Subspace Approximants and GCR Operators

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§1. Introduction and notation. In this paper we introduce the concept of subspace approximant and prove several results concerning subspace approximants. We also study some problems concerning GCR operators. All operators will be bounded linear operators on complex Hilbert spaces. We denote all operators from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{H}'$ by $\mathcal{B}(\mathcal{H}, \mathcal{H}')$, while $\mathcal{B}(\mathcal{H})$ will denote all operators from $\mathcal{H}$ to $\mathcal{H}$. We denote by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, $C^*(A)$ will denote the C^* -algebra generated by A ; that is, the smallest C^* -algebra containing A and I .

A C^* -algebra $\mathcal{A}$ is called CCR if for every irreducible representation π of $\mathcal{A}$ on a Hilbert space, and for every $A \in \mathcal{A}$, $\pi(A)$ is compact [7, 4.2.1]. A C^* -algebra $\mathcal{A}$ is called GCR if $\mathcal{A}$ has an increasing family of closed two-sided ideals $(\mathcal{I}_\rho)_{0 \leq \rho \leq \alpha}$ satisfying, $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_\alpha = \mathcal{A}$; if $\rho \leq \alpha$ is a limit ordinal, then $\mathcal{I}_\rho$ is the uniform closure of $\bigcup \{\mathcal{I}_{\rho'} : \rho' < \rho\}$ and $\mathcal{I}_{\rho+1}/\mathcal{I}_\rho$ is CCR [7, 4.3.4]. Equivalently, $\mathcal{A}$ is GCR if every irreducible representation of $\mathcal{A}$ contains a nonzero compact operator [15, 4.6.4]. An operator $A \in \mathcal{B}(\mathcal{H})$ is called GCR if $C^*(A)$ is a GCR C^* -algebra. We would like to thank W. Arveson for several stimulating conversations.

§2. Subspace approximants. We say that $F \in \mathcal{B}(\mathcal{H}')$ is a *subspace approximant* to $A \in \mathcal{B}(\mathcal{H})$ if for every $\epsilon > 0$ there is an operator $R_\epsilon \in \mathcal{B}(\mathcal{H}_\epsilon)$ and a unitary operator $U_\epsilon \in \mathcal{B}(\mathcