epsilon(-z) H_\#^z([0, 1] \times T) d\mathcal{L}^n z \in \mathbf{F}_{m+1, K, \epsilon}(U),$$

$$\mathbf{F}_{K, \epsilon}(T_\epsilon) \leq [1 + \text{Lip}(v | K)\epsilon]^{m+1} \mathbf{F}_K(T),$$

$$\mathbf{F}_{K, \epsilon}(H_\epsilon T) \leq [1 + \text{Lip}(v | K)\epsilon]^{m+2} 3\mathbf{F}_K(T);$$

moreover $\mathbf{F}_{K, \epsilon}(H_\epsilon T) \rightarrow 0$ and $\mathbf{F}_{K, \epsilon}(T_\epsilon - T) \rightarrow 0$ as $\epsilon \downarrow 0$.

Whenever $\alpha \in \mathbf{F}^m(U)$ we define the regularization $\alpha_\epsilon \in \mathbf{F}^m(U)$, and in case $m > 0$ the smoothing deformation cochain $H_\epsilon \alpha \in \mathbf{F}^{m-1}(U)$, by the formulae

$$\alpha_\epsilon(T) = \alpha(T_\epsilon) \quad \text{for} \quad T \in \mathbf{F}_m(U),$$

$$[H_\epsilon \alpha](S) = \alpha(H_\epsilon S) \quad \text{for} \quad S \in \mathbf{F}_{m-1}(U),$$

and infer the smoothing homotopy formulae for locally flat cochains

$$\alpha_\epsilon - \alpha = H_\epsilon(d\alpha) + d(H_\epsilon \alpha) \quad \text{if} \quad m > 0, \quad \alpha_\epsilon - \alpha = H_\epsilon(d\alpha) \quad \text{if} \quad m = 0.$$

The following proposition contains the smoothness of α_ϵ :

If ϕ and ψ are locally bounded Baire forms of degrees m and $m + 1$ on U associated with α as in 4.6, then

$$\alpha_\epsilon = (\phi_\epsilon)^\sim, \quad d(\alpha_\epsilon) = (\psi_\epsilon)^\sim \quad \text{and} \quad d(\phi_\epsilon) = \psi_\epsilon.$$

To prove the first conclusion we observe that α_* and $(\phi_*)^\sim$ are continuous on $\mathbf{F}_m(U)$ and, for every T in the dense subset $\mathbf{M}_m(U) \cap \mathbf{F}_m(U)$,

$$\begin{aligned}\alpha_*(T) &= \alpha(\mathcal{E}^n \wedge \xi_{T,*}) = \int \langle \xi_{T,*}(y), \phi(y) \rangle d\mathcal{E}^n y = \int T_x \langle \phi(y), \theta_*^m(x, y) \rangle d\mathcal{E}^n y \\ &= T_x \int \langle \phi(y), \theta_*^m(x, y) \rangle d\mathcal{E}^n y = T(\phi_*) = (\phi_*)^\sim(T).\end{aligned}$$

The first conclusion yields the second because ψ and 0 are associated with $d\alpha$. They jointly imply

$$[d(\phi_*)]^\sim = d[(\phi_*)^\sim] = d(\alpha_*) = (\psi_*)^\sim,$$

hence the third conclusion.

If ϕ is associated with α as in 4.6, $T \in \mathbf{F}_m(U) \cap \mathbf{M}_m(U)$ and ϕ is $\|T\|$ almost everywhere continuous, then

$$\alpha(T) = \lim_{\epsilon \downarrow 0} \alpha(T_\epsilon) = \lim_{\epsilon \downarrow 0} T(\phi_\epsilon) = \lim_{\epsilon \downarrow 0} \int \langle \vec{T}, \phi_\epsilon \rangle d\|T\| = \int \langle \vec{T}, \phi \rangle d\|T\|$$

by virtue of Lebesgue's bounded convergence theorem.

If $\rho \in \mathcal{E}^m(U)$, then $(\tilde{\rho})_* = (\rho_*)^\sim$.

If $\rho \in \mathcal{E}^m(U)$ and $m > 0$, then $H_* \tilde{\rho} = (H_* \rho)^\sim$ with

$$H_* \rho = \int \Phi_*(-z) \langle H^{z*} \rho \rangle_{[0,1]} d\mathcal{E}^n z \in \mathcal{E}^{m-1}(U).$$

In fact

$$\begin{aligned}[H_* \tilde{\rho}](S) &= \tilde{\rho}(H_* S) = [H_* S](\rho) \\ &= \int \Phi_*(-z) \langle [0, 1] \times S \rangle H^{z*} \rho d\mathcal{E}^n z \quad \text{for } S \in \mathbf{F}_{m-1}(U).\end{aligned}$$

4.8. *B has a neighborhood D in U such that if*

$$v \in \text{Hom} [\mathbf{H}_m(A, B; \mathbf{R}), \mathbf{R}],$$

then there exists $\rho \in \mathcal{E}^m(U)$ with $d\rho = 0$, $\text{spt } \rho \subset U \sim D$,

$$v(\chi) = T(\rho) \quad \text{whenever } T \in \chi \in \mathbf{H}_m(A, B; \mathbf{R}).$$

Proof. We choose G as in 4.3, use [5, 3.1.13] to construct

$$\begin{aligned}v \in \mathcal{E}^0(U) \quad \text{with} \quad 0 < v(x) < \text{dist}(x, \mathbf{R}^n \sim U) \quad \text{for } x \in U, \\ v(x) < \text{dist}(x, \mathbf{R}^n \sim G) \quad \text{for } x \in B,\end{aligned}$$

and let $D = U \cap \{x : \mathbf{B}[x, v(x)] \subset G\}$.

By 4.5 there exists $\gamma \in \mathbf{Z}^m(A, B)$ such that

$$v(\chi) = \gamma(T) \quad \text{whenever } T \in \chi \in \mathbf{H}_m(A, B; \mathbf{R}).$$

Then $f^*\gamma \in \mathbf{Z}^m(U, B)$ and 4.3 yields

$$\alpha \in \mathbf{Z}^m(U, G), \quad \beta \in \mathbf{F}^{m-1}(U, B) \quad \text{with} \quad f^*\gamma = \alpha + d\beta$$

in case $m > 0$, $f^*\gamma = \alpha$ in case $m = 0$. Representing α by locally bounded Baire forms ϕ and ψ as in 4.6, and using v to smooth α by the method of 4.7, we obtain $\alpha_\epsilon = (\phi_\epsilon)^\sim$ with

$$\begin{aligned} \phi_\epsilon \in \mathcal{E}^m(U), \quad \text{spt } \phi_\epsilon \subset U \sim D, \quad d\alpha = 0, \quad d(\phi_\epsilon) = 0, \\ \alpha_\epsilon - \alpha = d(H_\epsilon \alpha) \quad \text{and} \quad H_\epsilon \alpha \in \mathbf{F}^{m-1}(U, D) \quad \text{if} \quad m > 0, \end{aligned}$$

because $H^z(\{t : 0 \leq t \leq 1\} \times D) \subset G$ for $z \in \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$, hence $\text{spt } (H_\epsilon S) \subset G$ for $S \in \mathbf{F}_{m-1}(D)$; moreover $\alpha_\epsilon = \alpha$ if $m = 0$. We conclude that

$$v(\chi) = \gamma(T) = (f^*\gamma)(T) = \alpha(T) = \alpha_\epsilon(T) = T(\phi_\epsilon)$$

whenever $T \in \chi \in \mathbf{H}_m(A, B; \mathbf{R})$, and take $\rho = \phi_\epsilon$.

4.9. *A and B have neighborhoods U' and V' in U such that if*

$$\begin{aligned} \rho \in \mathcal{E}^m(U), \quad m > 0, \quad d\rho = 0, \quad \text{spt } \rho \subset U \sim V, \\ T(\rho) = 0 \quad \text{whenever} \quad T \in \mathbf{Z}_m(A, B), \end{aligned}$$

then there exists $\sigma \in \mathcal{E}^{m-1}(U')$ with $\rho \mid U' = d\sigma$, $\text{spt } \sigma \subset U' \sim V'$.

Proof. We define

$$\begin{aligned} k(t, x) &= tx + (1-t)f(x) \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad x \in U, \\ h(t, x) &= tx + (1-t)g(x) \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad x \in V, \\ U' &= \{x : k(t, x) \in U \quad \text{whenever} \quad 0 \leq t \leq 1\}, \end{aligned}$$

$$Z = \{x : k(t, x) \in V \quad \text{and} \quad h(t, x) \in U \cap V \quad \text{whenever} \quad 0 \leq t \leq 1\},$$

use the method of [5, 4.1.29] to construct a local Lipschitz neighborhood retract B' relatively closed in U such that $B \subset \text{Int } B' \subset B' \subset Z$, obtain from [5, 3.1.13] a function

$$\begin{aligned} v \in \mathcal{E}^0(U') \quad \text{with} \quad 0 < v(x) < \text{dist}(x, \mathbf{R}^n \sim U') \quad \text{for} \quad x \in U', \\ v(x) < \text{dist}(x, \mathbf{R}^n \sim \text{Int } B') \quad \text{for} \quad x \in B, \end{aligned}$$

and let $V' = U' \cap \{x : \mathbf{B}[x, v(x)] \subset \text{Int } B'\}$.

If $R \in \mathbf{Z}_m(U', Z)$, then $T = f_\# [R - h_\#([0, 1] \times \partial R)] \in \mathbf{Z}_m(A, B)$ and $0 = T(\rho) = [f_\# R](\rho)$; moreover

$$R - f_\# R = \partial k_\#([0, 1] \times R) + k_\#([0, 1] \times \partial R)$$

and $[R - f_\# R](\rho) = 0$; consequently $R(\rho) = 0$.

Thus $(\rho \mid U')^\sim \in \mathbf{B}^m(U', Z) \subset \mathbf{B}^m(U', B')$ and 4.3 yields

$$\alpha \in \mathbf{F}^{m-1}(U', B') \quad \text{with} \quad (\rho \mid U')^\sim = d\alpha.$$

Representing α by locally bounded Baire forms ϕ and ψ as in 4.6 with m replaced by $m - 1$, and using v to smooth α by the method of 4.7, we find that

$$\alpha_\epsilon = (\phi_\epsilon)^\sim, \quad \phi_\epsilon \in \mathcal{E}^{m-1}(U), \quad d(\alpha_\epsilon - \alpha) = d[H_\epsilon(d\alpha)],$$

$$H_\epsilon(d\alpha) = H_\epsilon[(\rho \mid U')^\sim] = [H_\epsilon(\rho \mid U')]^\sim$$

and $H_\epsilon(\rho \mid U') \in \mathcal{E}^{m-1}(U)$, hence

$$(\rho \mid U')^\sim = d[\alpha_\epsilon - H_\epsilon(d\alpha)] = d(\sigma^\sim) \quad \text{and} \quad \rho \mid U' = d\sigma$$

with $\sigma = \phi_\epsilon - H_\epsilon(\rho \mid U') \in \mathcal{E}^{m-1}(U)$. Moreover σ vanishes on V' , because ϕ and ρ vanish on $\mathcal{E}^n$ almost all of $\text{Int } B'$.

The preceding propositions 4.8 and 4.9 generalize two classical theorems of De Rham, first proved in [3]. They describe an isomorphism between

$$\text{Hom } [\mathbf{H}_m(A, B; \mathbf{R}), \mathbf{R}]$$

and the m^{th} cohomology group, with respect to exterior differentiation, of the cochain complex consisting of the germs at A of all differential forms of class ∞ with germ zero at B . (Two forms have the same germ at a set X if and only if they agree on some neighborhood of X .)

Next we will construct flat cochains and differential forms related to the minimization of $\int \Psi$ with homological constraints. For this purpose we dualize 3.8 by defining

$$\Psi_\Gamma(\alpha) = \inf \{ \mathbf{R} \cap \{r : \alpha(T) \leq r\Psi_\Gamma(T) \text{ for } T \in \Delta\} \}$$

whenever Γ, Δ are subsets of $\mathbf{F}_m(U)$ and α is a real valued function on Δ .

4.10. Suppose K is a compact subset of A ,

$$\Gamma = \mathbf{B}_{m,K}(A, B)$$

and Ψ is a convex positive integrand on U .

(1) For each $T \in \mathbf{F}_{m,K}(U) \sim \Gamma$ there exists

$$\lambda \in \mathbf{Z}_K^m(A, B) \quad \text{with} \quad \Psi_\Gamma(\lambda) = 1 \quad \text{and} \quad \lambda(T) = \Psi_\Gamma(T);$$

moreover, if $T \in \mathbf{Z}_{m,K}(A, B) \sim \Gamma$, then

$$\inf \{ \Psi_\Gamma(\lambda + \nu) : \nu \in \mathbf{B}_K^m(A, B) \} = \Psi_\Gamma(\lambda).$$

(2) If $\lambda \in \mathbf{F}_K^m(U)$, then $\Psi_\Gamma[\lambda \mid \mathbf{Z}_{m,K}(A, B)]$ equals the least number in

$$\{ \Psi_\Gamma(\lambda + \nu) : \nu \in \mathbf{B}_K^m(A, B) \}.$$

(3) For each $\lambda \in \mathbf{Z}_K^m(A, B)$ with

$$\inf \{ \Psi_\Gamma(\lambda + \nu) : \nu \in \mathbf{B}_K^m(A, B) \} = \Psi_\Gamma(\lambda) = 1,$$

there exists $T \in \mathbf{Z}_{m,K}(A, B)$ with $\lambda(T) = \Psi_\Gamma(T) = 1$.

(4) If $T \in \mathbf{F}_{m,K}(U)$, then

$$\Psi_{\Gamma}(T) = \sup \{T(\rho) : \rho \in \mathcal{S}^m(U), \quad d\rho = 0, \quad \text{spt } \rho \subset U \sim B$$

$$\text{and } K \subset \text{Int } \{x : \rho(x) \leq \Psi_x\}\}.$$

(5) If $T \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U)$, $\lambda \in \mathbf{Z}_K^m(A, B)$, $\lambda(T) = \int_T \Psi$,

$$\lambda(S) \leq \int_S \Psi \quad \text{whenever } S \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U),$$

then $\Psi_{\Gamma}(T) = \int_T \Psi$.

Proof of (1). The function $p = \Psi_{\Gamma} \mid \mathbf{F}_{m,K}(U)$ is convex and positively homogeneous by 3.8, with $p(T) > 0$ by 3.9. Applying the Hahn–Banach theorem we obtain

$$\lambda \in \text{Hom} [\mathbf{F}_{m,K}(U), \mathbf{R}] \quad \text{with } \lambda(T) = p(T) \quad \text{and } \lambda \leq p,$$

hence $\Psi_{\Gamma}(\lambda) = 1$. Assured by 3.8 that

$$p(S) \leq c\mathbf{F}_K(S) \quad \text{for } S \in \mathbf{F}_{m,K}(U), \quad p(S) = 0 \quad \text{for } S \in \Gamma,$$

we infer that λ is $\mathbf{F}_K$ continuous and $\lambda \in \mathbf{Z}_K^m(A, B)$.

Moreover, if $T \in \mathbf{Z}_{m,K}(A, B) \sim \Gamma$ and $\nu \in \mathbf{B}_K^m(A, B)$, then

$$\Psi_{\Gamma}(\lambda + \nu)p(T) \geq [\lambda + \nu](T) = \lambda(T) = p(T),$$

hence $\Psi_{\Gamma}(\lambda + \nu) \geq 1 = \Psi_{\Gamma}(\lambda)$.

Proof of (2). Let $\Omega = \mathbf{Z}_{m,K}(A, B)$, Clearly

$$\Psi_{\Gamma}(\lambda \mid \Omega) = \Psi_{\Gamma}[(\lambda + \nu) \mid \Omega] \leq \Psi_{\Gamma}(\lambda + \nu)$$

for all $\nu \in \mathbf{B}_K^m(A, B)$. Also, since $\lambda \mid \Omega \leq \Psi_{\Gamma}(\lambda \mid \Omega) \cdot (p \mid \Omega)$, the Hahn–Banach theorem yields $\mu \in \text{Hom} [\mathbf{F}_{m,K}(U), \mathbf{R}]$ with $\mu \mid \Omega = \lambda \mid \Omega$ and $\mu \leq \Psi_{\Gamma}(\lambda \mid \Omega) \cdot p$, hence

$$\mu - \lambda \in \mathbf{B}_K^m(A, B) \quad \text{and} \quad \Psi_{\Gamma}(\mu) = \Psi_{\Gamma}(\lambda \mid \Omega).$$

Proof of (3). Since $\Psi_{\Gamma}(S + R) = \Psi_{\Gamma}(S)$ and $\lambda(S + R) = \lambda(S)$ whenever $S \in \mathbf{Z}_{m,K}(A, B)$ and $R \in \Gamma$, there exist a convex positively homogeneous function q and a linear function κ on the finite dimensional vector space $\mathbf{H}_{m,K}(A, B; \mathbf{R})$ such that

$$\Psi_{\Gamma}(S) = q(\chi) \quad \text{and} \quad \lambda(S) = \kappa(\chi)$$

whenever $S \in \chi \in \mathbf{H}_{m,K}(A, B; \mathbf{R})$; furthermore $q(\chi) > 0$ unless $\chi = \Gamma$, the zero vector in $\mathbf{H}_{m,K}(A, B; \mathbf{R})$. We infer that the closed convex set

$$E = \mathbf{H}_{m,K}(A, B; \mathbf{R}) \cap \{\chi : q(\chi) \leq 1\}$$

is compact, choose $\tau \in E$ with $\kappa(\tau) = \sup \{\kappa(\chi) : \chi \in E\}$, take $T \in \tau$, then conclude $\Psi_{\Gamma}(T) = q(\tau) = 1$ and

$$\lambda(T) = \kappa(\tau) = \Psi_{\Gamma}[\lambda \mid \mathbf{Z}_{m,K}(A, B)] = \Psi_{\Gamma}(\lambda) = 1,$$

by virtue of (2).

Proof of (4). First we observe that if $\rho \in \mathcal{E}^m(U)$, $d\rho = 0$, $\text{spt } \rho \subset U \sim B$ and $\rho(x) \leq \Psi_x$ for $x \in K$, then

$$T(\rho) = Q(\rho) = \int \langle \vec{Q}(x), \rho(x) \rangle d \|\vec{Q}\| \quad x \leq \int \Psi[x, \vec{Q}(x)] d \|Q\| \quad x = \int_Q \Psi$$

whenever $Q \in \mathbf{M}_m(U)$ with $Q - T \in \Gamma$, hence $T(\rho) \leq \Psi_\Gamma(T)$.

Next we suppose $\Psi_\Gamma(T) > 0$. Given any $\delta > 0$ we use 3.10 to choose a compact Lipschitz neighborhood retract C in U with $K \subset \text{Int } C$, and a neighborhood D of B in W , so that

$$\Psi_{\mathbf{B}_{m,c}(U,D)}(T) > \Psi_\Gamma(T)/(1 + \delta),$$

construct local Lipschitz neighborhood retracts B' , B'' relatively closed in U for which

$$B \subset \text{Int } B'' \subset B'' \subset \text{Int } B' \subset B' \subset D,$$

apply (1) replacing Γ by $\Gamma' = \mathbf{B}_{m,c}(U, B')$ to find

$$\lambda \in \mathbf{Z}_c^m(U, B') \quad \text{with} \quad \Psi_{\Gamma'}(\lambda) = 1 \quad \text{and} \quad \lambda(T) = \Psi_{\Gamma'}(T),$$

apply 4.1(4) replacing B, D , by $B'', \text{Int } B'$ to find

$$\alpha \in \mathbf{Z}^m(U, B'') \quad \text{with} \quad \lambda = \alpha \upharpoonright C,$$

choose $v \in \mathcal{E}^0(U)$ with $0 < v(x) < \text{dist}(x, \mathbf{R}^n \sim U)$ for $x \in U$,

$$v(x) < \text{dist}(x, \mathbf{R}^n \sim \text{Int } B'') \quad \text{for} \quad x \in B,$$

$$v(x) < \text{dist}(x, \mathbf{R}^n \sim \text{Int } C) \quad \text{for} \quad x \in K,$$

represent α by locally bounded Baire forms ϕ, ψ on U as in 4.6, and use v to smooth α as in 4.7 to obtain $\alpha_\epsilon = (\phi_\epsilon)^\sim$ for $0 < \epsilon < 1$ with

$$\phi_\epsilon \in \mathcal{E}^m(U), \quad \text{spt } \phi_\epsilon \subset U \sim B, \quad d\alpha = 0, \quad d(\phi_\epsilon) = 0.$$

Noting that $\mathbf{F}_{K_1}(T_\epsilon - T) \rightarrow 0$ as $\epsilon \downarrow 0$, hence

$$T(\phi_\epsilon) = \alpha_\epsilon(T) = \alpha(T_\epsilon) \rightarrow \alpha(T) = \lambda(T) = \Psi_{\Gamma'}(T) > \Psi_\Gamma(T)/(1 + \delta)$$

as $\epsilon \downarrow 0$, we choose ϵ so small that

$$T(\phi_\epsilon) > \Psi_\Gamma(T)/(1 + \delta)$$

and also

$$\Psi[x + v(x)z, \langle \xi, \wedge_m DH_1^z(x) \rangle] \leq (1 + \delta)\Psi(x, \xi)$$

whenever $x \in C$, $z \in \mathbf{R}^n \cap \mathbf{B}(0, \epsilon)$, $\xi \in \wedge^m \mathbf{R}^n$. Evidently

$$\alpha(S) = \lambda(S) \leq \Psi_\Gamma(S) \leq \int_S \Psi$$

whenever $S \in \mathbf{F}_{m,c}(U) \cap \mathbf{M}_m(U)$. Given $\xi \in \wedge^m \mathbf{R}^n$ we take $S = (\mathcal{L}^n \llcorner X) \wedge \xi$

to infer that

$$\int_X \langle \xi, \phi(x) \rangle d\mathcal{L}^n x \leq \int_X \Psi(x, \xi) d\mathcal{L}^n x$$

whenever X is an $\mathcal{L}^n$ measurable subset of C , hence

$$\langle \xi, \phi(x) \rangle \leq \Psi(x, \xi) \quad \text{for } \mathcal{L}^n \text{ almost all } x \text{ in } C.$$

By continuity with respect to ξ it follows that, for $\mathcal{L}^n$ almost all x in C , $\langle \xi, \phi(x) \rangle \leq \Psi(x, \xi)$ for all ξ in $\wedge_m \mathbf{R}^n$. Now

$$Y = U \cap \{x : \mathbf{B}[x, v(x)] \subset \text{Int } C\}$$

is a neighborhood of K in U , and for each $x \in Y$ we obtain

$$\begin{aligned} \langle \xi, \phi_*(x) \rangle &= \int \Phi_*(-z) \langle \langle \xi, \wedge_m DH_1^*(x) \rangle, \phi[x + v(x)z] \rangle d\mathcal{L}^n z \\ &\leq \int \Phi_*(-z) \Psi[x + v(x)z, \langle \xi, \wedge_m DH_1^*(x) \rangle] d\mathcal{L}^n z \\ &\leq \int \Phi_*(-z) (1 + \delta) \Psi(x, \xi) d\mathcal{L}^n z = (1 + \delta) \Psi(x, \xi) \end{aligned}$$

whenever $\xi \in \wedge^m \mathbf{R}^n$, hence $\phi_*(x) \leq (1 + \delta) \Psi_x$. We conclude

$$\begin{aligned} \rho &= (1 + \delta)^{-1} \phi_* \in \mathcal{E}^m(U), \quad d\rho = 0, \quad \text{spt } \rho \subset U \sim B, \\ \rho(x) &\leq \Psi_x \quad \text{for } x \in Y, \quad T(\rho) \geq (1 + \delta)^{-2} \Psi_\Gamma(T). \end{aligned}$$

Proof of (5). If $Q \in \mathbf{M}_m(U)$ and $Q - T \in \Gamma$, then

$$\int_T \Psi = \lambda(T) = \lambda(Q) \leq \int_Q \Psi.$$

4.11. Suppose K is a compact subset of A ,

$$\Gamma = \mathbf{Z}_{m,K}(A, B)$$

and Ψ is a convex positive integrand on U .

(1) For each $T \in \mathbf{F}_{m,K}(U) \sim \Gamma$ there exists

$$\lambda \in \mathbf{B}_K^m(A, B) \text{ with } \Psi_\Gamma(\lambda) = 1 \text{ and } \lambda(T) = \Psi_\Gamma(T).$$

(2) If $T \in \mathbf{F}_{m,K}(U)$ and $m > 0$, then

$$\begin{aligned} \Psi_\Gamma(T) &= \sup \{T(d\sigma) : \sigma \in \mathcal{E}^{m-1}(U), \text{spt } \sigma \subset U \sim B \\ &\quad \text{and } K \subset \text{Int } \{x : d\sigma(x) \leq \Psi_x\}\}. \end{aligned}$$

(3) If $T \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U)$, $\lambda \in \mathbf{B}_K^m(A, B)$, $\lambda(T) = \int_T \Psi$,

$$\lambda(S) \leq \int_S \Psi \quad \text{whenever } S \in \mathbf{F}_{m,K}(U) \cap \mathbf{M}_m(U),$$

then $\Psi_\Gamma(T) = \int_T \Psi$.

One can prove 4.11(1) just like the first half of 4.10(1), and 4.11(3) like 4.10(5); in all cases λ annihilates Γ .

To verify 4.11(2) we modify the proof of 4.10(4) as follows:

Regarding the first part we observe that if $\sigma \in \mathcal{E}^{m-1}(U)$ with $\text{spt } \sigma \subset U \sim B$, then $T(d\sigma) = Q(d\sigma)$ whenever $Q = T \in \mathbf{Z}_{m,K}(A, B)$.

In the second part of the argument we choose C and D so that

$$\Psi_{\mathbf{Z}_{m,c}(U,D)}(T) > \Psi_{\mathbf{Z}_{m,K}(A,B)}(T)/(1 + \delta),$$

apply 4.11(1) with Γ replaced by $\Gamma' = \mathbf{Z}_{m,c}(U, B')$ to find $\lambda \in \mathbf{B}_c^m(U', B)$, apply 4.1(2) to find

$$\beta \in \mathbf{F}^{m-1}(U, B'') \quad \text{with} \quad \lambda = (d\beta) \llcorner C,$$

represent β by locally bounded Baire forms ω and ϕ of degrees $m-1$ and m , hence $d\beta = \alpha$ by ϕ and $\psi = 0$, smooth β to obtain $\beta_\epsilon = (\omega_\epsilon)^\sim$ with

$$\omega_\epsilon \in \mathcal{E}^{m-1}(U), \text{ spt } \omega_\epsilon \subset U \sim B, \quad d(\beta_\epsilon) = \alpha_\epsilon, \quad d(\omega_\epsilon) = \phi_\epsilon$$

and construct $\sigma = (1 + \delta)^{-1}\omega_\epsilon$ with $d\sigma = \rho$.

4.12. If Ψ is a convex nonnegative integrand on U , $T \in \mathbf{F}_m(A)$,

$$\Gamma = \mathbf{B}_m(A, B),$$

then there exists $\lambda \in \mathbf{Z}^m(A, B)$ with $\Psi_\Gamma(\lambda) \leq 1$, $\lambda(T) = \Psi_\Gamma(T)$.

4.13. If Ψ is a convex nonnegative integrand on U , $T \in \mathbf{F}_m(A)$,

$$\Gamma = \mathbf{Z}_m(A, B),$$

then there exists $\mu \in \mathbf{F}^{m-1}(A, B)$ with $\Psi_\Gamma(d\mu) \leq 1$, $\mu(\partial T) = \Psi_\Gamma(T)$.

To prove 4.12 and 4.13 we use the same method as for the first half of 4.10(1), applying the Hahn-Banach theorem with $p = \Psi_\Gamma \llcorner \mathbf{F}_m(A)$ to obtain a real valued linear function λ on $\mathbf{F}_m(A)$ which annihilates Γ and for which $\mathbf{F}_K(\lambda) < \infty$ whenever K is a compact subset of A ; moreover, if $\Gamma = \mathbf{Z}_m(A, B)$, then 4.4 yields $\mu \in \mathbf{F}^{m-1}(A, B)$ with $\lambda = d\mu$.

4.14. Suppose Ψ is a convex positive integrand on U .

By analogy with 4.10(4) and 4.11(2) it is natural to ask, for $T \in \mathbf{F}_m(A)$, whether $\Psi_{\mathbf{B}_m(A,B)}(T)$ equals the supremum of

$$\{T(\rho) : \rho \in \mathcal{E}^m(U), \quad d\rho = 0, \quad \text{spt } \rho \subset U \sim B, \quad A \subset \text{Int } \{x : \rho(x) \leq \Psi_x\}\},$$

and whether $\Psi_{\mathbf{Z}_m(A,B)}(T)$ equals the supremum of

$$\{T(d\sigma) : \sigma \in \mathcal{E}^{m-1}(U), \text{ spt } \sigma \subset U \sim B, A \subset \text{Int } \{x : d\sigma(x) \leq \Psi_x\}\}.$$

However, in case A is not compact, I cannot at present answer these questions affirmatively without strong restrictions on A, B, Ψ .

Since $\Psi_{\mathbf{B}_m(A, B)}$ is constant on each real homology class of (A, B) , it induces a convex nonnegative function q on $\mathbf{H}_m(A, B; \mathbf{R})$ such that

$$q(\chi) = \inf \left\{ \int_Q \Psi : Q \in \chi \right\} \quad \text{for } \chi \in H_m(A, B; \mathbf{R}).$$

If $T \in \tau \in \mathbf{H}_m(A, B; \mathbf{R})$, then the convex set

$$\mathbf{H}_m(A, B; \mathbf{R}) \cap \{\chi : q(\chi) \leq q(\tau)\}$$

has a supporting plane at τ corresponding to the flat cocycle λ constructed in 4.12.

4.15. In case Ψ is the m dimensional **mass integrand**,

$$\Psi(x, \xi) = ||\xi|| \quad \text{for } (x, \xi) \in U \times \wedge_m \mathbf{R}^n,$$

then $\int_Q \Psi = \mathbf{M}(Q)$ for $Q \in \mathbf{M}_m(U)$, and we denote

$$\Psi_{\mathbf{B}_m(A, B)} = \mathbf{F}_{A, B}^{\text{hom}}, \quad \Psi_{\mathbf{Z}_m(A, B)} = \mathbf{F}_{A, B}^{\text{abs}}.$$

Consequently, for $T \in \mathbf{F}_m(A)$,

$$\begin{aligned} \mathbf{F}_{A, B}^{\text{hom}}(T) &= \inf \{ \mathbf{M}(Q) : Q - T \in \mathbf{B}_m(A, B) \} \\ &= \inf \{ \mathbf{F}_K(Q) : Q - T \in \mathbf{B}_{m, K}(A, B) \text{ for some compact } K \subset A \}, \\ \mathbf{F}_{A, B}^{\text{abs}}(T) &= \inf \{ \mathbf{M}(Q) : Q - T \in \mathbf{Z}_m(A, B) \} \\ &= \inf \{ \mathbf{F}_K(Q) : Q - T \in \mathbf{Z}_{m, K}(A, B) \text{ for some compact } K \subset A \}. \end{aligned}$$

We omit the subscript “ B ” if $B = \emptyset$.

It seems that some important geometric properties of (A, B) are reflected by the shape of the convex set

$$H_m(A, B; \mathbf{R}) \cap \{\chi : \mathbf{F}_{A, B}^{\text{hom}}(T) \leq 1 \text{ for } T \in \chi\},$$

whose boundary often fails to be smooth.

For example suppose A_i corresponding to $i \in \{1, 2\}$ are compact Lipschitz neighborhood retracts in $\mathbf{R}^{n_i}$ and

$$T_i \in \mathbf{Z}_m(A_i) \text{ with } \mathbf{F}_{A_i}^{\text{hom}}(T_i) = \mathbf{M}(T_i) = 1, m \geq 2.$$

Choosing $a_i \in A_i$ we construct

$$S_\theta = (1 - \theta)T_1 \times [a_2] + \theta[a_1] \times T_2 \in \mathbf{Z}_m(A_1 \times A_2)$$

for $0 \leq \theta \leq 1$ and will verify that

$$\mathbf{F}_{A_1 \times A_2}^{\text{hom}}(S_\theta) = \mathbf{M}(S_\theta) = 1.$$

Clearly

$$\mathbf{F}_{A_1 \times A_2}^{\text{hom}}(S_\theta) \leq \mathbf{M}(S_\theta) \leq (1 - \theta)\mathbf{M}(T_1) + \theta\mathbf{M}(T_2) = 1.$$

Let

$$P_1(x, y) = x \quad \text{and} \quad P_2(x, y) = y \quad \text{for} \quad (x, y) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}.$$

If $\rho_i \in \mathcal{E}^m(U_i)$ with $d\rho_i = 0$ and $\mathbf{M}(\rho_i) \leq 1$ for some neighborhoods U_i of A_i in $\mathbf{R}^{n_i}$, then

$$\rho = P_1^* \rho_1 + P_2^* \rho_2 \in \mathcal{E}^m(U_1 \times U_2) \quad \text{with} \quad d\rho = 0$$

and $\mathbf{M}(\rho) \leq 1$, because [7, 9.15] implies

$$\begin{aligned} \langle \xi, \rho(x, y) \rangle &= \langle \langle \xi, \wedge_m P_1 \rangle, \rho_1(x) \rangle + \langle \langle \xi, \wedge_m P_2 \rangle, \rho_2(y) \rangle \\ &\leq |\langle \xi, \wedge_m P_1 \rangle| + |\langle \xi, \wedge_m P_2 \rangle| \leq |\xi| \end{aligned}$$

whenever $(x, y) \in U_1 \times U_2$, $\xi \in \wedge_m(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ and ξ is simple; moreover $S_\theta(\rho) = (1 - \theta)T_1(\rho_1) + T_2(\rho_2)$. Applying 4.10(4) to T_1 , T_2 and S_θ we infer that

$$\mathbf{F}_{A_1 \times A_2}^{\text{hom}}(S_\theta) \geq (1 - \theta)\mathbf{F}_{A_1}^{\text{hom}}(T_1) + \theta\mathbf{F}_{A_2}^{\text{hom}}(T_2) = 1.$$

Since $P_1^* S_\theta = (1 - \theta)T_1$ and $T_1 \notin \mathbf{B}_m(A_1)$, the cycles S_θ corresponding to $0 \leq \theta \leq 1$ belong to distinct homology classes, which form a nondegenerate line segment in

$$H_m(A_1 \times A_2; \mathbf{R}) \cap \{\chi : \mathbf{F}_{A_1 \times A_2}^{\text{hom}}(R) = 1 \quad \text{for} \quad R \in \chi\}.$$

In particular, for $m \geq 2$ the unit ball of $\mathbf{H}_m(\mathbf{S}^m \times \mathbf{S}^m; \mathbf{R})$ is a square.

If a compact connected group G acts isometrically on A , then in each real homology class of A there is a G invariant mass minimizing real flat cycle (constructible by 3.9 and averaging over G). In case A is a compact connected Lie group with a left and right invariant Riemannian metric, and Lie algebra $\mathfrak{A}$, then the left and right invariant cycles are dual (via [5, 1.5.2]) to the left and right invariant differential forms on A , hence unique in each real homology class, and $\mathbf{H}_m(A; \mathbf{R})$ is isometric to the subspace of $\wedge_m \mathfrak{A}$ (with the mass norm $\|\cdot\|$) consisting of all m -vectors invariant under the adjoint representation of A .

In particular, $\mathbf{H}_m[(\mathbf{S}^1)^k; \mathbf{R}]$ is isometric to $\wedge_m \mathbf{R}^k$ with the mass norm $\|\cdot\|$.

For any Kaehler manifold A with Kaehler form Ω it follows from Wirtinger's inequality that the positive cycles of type (p, p) in A are characterized by the equation

$$\mathbf{M}(T) = T(\Omega^p/p!),$$

which implies $\mathbf{F}_A^{\text{hom}}(T) = \mathbf{M}(T)$ because $d(\Omega^p/p!) = 0$ and $\mathbf{M}(\Omega^p/p!) = 1$. Consequently the convex set

$$\mathbf{H}_{2p}(A; \mathbf{R}) \cap \{\chi : \mathbf{F}_A^{\text{hom}}(T) \leq 1 \quad \text{for} \quad T \in \chi\}$$

has at all positive homology classes of type (p, p) and minimal mass 1 the common supporting plane

$$\mathbf{H}_{2p}(A; \mathbf{R}) \cap \{\chi : T(\Omega^p/p!) = 1 \quad \text{for} \quad T \in \chi\}.$$

We note that there exist homologically mass minimizing real flat chains which utterly lack local smoothness. To construct such an example we let

$$D = \mathbf{R}^2 \cap \{(s, t) : 0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi\},$$

$$h(s, t) = (e^{s1}, e^{t1}) \in \mathbf{C}^2 \quad \text{for } (s, t) \in D,$$

$$A = h(D) = \mathbf{S}^1 \times \mathbf{S}^1, \quad S = h_{\#} [(0, 0), (2\pi, 0)] \in \mathbf{Z}_1(A),$$

$$r_t(z, w) = (z, e^{t1}w) \in A \quad \text{for } (z, w) \in A \quad \text{and } 0 \leq t \leq 2\pi.$$

Clearly $\mathbf{F}_A^{\text{hom}}(S) = \mathbf{M}(S)$. Moreover, if k is any nonnegative $\mathcal{L}^1$ measurable function with $\int_0^{2\pi} k \, d\mathcal{L}^1 = 1$, then

$$T = \int_0^{2\pi} k(t)r_{t\#} S \, d\mathcal{L}^1 t \in \mathbf{Z}_1(A), \quad T - S \in \mathbf{B}_1(A),$$

and $\mathbf{F}_A^{\text{hom}}(T) = \mathbf{M}(T)$, but T is no smoother than k because

$$T = h_{\#}[(\mathcal{L}^2 \llcorner D) \wedge \zeta] \quad \text{with } \zeta(s, t) = (k(t), 0) \quad \text{for } (s, t) \in D.$$

Thus mass minimizing real chains can be much less regular than mass minimizing chains with coefficients in $\mathbf{Z}$ or $\mathbf{Z}_2$. (Compare [5, 5.3.17], [6]!) Still one might hope that in every real homology class of a pair of compact Lipschitz neighborhood retracts one could find at least one mass minimizing real flat chain with appropriate local smoothness.

5. Relations between real and integral minimizing chains. Assuming that A, B, U, V, f, g satisfy the conditions of §3 and Ψ is a convex parametric integrand of degree m on U , we will now make a comparative study of variational problems involving integral flat chains and real flat chains, respectively. Recalling [5, 4.4.1] we define

$$\mathfrak{z}_{m,K}(A, B) = \mathfrak{F}_{m,K}(U) \cap \mathfrak{z}_m(A, B),$$

$$\mathfrak{B}_{m,K}(A, B) = \mathfrak{F}_{m,K}(U) \cap \mathfrak{B}_m(A, B)$$

for every compact subset K of A , in analogy with 3.4. The main results of this section are the theorems 5.8, 5.10, 5.12. First we prove several lemmas concerning approximation by Lipschitz chains.

5.1. *If K is a compact subset of U , then the set of all m dimensional integral Lipschitz chains of K is an $\mathfrak{F}_K$ dense subgroup of $\mathfrak{F}_{m,K}(U)$.*

This proposition follows immediately from the definitions in 2.11 and [5, 4.1.24].

5.2. *For each compact Lipschitz neighborhood retract K in U there exists $\theta \in \mathbf{R}$ with the following property:*

If $F \in \mathbf{F}_{m,K}(U)$, $\delta > 0$ and either ∂F is an $m - 1$ dimensional Lipschitz chain of K or $m = 0$, then there exists $G \in \mathbf{F}_{m+1,K}(U)$ such that $F - \partial G$ is an m dimensional Lipschitz chain of K ,

$$\mathbf{M}(G) \leq \delta \quad \text{and} \quad \mathbf{M}(F - \partial G) \leq \delta + \theta \mathbf{M}(F).$$

Proof. We select a Lipschitzian retraction r of a neighborhood Z of K in U onto K , and let $\theta = [\text{Lip } (r)]^{m+1} 2n^{m+2}$.

Given F and δ as above we represent

$$F = F' + X + \partial Y \quad \text{with} \quad \mathbf{M}(X) + \mathbf{M}(Y) < \delta/2,$$

$$F' \in \mathbf{N}_{m,K}(U), \quad X \in \mathbf{F}_{m,K}(U), \quad Y \in \mathbf{F}_{m+1,K}(U),$$

and observe that $T = F' + X \in \mathbf{N}_{m,K}(U)$ with $\partial T = \partial F$. Using a positive number ϵ for which

$$\theta \in \mathbf{N}(T) < \delta/2 \quad \text{and} \quad \{x : \text{dist } (x, K) \leq 2n\epsilon\} \subset Z,$$

we construct P, Q, S , as in [5, 4.2.9] and infer

$$T = r_* T = r_*(P + Q) + \partial r_* S,$$

$$\mathbf{M}[r_*(P + Q)] \leq \theta[\mathbf{M}(T) + \epsilon \mathbf{M}(\partial T)] \leq \theta \mathbf{M}(T) + \delta/2,$$

$$\mathbf{M}(r_* S) \leq \theta \epsilon \mathbf{M}(T) \leq \delta/2,$$

hence $G = Y + r_* S \in \mathbf{F}_{m+1,K}(U)$, $\mathbf{M}(G) < \delta$ and

$$F - \partial G = T - \partial r_* S = r_* P + r_* Q.$$

Clearly $r_* P$ is an m dimensional Lipschitz chain of K . Moreover, either $m = 0$ and $Q = 0$, or ∂T is an $m - 1$ dimensional Lipschitz chain of K , hence ∂T is a Lipschitz chain of some $m - 1$ rectifiable subset C of K , $\mathbf{W}_{n-m}'' \times C$ is a countably $n - 1$ rectifiable subset of $\mathbf{R}^n \times \mathbf{R}^n$,

$$\mathbf{R}^n \cap \{\eta : \mathbf{W}_{n-m}'' \cap \tau_\eta(C) \neq \emptyset\} = \{x - y : (x, y) \in \mathbf{W}_{n-m}'' \times C\}$$

has $\mathcal{L}^n$ measure 0, the point a can be chosen so that

$$\tau_a(C) \subset \mathbf{R}^n \sim \mathbf{W}_{n-m}'',$$

and then $Q, r_* Q$ are m dimensional Lipschitz chains of Z, K . Finally we note that in case $\mathbf{M}(F) < \infty$ we can take $F' = F, X = 0, Y = 0$ and $T = F$.

Corollary. *If $F \in \mathbf{F}_m(A)$ and either ∂F is an $m - 1$ dimensional Lipschitz chain of A or $m = 0$, then there exists $H \in \mathbf{F}_{m+1}(A)$ such that $F - \partial H$ is an m dimensional Lipschitz chain of A .*

Proof. We note that $f_* G \in \mathbf{F}_{m+1}(A)$ and $F - \partial f_* G = f_*(F - \partial G)$.

5.3. *If Ψ has the real Lipschitz chain approximation property with respect to the compact Lipschitz neighborhood retract K in U , $F \in \mathbf{N}_{m,K}(U)$, $\delta > 0$ and either ∂F is an $m - 1$ dimensional Lipschitz chain of K or $m = 0$, then there exists $G \in \mathbf{N}_{m+1,K}(U)$ such that $F - \partial G$ is an m dimensional Lipschitz chain of K ,*

$$\mathbf{M}(G) < \delta \quad \text{and} \quad \left| \int_{F - \partial G} \Psi - \int_F \Psi \right| < \delta.$$

Proof. We let $c = \sup \{ |\Psi(x, \xi)| : x \in K, \xi \in \wedge_m \mathbf{R}^n, \|\xi\| \leq 1 \}$, choose θ according to 5.2, recall 2.11 to obtain an m dimensional Lipschitz chain S of K for which

$$4(c\theta + 1)\mathbf{F}_K(F - S) < \delta \quad \text{and} \quad \left| \int_F \Psi - \int_S \Psi \right| < \delta/2,$$

select $X \in \mathbf{F}_{m,K}(U)$ and $Y \in \mathbf{F}_{m+1,K}(U)$ so that

$$F - S = X + \partial Y \quad \text{and} \quad 4(c\theta + 1)[\mathbf{M}(X) + \mathbf{M}(Y)] < \delta,$$

observe that either $\partial X = \partial S - \partial F$ is an $m - 1$ dimensional Lipschitz chain of K or $m = 0$, apply 5.2 with F replaced by X to find $Y' \in \mathbf{F}_{m+1,K}(U)$ such that $X - \partial Y'$ is an m dimensional Lipschitz chain of K ,

$$\mathbf{M}(Y') \leq \delta/2 \quad \text{and} \quad \mathbf{M}(X - \partial Y') \leq \delta/(4c) + \theta\mathbf{M}(X),$$

infer that $Y + Y' \in \mathbf{N}_{m+1,K}(U)$, $F - \partial(Y + Y') = S + (X - \partial Y')$,

$$\left| \int_{X - \partial Y'} \Psi \right| \leq \delta/4 + c\theta\mathbf{M}(X) < \delta/2,$$

and take $G = Y + Y'$.

5.4. If $r \in \mathbf{R}^N$, $\epsilon > 0$, $\Omega \subset \mathbf{Q}^N$ and

$$r \bullet \omega = 0 \quad \text{whenever} \quad \omega \in \Omega,$$

then there exists $\rho \in \mathbf{Q}^N$ with $|r - \rho| < \epsilon$ and

$$\rho \bullet \omega = 0 \quad \text{whenever} \quad \omega \in \Omega.$$

Proof. Viewing the real field $\mathbf{R}$ as a vectorspace over the rational field $\mathbf{Q}$, we choose a $\mathbf{Q}$ base $s_1, \dots, s_k$ for the $\mathbf{Q}$ span of $r_1, \dots, r_N$, express

$$r_i = \sum_{j=1}^k \alpha_{i,j} s_j \quad \text{with} \quad \alpha_{i,j} \in \mathbf{Q},$$

and infer that, for all $\omega \in \Omega$,

$$0 = r \bullet \omega = \sum_{i=1}^N r_i \omega_i = \sum_{j=1}^k \sum_{i=1}^N \alpha_{i,j} \omega_i s_j,$$

hence

$$0 = \sum_{i=1}^N \alpha_{i,j} \omega_i \quad \text{for} \quad j = 1, \dots, k.$$

Then we select

$$\sigma \in \mathbf{Q}^k \quad \text{with} \quad \sum_{i=1}^N \sum_{j=1}^k |\alpha_{i,j}| \cdot |s_j - \sigma_j| < \epsilon$$

and take

$$\rho_i = \sum_{j=1}^k \alpha_{i,j} \sigma_j \quad \text{for} \quad i = 1, \dots, N.$$

5.5. *If S and T are m and $m + 1$ dimensional Lipschitz chains of U , and if $\delta > 0$, then there exist m and $m + 1$ dimensional integral Lipschitz chains S' and T' of U , and a positive integer k such that*

$$\begin{aligned} \text{spt } S' &\subset \text{spt } S, & \mathbf{N}(S - k^{-1}S') &< \delta, \\ \text{spt } T' &\subset \text{spt } T, & \mathbf{N}(T - k^{-1}T') &< \delta, \\ \text{spt } (S' + \partial T') &\subset \text{spt } (S + \partial T). \end{aligned}$$

Proof. We represent

$$S = \sum_{i=1}^I s_i \alpha_i \# P \quad \text{and} \quad T = \sum_{j=1}^J t_j \beta_j \# Q,$$

where P and Q are m and $m + 1$ dimensional oriented simplexes in $\mathbf{R}^m$ and $\mathbf{R}^{m+1}$, I and J are positive integers, $\alpha_i : \mathbf{R}^m \rightarrow U$ and $\beta_j : \mathbf{R}^{m+1} \rightarrow U$ are Lipschitzian maps, $s_i \in \mathbf{R}$ and $t_j \in \mathbf{R}$ for $i = 1, \dots, I$ and $j = 1, \dots, J$. Observing that

$$X = \bigcup_{i=1}^I \alpha_i(\text{spt } P) \cup \bigcup_{j=1}^J \beta_j(\text{spt } \partial Q)$$

is an m rectifiable subset of $\mathbf{R}^n$, we use [5, 3.2.25] to obtain an $\mathcal{H}^m \llcorner X$ measurable function ξ such that, for $\mathcal{H}^m$ almost all x in X , $\xi(x)$ is a simple m vector of $\mathbf{R}^n$, $|\xi(x)| = 1$ and $\text{Tan}^m(\mathcal{H}^m \llcorner X, x)$ is the m dimensional vectorsubspace of $\mathbf{R}^n$ associated with $\xi(x)$. Then we apply [5, 4.1.25] to find $\mathcal{H}^m \llcorner X$ measurable integer valued functions λ_i and μ_j such that

$$\begin{aligned} \alpha_i \# P &= (\mathcal{H}^m \llcorner X) \wedge \lambda_i \xi \quad \text{for } i = 1, \dots, I, \\ \beta_j \# (\partial Q) &= (\mathcal{H}^m \llcorner X) \wedge \mu_j \xi \quad \text{for } j = 1, \dots, J. \end{aligned}$$

Similarly, since

$$Y = \bigcup_{j=1}^J \beta_j(\text{spt } Q)$$

is an $m + 1$ rectifiable subset of $\mathbf{R}^n$, we obtain an $\mathcal{H}^{m+1} \llcorner Y$ measurable function η such that, for $\mathcal{H}^{m+1}$ almost all y in Y , $\eta(y)$ is a simple $m + 1$ vector of $\mathbf{R}^n$, $|\eta(y)| = 1$ and $\text{Tan}^{m+1}(\mathcal{H}^{m+1} \llcorner Y, y)$ is the $m + 1$ dimensional vectorsubspace associated with $\eta(y)$. We also find $\mathcal{H}^{m+1} \llcorner Y$ measurable integer valued functions μ_j such that

$$\beta_j \# Q = (\mathcal{H}^{m+1} \llcorner Y) \wedge \mu_j \eta \quad \text{for } j = 1, \dots, J.$$

Using the covering relation

$$C = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^n, 0 < r \in \mathbf{R}\}$$

we define D as the set of all points x such that

$$\Theta^m(\mathcal{H}^m \llcorner X, x) = 1$$

and the functions $\xi, \lambda_1, \dots, \lambda_I, \mu_1, \dots, \mu_J$ are $(\mathcal{H}^m \llcorner X, C)$ approximately

continuous at x . From [5, 3.2.19, 2.8.18, 2.9.13] we see that $\mathcal{H}^m(X \sim D) = 0$. Similarly we define E as the set of all points y such that

$$\Theta^{m+1}(\mathcal{H}^{m+1} \llcorner Y, y) = 1$$

and the functions $\eta, \nu_1, \dots, \nu_J$ are $(\mathcal{H}^{m+1} \llcorner Y, C)$ approximately continuous at y , and see that $\mathcal{H}^{m+1}(Y \sim E) = 0$.

The point $(s, t) \in \mathbf{R}^I \times \mathbf{R}^J$ satisfies the following linear homogeneous equations with integral coefficients:

$$\begin{aligned} \sum_{i=1}^I s_i \lambda_i(x) &= 0 \quad \text{for } x \in D \sim \text{spt } S, \\ \sum_{j=1}^J t_j \nu_j(y) &= 0 \quad \text{for } y \in E \sim \text{spt } T, \\ \sum_{i=1}^I s_i \lambda_i(x) + \sum_{j=1}^J t_j \mu_j(x) &= 0 \quad \text{for } x \in D \sim \text{spt } (S + \partial T). \end{aligned}$$

From 5.4 we obtain $(\sigma, \tau) \in \mathbf{Q}^I \times \mathbf{Q}^J$ such that the preceding linear equations hold with (s, t) replaced by (σ, τ) ,

$$\sum_{i=1}^I |s_i - \sigma_i| \mathbf{N}(\alpha_i \# P) < \delta \quad \text{and} \quad \sum_{j=1}^J |t_j - \tau_j| \mathbf{N}(\beta_j \# Q) < \delta.$$

Choosing a positive integer k so that $k\sigma_1, \dots, k\sigma_I, k\tau_1, \dots, k\tau_J$ are integers, we take

$$S' = \sum_{i=1}^I k\sigma_i \alpha_i \# P \quad \text{and} \quad T' = \sum_{j=1}^J k\tau_j \beta_j \# Q.$$

5.6. *If X is an m dimensional Lipschitz chain of U ,*

$$X \in \mathbf{Z}_m(A, B) \quad \text{and} \quad \delta > 0,$$

then there exist an m dimensional integral Lipschitz chain X' of U and a positive integer k such that

$$X' \in \mathfrak{Z}_m(A, B), \quad \text{spt } X' \subset \text{spt } X, \quad \mathbf{N}(X - k^{-1}X') < \delta.$$

Proof. Applying 5.5 with $S = 0$, $T = X$, and m replaced by $m - 1$, we take $X' = T'$.

5.7. *If X is an m dimensional Lipschitz chain of A , either ∂X is an $m - 1$ dimensional Lipschitz chain of B or $m = 0$,*

$$X \in \mathbf{B}_m(A, B) \quad \text{and} \quad \delta > 0,$$

then there exist an m dimensional integral Lipschitz chain X' of U and a positive integer k such that

$$X' \in \mathfrak{B}_m(A, B), \quad \text{spt } X' \subset \text{spt } X, \quad \mathbf{N}(X - k^{-1}X') < \delta.$$

Proof. We choose $Y \in \mathbf{F}_{m+1}(A)$ so that $X - \partial Y \in \mathbf{F}_m(B)$, apply the corollary of 5.2 with F and A replaced by $X - \partial Y$ and B to find $Y' \in \mathbf{F}_{m+1}(B)$ such that

$$S = X - \partial Y - \partial Y'$$

is an m dimensional Lipschitz chain of B , apply the corollary of 5.2 with F and m replaced by $Y + Y'$ and $m + 1$ to find $Z \in \mathbf{F}_{m+2}(A)$ such that

$$T = Y + Y' + \partial Z$$

is an $m + 2$ dimensional Lipschitz chain of A , note that $X = S + \partial T$, obtain S' , T' , k from 5.5 with δ replaced by $\delta/2$, and take $X' = S' + \partial T'$.

5.8. Suppose K is a compact subset of A , K and $K \cap B$ are Lipschitz neighborhood retracts in U , and Ψ has the Lipschitz chain approximation property with respect to K .

If $T \in \mathfrak{F}_{m,K}(U)$, then

$$\Psi_{\mathbf{Z}_{m,K}(A,B)}(T) = \inf \{k^{-1}\Psi_{\mathfrak{Z}_{m,K}(A,B)}(kT) : 0 < k \in \mathbf{Z}\}$$

and

$$\Psi_{\mathbf{B}_{m,K}(A,B)}(T) = \inf \{k^{-1}\Psi_{\mathfrak{B}_{m,K}(A,B)}(kT) : 0 < k \in \mathbf{Z}\}.$$

Proof. In order to derive both conclusions of this theorem simultaneously, we denote

$$\text{either (1) } \Gamma = \mathbf{Z}_{m,K}(A, B) \quad \text{and} \quad \Delta = \mathfrak{Z}_{m,K}(A, B),$$

$$\text{or (2) } \Gamma = \mathbf{B}_{m,K}(A, B) \quad \text{and} \quad \Delta = \mathfrak{B}_{m,K}(A, B).$$

We also let $c = \sup \{|\Psi(x, \xi)| : x \in K, \xi \in \wedge_m \mathbf{R}^n, \|\xi\| \leq 1\}$.

Given any $\delta > 0$ we use 5.1 to represent

$$T = T' + X_1 + \partial Y_1$$

where T' is an m dimensional integral Lipschitz chain of K and

$$X_1 \in \mathfrak{F}_{m,K}(U), \quad Y_1 \in \mathfrak{F}_{m+1,K}(U), \quad \mathbf{M}(X_1) + \mathbf{M}(Y_1) < \delta,$$

recall 3.8 to select $R \in \Gamma$ so that

$$\mathbf{M}(T + R) < \infty \quad \text{and} \quad \int_{T+R} \Psi \leq \delta + \Psi_\Gamma(T),$$

apply 5.2 with F , m , K replaced by ∂R , $m - 1$, $B \cap K$ to find $X_2 \in \mathbf{F}_{m,B \cap K}(U)$ such that $\mathbf{M}(X_2) < \delta$ and either $m = 0$ or $\partial R - \partial X_2$ is an $m - 1$ dimensional Lipschitz chain of $B \cap K$, infer that

$$Q = T' + \partial Y_1 + R - X_2 = T + R - X_1 - X_2,$$

$$\int_Q \Psi \leq c\mathbf{M}(X_1 + X_2) + \int_{T+R} \Psi \leq (2c + 1)\delta + \Psi_\Gamma(T)$$

and either $\partial Q = \partial T' + \partial R - \partial X_2$ is an $m - 1$ dimensional Lipschitz chain of K or $m = 0$, apply 5.3 to find

$$Y_2 \in \mathbf{N}_{m+1,K}(U)$$

such that $Q - \partial Y_2$ is an m dimensional Lipschitz chain of K and

$$\int_{Q - \partial Y_2} \Psi \leq \delta + \int_Q \Psi \leq (2c + 2) \delta + \Psi_\Gamma(T),$$

observe that $X = Q - \partial Y_2 - T' = R - X_2 + \partial(Y_1 - Y_2) \in \Gamma$, use 5.6 in case (1), or 5.7 in case (2), to obtain

$$X' \in \Delta \quad \text{and} \quad 0 < k \in \mathbf{Z} \quad \text{with} \quad \mathbf{N}(Q - \partial Y_2 - T' - k^{-1}X') < \delta,$$

and conclude $X' - \partial(kY_1) \in \Delta$,

$$\int_{T' + k^{-1}X'} \Psi \leq c\delta + \int_{Q - \partial Y_2} \Psi \leq (3c + 2)\delta + \Psi_\Gamma(T),$$

$$kT + X' - \partial(kY_1) = k(T' + k^{-1}X' + X_1),$$

$$k^{-1}\Psi_\Delta(kT) \leq k^{-1} \int_{kT + X' - \partial(kY_1)} \Psi$$

$$= \int_{T' + k^{-1}X' + X_1} \Psi \leq c\mathbf{M}(X_1) + \int_{T' + k^{-1}X'} \Psi \leq (4c + 2)\delta + \Psi_\Gamma(T).$$

Accordingly $\inf \{k^{-1}\Psi_\Delta(kT) : 0 < k \in \mathbf{Z}\} \leq \Psi_\Gamma(T)$. The opposite inequality holds because $\Delta \subset \Gamma$ and $\Psi_\Gamma(T) \leq \Psi_\Delta(T)$.

5.9. If Z is an open subset of $\mathbf{R}^n$, C is an $m + 1$ dimensional submanifold of class 1 of Z , C is relatively closed in Z , γ is an $m + 1$ vectorfield orienting C , τ is an integer valued $\mathcal{H}^{m+1}$ measurable function on C and

$$H = (\mathcal{H}^{m+1} \llcorner C) \wedge \tau\gamma \in \mathbf{I}_{m+1}^{loc}(Z),$$

then $H_i = [\mathcal{H}^{m+1} \llcorner C \cap \{x : \tau(x) \geq i\}] \wedge \gamma \in \mathbf{I}_{m+1}^{loc}(Z)$ for $i \in \mathbf{Z}$,

$$\partial H = \sum_{i \in \mathbf{Z}} \partial H_i,$$

$$\partial H_i = (\mathcal{H}^m \llcorner Y_i) \wedge \eta \quad \text{for} \quad i \in \mathbf{Z},$$

where the sets Y_i are $\mathcal{H}^m$ measurable subsets of C , with union

$$Y = \bigcup_{i \in \mathbf{Z}} Y_i$$

such that $Y \cap E$ is $(\mathcal{H}^m, m)$ rectifiable for each compact subset E of Z , and where η is an $\mathcal{H}^m$ measurable function on Y such that, for $\mathcal{H}^m$ almost all x in Y ,

$$\eta(x) \text{ is a simple } m \text{ vector of } \mathbf{R}^n, |\eta(x)| = 1,$$

$$\text{Tan}^m(\mathcal{H}^m \llcorner Y, x) \text{ is associated with } \eta(x).$$

Verification of the preceding proposition is easily reduced by means of coordinate systems and [5, 4.1.30] to the special case when $n = m + 1$ and $C = \mathbf{Z}$. But then [5, 4.5.9(13)] implies

$$\partial H = \sum_{i \in \mathbf{Z}} \partial H_i \quad \text{and} \quad ||\partial H|| = \sum_{i \in \mathbf{Z}} ||\partial H_i||,$$

for each integer i [5, 4.5.6] implies

$$\partial H_i = (\mathcal{H}^m \llcorner Y_i) \wedge * \mathbf{n}(C_i, \cdot) \quad \text{with}$$

$$C_i = \{x : \tau(x) \geq i\} \quad \text{and} \quad Y_i = \{x : \mathbf{n}(C_i, x) \in \mathbf{S}^m\},$$

for any two integers $i > j$ the fact that $C_i \subset C_j$ implies

$$\mathbf{n}(C_i, x) = \mathbf{n}(C_j, x) \quad \text{whenever} \quad x \in Y_i \cap Y_j,$$

and hence the function η can be defined so that

$$\eta(x) = * \mathbf{n}(C_i, x) \quad \text{whenever} \quad i \in \mathbf{Z} \quad \text{and} \quad x \in Y_i.$$

In the proof of 5.10 we will use the fact that *in the existence theorem 3.9 one may replace $\mathbf{F}_{m,K}(U)$, $\mathbf{Z}_{m,K}(A, B)$, $\mathbf{B}_{m,K}(A, B)$ by $\mathcal{F}_{m,K}(U)$, $\mathcal{B}_{m,K}(A, B)$, $\mathcal{G}_{m,K}(A, B)$* . This follows readily from [5, 4.4.2, 4.4.4, 5.1.6]. Similarly the continuity theorem 3.10 can be adapted to integral flat chains. It is also important to remember from [5, 4.2.16] that integral flat chains with finite mass are rectifiable currents.

5.10. *Suppose A is an $m + 1$ dimensional orientable submanifold of class 1 of U , K is a compact subset of A , K and $K \cap B$ are Lipschitz neighborhood retracts in U , Ψ is a positive convex even integrand on U and Ψ has the Lipschitz chain approximation property with respect to K .*

If $T \in \mathcal{F}_{m,K}(U)$, then

$$\Psi_{\mathbf{B}_{m,K}(A,B)}(T) = \Psi_{\mathcal{G}_{m,K}(A,B)}(T).$$

Proof. Letting $\Delta = \mathcal{G}_{m,K}(A, B)$ we choose

$$Q \in \mathcal{R}_{m,K}(U) \quad \text{with} \quad Q - T \in \Delta, \quad \Psi_\Delta(T) = \int_Q \Psi$$

and recall [5, 4.1.28] to represent

$$Q = (\mathcal{H}^m \llcorner X) \wedge q\xi$$

where X is an $\mathcal{H}^m$ measurable ($\mathcal{H}^m, m$) rectifiable subset of K and q, ξ are $\mathcal{H}^m$ measurable functions such that

$$0 < q(x) \in \mathbf{Z}, \quad \xi(x) \text{ is a simple } m \text{ vector of } \mathbf{R}^n, \quad |\xi(x)| = 1,$$

$$\text{Tan}^m (\mathcal{H}^m \llcorner X, x) \text{ is associated with } \xi(x)$$

for $\mathcal{H}^m$ almost all x in X , and $q(x) = 0$ for $x \in U \sim X$. We note that $Q \llcorner B = 0$, because $Q \llcorner B \in \Delta$ and Ψ is positive.

We will prove that $\Psi_\Delta(kQ) = \int_{kQ} \Psi$ for $0 < k \in \mathbf{Z}$.

Given $0 < k \in \mathbf{Z}$ and $R \in \Delta \cap \mathcal{R}_{m,k}(U)$ we select

$$S \in \mathcal{R}_{m+1}(U) \quad \text{with} \quad \text{spt } S \subset A, \quad \text{spt } (R - \partial S) \subset B,$$

represent $S = (\mathcal{H}^{m+1} \llcorner A) \wedge \sigma\alpha$ where α is an $m+1$ vector-field orienting A and σ is an integer valued $\mathcal{H}^{m+1}$ measurable function on A with compact support in A , define

$$A_i = A \cap \{x : \sigma(x) \geq i\}, \quad S_i = (\mathcal{H}^{m+1} \llcorner A_i) \wedge \alpha$$

for $i \in \mathbf{Z}$ and infer that

$$\partial S_i \in \mathcal{B}_m(A, \emptyset)$$

because either $i > 0$ and $A_i \subset \text{spt } \sigma$, or $i \leq 0$ and $A \sim A_i \subset \text{spt } \sigma$,

$$S_i = (\mathcal{H}^{m+1} \llcorner A) \wedge \alpha - (\mathcal{H}^{m+1} \llcorner A \sim A_i) \wedge \alpha,$$

$$\partial S_i = -\partial[(\mathcal{H}^{m+1} \llcorner A \sim A_i) \wedge \alpha]$$

since $\text{spt } \partial[(\mathcal{H}^{m+1} \llcorner A) \wedge \alpha] \subset (U \sim A) \cap \text{Clos } A = \emptyset$ by [5, 4.1.31]. Taking $Z = U \sim B$, $C = A \sim B$, $\gamma = \alpha \llcorner C$, $\tau = \sigma \llcorner C$ we apply 5.9 to the restriction H of S to Z , as we may because S is a rectifiable current and ∂H equals the restriction of the rectifiable current R to Z . We infer that the restriction of R to Z equals

$$\sum_{i \in \mathbf{Z}} \partial H_i = \sum_{i \in \mathbf{Z}} (\mathcal{H}^m \llcorner Y_i) \wedge \eta$$

and that

$$\sum_{i \in \mathbf{Z}} \|\partial H_i\| \llcorner E = \sum_{i \in \mathbf{Z}} \mathcal{H}^m(Y_i \cap E) = \|R\| \llcorner E \leq \mathbf{M}(R) < \infty$$

for every Borel subset E of Z , hence each ∂H_i is the restriction to Z of a unique current

$$R_i \in \mathcal{R}_m(U) \quad \text{with} \quad \text{spt } R_i \subset \text{spt } R \subset K \quad \text{and} \quad R_i \llcorner B = 0,$$

representable in the form

$$R_i = (\mathcal{H}^m \llcorner Y_i) \wedge \eta = (\mathcal{H}^m \llcorner Y) \wedge r_i \eta$$

where $r_i(x) = 1$ for $x \in Y_i$, $r_i(x) = 0$ for $x \in U \sim Y_i$, and then

$$R \llcorner Z = \sum_{i \in \mathbf{Z}} R_i = (\mathcal{H}^m \llcorner Y) \wedge \sum_{i \in \mathbf{Z}} r_i \eta.$$

Since H_i equals the restriction of S_i to Z , ∂H_i equals the restriction of ∂S_i to Z , and we find that

$$\text{spt } (R_i - \partial S_i) \subset B, \quad R_i \in \Delta.$$

Observing that $X \cup Y$ is $(\mathcal{H}^m, m)$ rectifiable we see from [5, 3.2.19] that X , $X \cup Y$, Y have the same approximate tangent spaces at x and

$$\xi(x) = \pm \eta(x) \quad \text{for} \quad \mathcal{H}^m \quad \text{almost all } x \text{ in } X \cap Y.$$

We define

$$\begin{aligned}\lambda(x) &= -1 \text{ in case } x \in X \cap Y, q(x) > 0 \text{ and } \xi(x) = -\eta(x), \\ \lambda(x) &= 1 \text{ for all other } x \text{ in } X \cup Y.\end{aligned}$$

We also define

$$\begin{aligned}\omega(x) &= \Psi[x, \xi(x)] \text{ for } \mathcal{H}^m \text{ almost all } x \text{ in } X, \\ \omega(x) &= \Psi[x, \eta(x)] \text{ for } \mathcal{H}^m \text{ almost all } x \text{ in } Y \sim X,\end{aligned}$$

and infer, using the evenness of Ψ , that

$$\omega(x) = \Psi[x, \eta(x)] \text{ for } \mathcal{H}^m \text{ almost all } x \text{ in } Y.$$

For every integer i

$$\int_{X \cup Y} q \omega \, d\mathcal{H}^m = \int_Q \Psi = \Psi_\Delta(T) = \Psi_\Delta(Q) \leq \int_{Q+R_i} \Psi = \int_{X \cup Y} (q + \lambda r_i) \omega \, d\mathcal{H}^m,$$

because $q(x) + \lambda(x)r_i(x) \geq 0$ for $\mathcal{H}^m$ almost all x in $X \cup Y$, and it follows that

$$\int_{X \cup Y} \lambda r_i \omega \, d\mathcal{H}^m \geq 0.$$

Consequently

$$\begin{aligned}\int_{kQ} \Psi &= \int_{X \cup Y} kq \omega \, d\mathcal{H}^m \leq \int_{X \cup Y} kq \omega \, d\mathcal{H}^m + \sum_{i \in \mathbb{Z}} \int_{X \cup Y} \lambda r_i \omega \, d\mathcal{H}^m \\ &\leq \int_{X \cup Y} |kq + \sum_{i \in \mathbb{Z}} \lambda r_i| \omega \, d\mathcal{H}^m \\ &= \int_{kQ+R_{\mathbb{L}\mathbb{Z}}} \Psi = \int_{(kQ+R)_{\mathbb{L}\mathbb{Z}}} \Psi \leq \int_{kQ+R} \Psi.\end{aligned}$$

We infer that

$$\Psi_\Delta(kT) = \Psi_\Delta(kQ) = \int_{kQ} \Psi = k \int_Q \Psi = k\Psi_\Delta(T)$$

for every positive integer k , and deduce the conclusion of theorem 5.10 from theorem 5.8.

I do not know whether evenness of Ψ is essential for 5.10. The example of a real projective line in a real projective plane shows that the orientability hypothesis cannot be deleted from 5.10, while the example of a real projective line in a 3 dimensional real projective space shows that the codimension of T in A cannot be raised from 1 to 2. Moreover the example due to Frederick J. Almgren, Jr. described in 5.11 illustrates how real and integral homological minima can differ even in a compact 3 dimensional orientable manifold whose homology groups are torsionfree, and how the infimum in the second conclusion of 5.8 need not be reached for any positive integer k .

5.11. On the simply connected 3 dimensional manifold

$$\tilde{A} = \{(x, y, z) : x \in \mathbf{R}, y \in \mathbf{R}, z \in \mathbf{C}, y^2 + |z|^2 = 1\} \simeq \mathbf{R} \times \mathbf{S}^2$$

the additive group of $\mathbf{Z}$ acts discretely through the orientation preserving diffeomorphisms α_ν defined by the formula

$$\alpha_\nu(x, y, z) = (x + \nu 2\pi, y, ze^{\nu i}) \text{ for } \nu \in \mathbf{Z} \text{ and } (x, y, z) \in \tilde{A}.$$

Associating with each point of $\tilde{A}$ its orbit under this action, we obtain a covering map ω of $\tilde{A}$ onto a compact 3 dimensional orientable manifold A of class ∞ . We let

$$X(x, y, z) = x, Y(x, y, z) = y, Z(x, y, z) = z \text{ for } (x, y, z) \in \tilde{A},$$

consider the parametric integrand

$$\tilde{\Psi} = (1 + Y^2)(|dX|^2 + |dY|^2 + |dZ|^2)^{1/2}$$

of degree 1 on $\tilde{A}$, note that $\alpha_\nu^* \tilde{\Psi} = \tilde{\Psi}$ for $\nu \in \mathbf{Z}$, and infer the existence of an integrand Ψ on A such that

$$\tilde{\Psi} = \omega^* \Psi.$$

Moreover $\alpha_\nu^*(dX) = dX$ for $\nu \in \mathbf{Z}$, hence there exists

$$\Xi \in \mathcal{D}^1(A) \text{ with } dX = \omega^* \Xi.$$

Since $dX \leq \tilde{\Psi}$ and $d(dX) = 0$, it follows that

$$\Xi \leq \Psi \text{ and } d\Xi = 0.$$

We observe that

$$T = \omega_* [(0, 1, 0), (2\pi, 1, 0)] \in \mathbf{I}_1(A) \cap \mathfrak{I}_1(A),$$

because $\partial T = [\omega(2\pi, 1, 0)] - [\omega(0, 1, 0)] = 0$, and that

$$\Psi_{\mathbf{B}_1(A)}(T) \geq T(\Xi) = 2\pi.$$

Also, letting μ be the Haar measure of the multiplicative group $\mathbf{S}^1$ with $\mu(\mathbf{S}^1) = 1$, we see that

$$S = \int_{\mathbf{S}^1} \omega_* [(0, 0, z), (2\pi, 0, z)] d\mu z \in \mathbf{N}_1(A) \cap \mathbf{Z}_1(A),$$

because $\omega(2\pi, 0, z) = \omega(0, 0, ze^{-i})$ for $z \in \mathbf{S}^1$, hence

$$\partial S = \int_{\mathbf{S}^1} [\omega(0, 0, ze^{-i})] d\mu z - \int_{\mathbf{S}^1} [\omega(0, 0, z)] d\mu z = 0.$$

Since $S(\Xi) = 2\pi = T(\Xi)$ and $\dim \mathbf{H}_1(A; \mathbf{R}) = 1$, we infer that $S = T \in \mathbf{B}_1(A)$ and

$$\Psi_{\mathbf{B}_1(A)}(T) \leq \int_S \Psi \leq \int_{\mathbf{S}^1} 2\pi d\mu z = 2\pi.$$

We conclude $\Psi_{\mathbf{B}_1(A)}(T) = \int_S \Psi = 2\pi$. Next we will prove that

$$k^{-1}\Psi_{\mathbf{B}_1(A)}(kT) > \Psi_{\mathbf{B}_1(A)}(T) \quad \text{for } 0 < k \in \mathbf{Z}.$$

Otherwise we could find a positive integer k and a cycle

$$Q \in \mathcal{R}_1(A) \quad \text{with} \quad Q - kT \in \mathcal{R}_1(A) \quad \text{and} \quad \int_Q \Psi = k2\pi,$$

apply [5, 4.2.25] to decompose

$$Q = \sum_{i=1}^{\infty} Q_i \quad \text{with} \quad \int_Q \Psi = \sum_{i=1}^{\infty} \int_{Q_i} \Psi,$$

where each Q_i is an oriented simple closed curve of finite length in A , lift Q_i relative to the covering map ω to an oriented simple curve R_i of finite length in $\tilde{A}$ with endpoints r_i and s_i , so that

$$Q_i = \omega_* R_i \quad \text{and} \quad \partial R_i = [s_i] - [r_i],$$

observe that $\omega(s_i) = \omega(r_i)$ because $\partial Q_i = 0$, choose

$$\nu_i \in \mathbf{Z} \quad \text{with} \quad X(s_i) = X(r_i) + \nu_i 2\pi,$$

$$Y(s_i) = Y(r_i), \quad Z(s_i) = Z(r_i) \exp(\nu_i i),$$

$$Q_i(\Xi) = R_i(dX) = X(s_i) - X(r_i) = \nu_i 2\pi,$$

use the fact that $\Xi \leq \Psi$ and

$$\sum_{i=1}^{\infty} Q_i(\Xi) = Q(\Xi) = kT(\Xi) = k2\pi = \sum_{i=1}^{\infty} \int_{Q_i} \Psi$$

to infer $Q_i(\Xi) = \int_{Q_i} \Psi$ and

$$R_i(dX) = \int_{R_i} \tilde{\Psi} = \int (1 + Y^2) d||R_i|| \quad \text{for } 0 < j \in \mathbf{Z},$$

hence R_i is an oriented line segment on which Y vanishes and Z is constant, so that

$$Y(s_i) = 0, \quad Z(s_i) \neq 0, \quad \exp(\nu_i i) = 1, \quad \nu_i = 0,$$

and would conclude $0 < k2\pi = \sum_{i=1}^{\infty} Q_i(\Xi) = \sum_{i=1}^{\infty} \nu_i 2\pi = 0$.

We note furthermore that the relation

$$\{(\omega(x, y, z), (e^{ix}, y, ze^{-ix/(2\pi)})) : (x, y, z) \in \tilde{A}\}$$

is a diffeomorphism mapping A onto $\mathbf{S}^1 \times \mathbf{S}^2$.

5.12. *If A is compact, $m = 1$, Ψ is a positive convex even integrand on U and $T \in \mathfrak{F}_{1,A}(U)$, then*

$$\Psi_{\mathbf{Z}_1(A, B)}(T) = \Psi_{\mathfrak{d}_1(A, B)}(T).$$

Proof. Representing A as intersection of a decreasing sequence of compact subsets A_j of U whose boundaries are $n - 1$ dimensional submanifolds of class ∞ of U , we infer from the integral flat analogue of 3.10 that

$$\Psi_{\partial_1(A_j, B)}(T) \uparrow \Psi_{\partial_1(A, B)}(T) \quad \text{as } j \uparrow \infty,$$

and from 2.12, 2.13, 5.8 that

$$\Psi_{\mathbf{z}_1(A_j, B)}(T) = \inf \{k^{-1}\Psi_{\partial_1(A_j, B)}(kT) : 0 < k \in \mathbf{Z}\}$$

for $0 < j \in \mathbf{Z}$. Inasmuch as $\Psi_{\mathbf{z}_1(A, B)}(T) \geq \Psi_{\mathbf{z}_1(A_j, B)}(T)$, we will complete the proof by showing that

$$k^{-1}\Psi_{\partial_1(A_j, B)}(kT) = \Psi_{\partial_1(A_j, B)}(T)$$

for all positive integers j and k .

Letting $\Delta = \partial_1(A_j, B)$ we choose

$$Q \in \mathcal{R}_1(U) \quad \text{with} \quad Q - T \in \Delta \quad \text{and} \quad \Psi_\Delta(T) = \int_Q \Psi,$$

and we will verify that

$$\int_{kQ+R} \Psi \geq \int_{kQ} \Psi \quad \text{for } R \in \Delta \cap \mathcal{R}_1(U).$$

Given such a current R and any $\epsilon > 0$ we select

$$F \in \mathcal{R}_{0,B}(U), \quad G \in \mathcal{R}_{1,B}(U) \quad \text{with} \quad \partial R = F + \partial G, \quad \int_G \Psi < \epsilon,$$

hence $D = R - G \in \Delta \cap \mathcal{I}_1(U)$. Using [5, 4.1.28] we represent

$$Q = (\mathcal{C}^1 \llcorner X) \wedge q\xi \quad \text{and} \quad D = (\mathcal{C}^1 \llcorner Y) \wedge r\eta,$$

where X and Y are $\mathcal{C}^1$ measurable ($\mathcal{C}^1, 1$) rectifiable subsets of A_j ; q, ξ and r, η are $\mathcal{C}^1$ measurable functions on U ;

$$0 < q(x) \in \mathbf{Z}, \quad \xi(x) \in \mathbf{S}^{n-1}, \quad \text{Tan}^1(\mathcal{C}^1 \llcorner X, x) = \mathbf{R}\xi(x) \quad \text{for } x \in X;$$

$$0 < r(x) \in \mathbf{Z}, \quad \eta(x) \in \mathbf{S}^{n-1}, \quad \text{Tan}^1(\mathcal{C}^1 \llcorner Y, x) = \mathbf{R}\eta(x) \quad \text{for } x \in Y;$$

$$q(x) = 0 \quad \text{for } x \in U \sim X; \quad r(x) = 0 \quad \text{for } x \in U \sim Y.$$

We also apply [5, 4.2.25] to decompose

$$D = \sum_{i \in \mathbf{Z}} R_i \quad \text{with} \quad \int_D \Psi = \sum_{i \in \mathbf{Z}} \int_{R_i} \Psi,$$

$$R_i = (\mathcal{C}^1 \llcorner Y) \wedge r_i\eta \quad \text{and} \quad \text{spt } \partial R_i \subset \text{spt } \partial D$$

hence $R_i \in \Delta$ for $i \in \mathbf{Z}$, where each R_i is an oriented simple curve of finite length in A_j and r_i is the characteristic function of an $\mathcal{C}^1$ measurable subset of Y . Then we proceed as in the proof of 5.10 to infer

$$\xi(x) = \pm \eta(x) \quad \text{for } \mathcal{C}^1 \text{ almost all } x \text{ in } X \cap Y,$$

to define λ and ω , and eventually to conclude

$$\begin{aligned} \int_{kQ} \Psi &\leq \int_{X \cup Y} |kq + \sum_{i \in \mathbf{Z}} \lambda r_i| \omega d\mathcal{H}^1 = \int_{kQ+D} \Psi \\ &\leq \int_G \Psi + \int_{kQ+R} \Psi < \epsilon + \int_{kQ+R} \Psi. \end{aligned}$$

Thus $\Psi_\Delta(kT) = \Psi_\Delta(kQ) = \int_{kQ} \Psi = k \int_Q \Psi = k\Psi_\Delta(T)$.

The example constructed in 5.13 shows that the equation

$$\Psi_{\mathbf{Z}_m, \kappa(A, B)}(T) = \Psi_{\mathfrak{z}_m, \kappa(A, B)}(T)$$

is sometimes false for $m = 2$. I do not know whether this equation is always true under the hypotheses of 5.10. Of course it becomes a corollary of 5.10 if also $\mathbf{H}_m(A, B; \mathbf{R}) = \{0\}$, for example when $A = \mathbf{R}^{m+1}$ and $B = \emptyset$. (This special result can be obtained alternately by the method of [8, 3.3] with use of the fact that $\mathbf{H}_{m-1}(P; \mathbf{Z})$ is torsionfree for every trianguable subset P of $\mathbf{R}^{m+1}$.)

5.13. Here we consider the space A of all orbits of the discrete action of $\mathbf{Z}$ on $\mathbf{R}^3$ through the isometries α_ν defined by the formula

$$\alpha_\nu(x) = (x_1 + \nu, x_2, (-1)^\nu x_3) \quad \text{for } \nu \in \mathbf{Z} \text{ and } x \in \mathbf{R}^3.$$

The covering map ω , which associates with each point of $\mathbf{R}^3$ its orbit under this action, induces a locally Euclidean Riemannian metric on A . We let Ψ be the 2 dimensional mass integrand on A , and we identify $\mathbf{R}^3$ with $\mathbf{R} \times \mathbf{R}^2$ to define

$$T_r = \omega_\#[\delta_1 \times (\mathbf{E}^2 \llcorner \{z : |z| < r\})] \in \mathbf{I}_2(A)$$

for $0 < r \in \mathbf{R}$. We will show that

$$\Psi_{\mathbf{Z}_2(A)}(T_r) \leq \pi r \quad \text{and} \quad \Psi_{\mathfrak{z}_2(A)}(T_r) = \pi r^2.$$

The preceding (rough) estimate for $\Psi_{\mathbf{Z}_2(A)}(T_r)$ holds because

$$S_r = \omega_\#([0, 1] \times \partial(\mathbf{E}^2 \llcorner \{z : |z| < r\})) \in \mathbf{I}_2(A)$$

with $\partial S_r = 2\partial T_r$, hence $\Psi_{\mathfrak{z}_2(A)}(2T) \leq \mathbf{M}(S_r) = 2\pi r$.

Clearly $\Psi_{\mathfrak{z}_2(A)}(T_r) \leq \mathbf{M}(T_r) = \pi r^2$. In order to prove the opposite inequality we observe first that there exists a function u on A such that

$$u[\omega(y, z)] = |z| \quad \text{for } (y, z) \in \mathbf{R} \times \mathbf{R}^2,$$

hence $\text{Lip}(u) = 1$, that $\partial T_r \in \mathfrak{z}_1(K_r)$ where $K_r = \{a : u(a) = r\}$ is a Klein bottle imbedded in A , and that

$$\mathbf{M}(R + \partial T_r) \geq \mathbf{M}(\partial T_r) = 2\pi r \quad \text{for } R \in \mathfrak{B}_1(K_r).$$

In view of [5, 5.3.20] this assertion follows from the statement that in K , every oriented closed geodesic G of length less than $2\pi r$ is integrally homologous to an integral multiple of $\omega_\#[(0, r, 0), (1, r, 0)]$, which may be verified through study of the locally isometric covering map γ_r of $\mathbf{R}^2$ onto K , given by the formula

$$\gamma_r(s, t) = \omega(s, r \cos(t/r), r \sin(t/r)) \quad \text{for } (s, t) \in \mathbf{R}^2;$$

one finds that $\gamma_r^{-1}(\text{spt } G)$ is the union of a family of parallel straight lines in $\mathbf{R}^2$, that $G = \gamma_{r\#}H$ for some oriented line segment H of length less than $2\pi r$, hence that H is horizontal and

$$G = \omega_{\#}[(0, r, 0), (\nu, r, 0)] \in \mathfrak{B}_1(K_r) \quad \text{for some } \nu \in \mathbf{Z}.$$

Now suppose $Q \in \mathbf{I}_2(A)$ with $\partial Q = \partial T_r$. For $0 < \rho < r$ there exists a Lipschitzian retraction λ_{ρ} of $C_{\rho} = \{a : u(a) \geq \rho\}$ onto K_{ρ} such that $\lambda_{\rho\#}(\partial T_r) = \partial T_{\rho}$, and

$$\begin{aligned} \langle Q, u, \rho- \rangle &= \lambda_{\rho\#} \langle Q, u, \rho- \rangle \\ &= \lambda_{\rho\#}[(\partial Q) \lfloor C_{\rho} - \partial(Q \lfloor C_{\rho})] = \partial T_{\rho} - \partial \lambda_{\rho\#}(Q \lfloor C_{\rho}), \\ \langle Q, u, \rho- \rangle - \partial T_{\rho} &\in \mathfrak{B}_1(K_{\rho}), \quad \mathbf{M}\langle Q, u, \rho- \rangle \geq 2\pi\rho. \end{aligned}$$

We conclude that

$$\mathbf{M}(Q) \geq \int_0^r \mathbf{M}\langle Q, u, \rho- \rangle d\mathcal{L}^1 \rho \geq \int_0^r 2\pi\rho d\mathcal{L}^1 \rho = \pi r^2.$$

6. On certain differential forms defining flat cochains. In 6.2 we will show that the condition 4.6(4)', characteristic of a pair of differential forms associated with a locally flat cochain, is a consequence of continuity and differentiability hypotheses allowing fairly large exceptional sets. As an application of this result we will give a simple proof of the theorem of [2] asserting the area minimizing property of the join of $\mathbf{S}^3 \times \mathbf{S}^3$ to the origin in $\mathbf{R}^4 \times \mathbf{R}^4$.

First we establish an estimate concerning functions which are additive with respect to cubical subdivision.

6.1. Suppose:

(1) To each nonnegative integer i corresponds a family H_i of compact n dimensional cubes with side length $r2^{-i}$, such that H_0 consists of a single cube C , and H_{i+1} consists of the cubes obtained by subdividing each member of H_i into 2^n congruent parts.

(2) λ is a real valued function on $\cup \{H_i : 0 \leq i \in \mathbf{Z}\}$ such that

$$\lambda(S) = \sum_{S \supset T \in H_{i+1}} \lambda(T) \quad \text{for } S \in H_i.$$

(3) To each nonnegative integer j correspond a subset E_j of C and finite non-negative numbers β_j, γ_j such that

$$\limsup_{\nu \rightarrow \infty} \left\{ (r2^{-i})^{-\beta_i} |\lambda(S)| : x \in S \in \bigcup_{i=\nu}^{\infty} H_i \right\} \leq \gamma_i \quad \text{for } x \in E_i.$$

(4) $C = \cup \{E_j : 0 \leq j \in \mathbf{Z}\}$.

Then

$$|\lambda(C)| \leq \sum_{i=0}^{\infty} 2^{n+2\beta_i} \alpha(\beta_i)^{-1} \gamma_i \mathfrak{C}^{\beta_i}(E_i).$$

Proof. We will use the covering lemma:

If $0 \leq p \in \mathbf{Z}$, $0 \leq q \in \mathbf{Z}$, $q \leq p$, $F \subset \bigcup_{i=a}^p H_i$ and $\bigcup F = S \in H_a$, then there exists $G \subset F$ with $\bigcup G = S$ and $\sum_{T \in G} \lambda(T) = \lambda(S)$.

This lemma follows readily from (1) and (2) by downward induction with respect to q , starting with the trivial case where $q = p$.

Suppose $\delta_0, \delta_1, \delta_2, \dots$ are arbitrary positive numbers.

For $0 \leq \nu \in \mathbf{Z}$ we define $E_{i,\nu}$ as the set of all points x in E_i such that

$$(r2^{-i})^{-\beta_i} |\lambda(S)| \leq \gamma_i + \delta_i \quad \text{whenever} \quad x \in S \in \bigcup_{i=\nu}^{\infty} H_i.$$

We also define $E_{i,-1} = \emptyset$. We cover $E_{i,\nu} \sim E_{i,\nu-1}$ by a countable family $\Delta_{i,\nu}$ consisting of open sets with diameters less than $r2^{-\nu}$ such that

$$\sum_{D \in \Delta_{i,\nu}} \alpha(\beta_i) 2^{-\beta_i} (\text{diam } D)^{\beta_i} \leq \mathfrak{H}^{\beta_i}(E_{i,\nu} \sim E_{i,\nu-1}) + 2^{-\nu} \delta_i.$$

To each $D \in \Delta_{i,\nu}$ corresponds a unique integer $i \geq \nu$ such that $r2^{-i-1} \leq \text{diam } D < r2^{-i}$; then the family

$$\Omega_{i,\nu}(D) = H_i \cap \{T : T \cap D \cap E_{i,\nu} \neq \emptyset\}$$

has at most 2^n elements, hence

$$\begin{aligned} \sum_{T \in \Omega_{i,\nu}(D)} |\lambda(T)| &\leq 2^n (r2^{-i})^{\beta_i} (\gamma_i + \delta_i) \\ &\leq 2^n (2 \text{diam } D)^{\beta_i} (\gamma_i + \delta_i) = 2^{n+\beta_i} (\text{diam } D)^{\beta_i} (\gamma_i + \delta_i); \end{aligned}$$

moreover the set $B_{i,\nu}(D) = C \sim \bigcup [H_i \sim \Omega_{i,\nu}(D)]$ is open relative to C , and $D \cap E_{i,\nu} \subset B_{i,\nu}(D) \subset \bigcup \Omega_{i,\nu}(D)$. Observing that

$$C = \bigcup_{i=0}^{\infty} \bigcup_{\nu=0}^{\infty} \bigcup_{D \in \Delta_{i,\nu}} B_{i,\nu}(D)$$

we choose $\Upsilon_{i,\nu} \subset \Delta_{i,\nu}$ so that $\bigcup_{i=0}^{\infty} \bigcup_{\nu=0}^{\infty} \Upsilon_{i,\nu}$ is finite and

$$C = \bigcup_{i=0}^{\infty} \bigcup_{\nu=0}^{\infty} \bigcup_{D \in \Upsilon_{i,\nu}} B_{i,\nu}(D),$$

hence $F = \bigcup_{i=0}^{\infty} \bigcup_{\nu=0}^{\infty} \bigcup_{D \in \Upsilon_{i,\nu}} \Omega_{i,\nu}(D)$ is finite subfamily of $\bigcup \{H_i : 0 \leq i \in \mathbf{Z}\}$ with $\bigcup F = C$. Our covering lemma yields $G \subset F$ with $\bigcup G = C$ and $\sum_{T \in G} \lambda(T) = \lambda(C)$. We infer that

$$\begin{aligned} |\lambda(C)| &\leq \sum_{T \in G} |\lambda(T)| \leq \sum_{T \in F} |\lambda(T)| \leq \sum_{i=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{D \in \Upsilon_{i,\nu}} \sum_{T \in \Omega_{i,\nu}(D)} |\lambda(T)| \\ &\leq \sum_{i=1}^{\infty} \sum_{\nu=0}^{\infty} \sum_{D \in \Delta_{i,\nu}} 2^{n+\beta_i} (\text{diam } D)^{\beta_i} (\gamma_i + \delta_i) \\ &\leq \sum_{i=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{n+2\beta_i} (\gamma_i + \delta_i) \alpha(\beta_i)^{-1} [\mathfrak{H}^{\beta_i}(E_{i,\nu} \sim E_{i,\nu-1}) + 2^{-\nu} \delta_i] \\ &= \sum_{i=0}^{\infty} 2^{n+2\beta_i} (\gamma_i + \delta_i) \alpha(\beta_i)^{-1} [\mathfrak{H}^{\beta_i}(E_i) + 2\delta_i] \end{aligned}$$

and deduce the conclusion of 6.1 from the arbitrariness of the positive numbers δ_j .

The following proposition 6.2 is a form of Stokes' Theorem allowing exceptional sets, which include both the extended Fundamental Theorem of the Calculus in [10, Corollary of VI (7.9)] and the extended Cauchy Integral Theorem in [1], [10, VI (5.3)]. The hypothesis (II) implies through Stepanoff's Theorem [5, 3.1.9] that for $\mathcal{L}^n$ almost all x in U the differential $D\phi(x)$ exists and the exterior derivative $d\phi(x)$ can be defined by the standard formula given in [5, 4.1.6]. The hypothesis (III) assures $\mathcal{L}^n \llcorner K$ almost boundedness of $d\phi$, but not of $D\phi$, as seen from the example where $m = n = 1$ and $\phi = u \wedge DX_1$ with

$$u(x) = x^2 \sin(x^{-2}) \quad \text{for } 0 \neq x \in \mathbf{R}, \quad u(0) = 0.$$

6.2. Suppose U is an open subset of $\mathbf{R}^n$ and ϕ is an $\mathcal{L}^n \llcorner U$ measurable differential form of degree m on U satisfying the three hypotheses:

(I) ϕ is continuous at $\mathcal{H}^{n-1}$ almost all points of U .

(II) $F_k \subset U$ with $\mathcal{H}^{n-1}(F_k) < \infty$ for $0 < k \in \mathbf{Z}$, and

$$\limsup_{y \rightarrow x} \|\phi(y) - \phi(x)\|/|y - x| < \infty \quad \text{for } x \in U \sim \bigcup_{k=1}^{\infty} F_k$$

(III) To each compact $K \subset U$ corresponds $\mu_K \in \mathbf{R}$ with

$$\sup \{ \|\phi(x)\|, \|d\phi(x)\| \} \leq \mu_K \quad \text{for } \mathcal{L}^n \text{ almost all } x \text{ in } K.$$

Then there exists a unique $\alpha \in \mathbf{F}^m(U)$ such that the conditions (1), (2), (3), (4) of 4.6 hold with $\psi = d\phi$.

Proof. First we will verify the lemma:

If $m = n - 1$, C is a compact n dimensional cube contained in U and $Q = \mathbf{E}^n \llcorner C$, then $(\partial Q)(\phi) = Q(d\phi)$.

Observing that $\|\phi(x)\| \leq \mu_C$ for $\mathcal{H}^{n-1}$ almost all x in C , we interpret

$$(\partial Q)(\phi) = \int \langle \vec{\partial Q}, \phi \rangle d\|\partial Q\|, \quad Q(d\phi) = \int \langle \vec{Q}, d\phi \rangle d\|Q\|.$$

We will apply 6.1 with

$$\lambda(S) = [\partial(\mathbf{E}^n \llcorner S)](\phi) - [\mathbf{E}^n \llcorner S](d\phi) \quad \text{for } S \in \bigcup_{i=0}^{\infty} H_i$$

and with E_i , β_i , γ_i defined as follows:

The set E_0 consists of all points x in C such that ϕ is differentiable at x and $d\phi$ is $\mathcal{H}^n$ approximately continuous at x . We see from [5, 3.1.9, 2.9.13] that $\mathcal{H}^n(E_0) = \mathcal{H}^n(C) < \infty$.

With each $x \in E_0$ we associate

$$\sigma_x \in \mathcal{E}^{n-1}(U), \quad \sigma_x(y) = \phi(x) + \langle y - x, D\phi(x) \rangle \quad \text{for } y \in U,$$

and compute $D\sigma_x(y) = D\phi(x)$, $d\sigma_x(y) = d\phi(x)$ for $y \in U$.

For any $x \in E_0$ and $\epsilon > 0$ there exists $\nu \in \mathbf{Z}$ such that

$$||\phi(y) - \sigma_x(y)|| \leq \epsilon |y - x| \quad \text{for } y \in S,$$

$$\int_S ||d\phi(y) - d\phi(x)|| d\mathcal{H}^n y \leq \epsilon \mathcal{H}^n(S)$$

whenever $\nu \leq i \in \mathbf{Z}$ and $x \in S \in H_i$, hence

$$\begin{aligned} |\lambda(S)| &= |[\partial(\mathbf{E}^\nu \llcorner S)](\phi - \sigma_x) - [\mathbf{E}^\nu \llcorner S](d\phi - d\sigma_x)| \\ &\leq \int_{Bdr_\nu S} ||\phi - \sigma_x|| d\mathcal{H}^{n-1} + \int_S ||d\phi - d\sigma_x|| d\mathcal{H}^n \\ &\leq \epsilon n r 2^{-i} 2n(r 2^{-i})^{n-1} + \epsilon(r 2^{-i})^n = \epsilon(2n^2 + 1)(r 2^{-i})^n. \end{aligned}$$

Accordingly we take $\beta_0 = n$ and $\gamma_0 = 0$.

The set E_{2k} corresponding to $0 < k \in \mathbf{Z}$ consists of all points x in $C \sim E_0$ such that

$$\limsup_{y \rightarrow x} ||\phi(y) - \phi(x)||/|y - x| < k.$$

We note that $\mathcal{H}^n(E_{2k}) \leq \mathcal{H}^n(C \sim E_0) = 0$.

With each $x \in U$ we associate

$$\rho_x \in \mathcal{S}^{n-1}(U), \quad \rho_x(y) = \phi(x) \quad \text{for } y \in U,$$

and compute $D\rho_x(y) = 0$, $d\rho_x(y) = 0$ for $y \in U$.

For any $x \in E_{2k}$ there exists $\nu \in \mathbf{Z}$ such that

$$||\phi(y) - \phi(x)|| \leq k |y - x| \quad \text{for } y \in S$$

whenever $\nu \leq i \in \mathbf{Z}$ and $x \in S \in H_i$, hence

$$\begin{aligned} |\lambda(S)| &= |[\partial(\mathbf{E}^\nu \llcorner S)](\phi - \rho_x) - (\mathbf{E}^\nu \llcorner S)(d\phi)| \\ &\leq \int_{Bdr_\nu S} ||\phi - \rho_x|| d\mathcal{H}^{n-1} + \int_S ||d\phi|| d\mathcal{H}^n \\ &\leq k n r 2^{-i} 2n(r 2^{-i})^{n-1} + \mu_C(r 2^{-i})^n = (k 2n^2 + \mu_C)(r 2^{-i})^n. \end{aligned}$$

Accordingly we take $\beta_{2k} = n$ and $\gamma_{2k} = k 2n^2 + \mu_C$.

The set E_{2k+1} corresponding to $0 < k \in \mathbf{Z}$ consists of all points x in $C \cap F_k$ such that ϕ is continuous at x . We note that

$$\mathcal{H}^{n-1}(E_{2k+1}) \leq \mathcal{H}^{n-1}(F_k) < \infty.$$

For any $x \in E_{2k+1}$ and $\epsilon > 0$ there exists $\nu \in \mathbf{Z}$ such that

$$r 2^{-i} < \epsilon, \quad ||\phi(y) - \phi(x)|| \leq \epsilon \quad \text{for } y \in S$$

whenever $\nu \leq i \in \mathbf{Z}$ and $x \in S \in H_i$, hence

$$\begin{aligned} |\lambda(S)| &= |[\partial(\mathbf{E}^\nu \llcorner S)](\phi - \rho_x) - [\mathbf{E}^\nu \llcorner S](d\phi)| \\ &\leq \epsilon 2n(r 2^{-i})^{n-1} + \mu_C(r 2^{-i})^n \leq \epsilon(2n + \mu_C)(r 2^{-i})^{n-1}. \end{aligned}$$

Accordingly we take $\beta_{2k+1} = n - 1$ and $\gamma_{2k+1} = 0$.

The set E_1 consists of all points x in C such that ϕ is not continuous at x . We note that $\mathcal{H}^{n-1}(E_1) = 0$.

If $0 \leq i \in \mathbf{Z}$ and $S \in H_i$, then

$$\begin{aligned} |\lambda(S)| &\leq \int_{\text{Boundary } S} \|\phi\| d\mathcal{H}^{n-1} + \int_S \|d\phi\| d\mathcal{H}^n \\ &\leq \mu_C 2n(r2^{-i})^{n-1} + \mu_C (r2^{-i})^n \leq \mu_C (2n + r)(r2^{-i})^{n-1}. \end{aligned}$$

Accordingly we take $\beta_1 = n - 1$ and $\gamma_1 = \mu_C(2n + r)$.

Observing that $\gamma_j \mathcal{H}^{\beta_j}(E_j) = 0$ for $0 \leq j \in \mathbf{Z}$, we infer from 6.1 that $\lambda(C) = 0$, which confirms our lemma.

Next we only assume $m \leq n - 1$ and verify the condition (4)' of 4.6 with $\psi = d\phi$. Covering $\text{spt } \omega$ by the interiors of finitely many compact cubes contained in U and using a partition of unity, we reduce the problem to the special case where there exists a compact n dimensional cube C such that

$$C \subset U \quad \text{and} \quad \text{spt } \omega \subset \text{Int } C.$$

Then $\phi \wedge \omega$ is an $\mathcal{E}^n \llcorner U$ measurable differential form of degree $m + (n - m - 1) = n - 1$ on U ,

$$d(\phi \wedge \omega)(x) = (d\phi)(x) \wedge \omega(x) + (-1)^m \phi(x) \wedge d\omega(x)$$

for $x \in \text{dmn } D\phi$, the hypotheses (I), (II), (III) of 6.2 hold with ϕ and μ_K replaced by $\phi \wedge \omega$ and $2\mu_K \mathbf{F}_C(\omega)$, and

$$\text{spt } (\phi \wedge \omega) \subset \text{Int } C \subset U \sim \text{spt } \partial(\mathbf{E}^n \llcorner C),$$

hence our lemma implies

$$0 = (\mathbf{E}^n \llcorner C)[d(\phi \wedge \omega)] = \mathbf{E}^n[d(\phi \wedge \omega)] = \mathbf{E}^n[(d\phi) \wedge \omega + (-1)^m \phi \wedge d\omega].$$

Finally we note that in case $m \geq n$ the proposition 6.2 is trivial.

6.3. We will use the preceding theorem to show that for each integer $\kappa \geq 3$ the $2\kappa + 1$ dimensional oriented cone

$$T = \partial(\mathbf{E}^{\kappa+1} \times \mathbf{E}^{\kappa+1}) \llcorner \{(x, y) : x \in \mathbf{R}^{\kappa+1}, y \in \mathbf{R}^{\kappa+1}, |x| < |y|\}$$

is absolutely area minimizing with respect to $\mathbf{R}^{\kappa+1} \times \mathbf{R}^{\kappa+1}$, even though

$$\text{spt } T = \{(x, y) : x \in \mathbf{R}^{\kappa+1}, y \in \mathbf{R}^{\kappa+1}, |x| = |y|\}$$

fails to be a manifold. This fact was proved first in [2], then more simply in [9]. [The following modified method could be applied to all the examples discovered in [9], because the formulae (4.3) and (4.4) of [9] imply that $\pi^\# \omega \wedge d\Omega = 0$ for any $\omega \in \mathcal{E}^{n-1}(X/G)$.]

For $(x, y) \in U = \mathbf{R}^{\kappa+1} \times \mathbf{R}^{\kappa+1}$ we let

$$p(x, y) = x, \quad q(x, y) = y, \quad r(x, y) = (|x|^2 + |y|^2)^{1/2},$$

in case $r(x, y) > 0$ we define $\theta(x, y)$ so that $0 \leq \theta(x, y) \leq \pi/2$ and

$$|x| = r(x, y) \cos [\theta(x, y)], \quad |y| = r(x, y) \sin [\theta(x, y)].$$

The restrictions of p, q, r, θ to $Z = U \cap \{(x, y) : x \neq 0, y \neq 0\}$ are of class ∞ and $\mathfrak{H}^{2\kappa+1}(U \sim Z) = 0$. Using the solid angle form $\Upsilon \in \mathfrak{E}^\kappa(\mathbf{R}^{\kappa+1} \sim \{0\})$ given by the formula

$$\Upsilon(x) = |x|^{-\kappa-1}(X_1 \wedge \cdots \wedge X_{\kappa+1}) \lrcorner x \quad \text{for } x \in \mathbf{R}^{\kappa+1} \sim \{0\},$$

with $||\Upsilon(x)|| = |x|^{-\kappa}$ and $d\Upsilon(x) = 0$, we construct

$$\Omega = p^* \Upsilon \wedge q^* \Upsilon \in \mathfrak{E}^{2\kappa}(Z)$$

with $d\Omega = 0$ and $||\Omega|| = |p|^{-\kappa} |q|^{-\kappa} = [r^2 \sin(2\theta)/2]^{-\kappa}$.

If $(x, y) \in Z \cap \text{spt } T$, then $\theta(x, y) = \pi/4$, the $2\kappa + 1$ vector $\vec{T}(x, y)$ is dual in the sense of [5, 1.5.2] to the covector $r(x, y) d\theta(x, y)$ and

$$\langle \vec{T}(x, y), (-1)^{\kappa+1} 2^{-\kappa} r(x, y)^{2\kappa} dr(x) \wedge \Omega(x, y) \rangle = 1.$$

Using a real number λ to be specified later we define

$$u = (2\kappa + 1)^{-1} r^{2\kappa+1} [\sin(2\theta)]^\lambda \in \mathfrak{E}^0(Z),$$

$$\phi = (-1)^{\kappa+1} 2^{-\kappa} du \wedge \Omega \in \mathfrak{E}^{2\kappa+1}(Z),$$

$\gamma = 2\lambda/(2\kappa + 1)$ and compute $d\phi = 0$,

$$du = r^{2\kappa} [\sin(2\theta)]^{\lambda-1} [\sin(2\theta) dr + \gamma \cos(2\theta) r d\theta],$$

$$\begin{aligned} ||\phi|| &= ||du|| 2^{-\kappa} ||\Omega|| = ||du|| r^{-2\kappa} [\sin(2\theta)]^{-\kappa} \\ &= [\sin(2\theta)]^{\lambda-1-\kappa} ((1 - \gamma^2) [\sin(2\theta)]^2 + \gamma^2)^{1/2}. \end{aligned}$$

In order to make $||\phi|| \leq 1$ we must choose λ so that

$$v(t) = t^{\lambda-1-\kappa} [(1 - \gamma^2)t + \gamma^2] \leq 1 \quad \text{for } 0 < t \leq 1.$$

If $\lambda > \kappa + 1/2$, then $\gamma > 1$ and

$$\begin{aligned} t^{\kappa+2-\lambda} v'(t) &= (1 - \gamma^2)(\lambda - \kappa)t + \gamma^2(\lambda - \kappa - 1) \geq \lambda - \kappa - \gamma^2 \\ &= \lambda - \kappa - \left(\frac{2\lambda}{2\kappa + 1}\right)^2 = \left(\frac{2\kappa + 1}{4}\right)^2 - \kappa - \left(\frac{2\lambda}{2\kappa + 1} - \frac{2\kappa + 1}{4}\right)^2 \end{aligned}$$

for $0 < t \leq 1$. Observing that $(2\kappa + 1)^2 > 16\kappa$ because $\kappa \geq 3$, and taking $\lambda = (2\kappa + 1)^2/8$, we infer $\lambda > 2\kappa > \kappa + 1/2$ and $v'(t) > 0$ for $0 < t \leq 1$, hence $v(t) < v(1) = 1$ for $0 < t < 1$. Thus

$$||\phi(x, y)|| \leq 1 \quad \text{for } (x, y) \in Z$$

and the inequality is strict unless $|x| = |y|$.

To the differential form ϕ corresponds according to 6.2 a flat cochain $\alpha \in \mathbf{F}^{2\kappa+1}(U)$ such that the conditions (1), (2), (3), (4) of 4.6 hold with $\psi = 0$, hence $d\alpha = 0$. For every compact subset K of U and every $Q \in \mathbf{M}_{2\kappa+1}(U)$ such that

$$Q - (T \lrcorner K) \in \mathbf{Z}_{2\kappa+1}(U) = \mathbf{B}_{2\kappa+1}(U)$$

we conclude, recalling [5, 4.1.21] and 4.7, that

$$\mathbf{M}(Q) \geq \int \langle \vec{Q}, \phi \rangle d||Q|| = \alpha(Q) = \alpha(T \llcorner K) = \int_K \langle \vec{T}, \phi \rangle d||T|| = \mathbf{M}(T \llcorner K)$$

and the inequality is strict unless $\text{spt } Q \subset U \cap \{(x, y) : |x| = |y|\}$, which occurs only if $Q = T \llcorner K$.

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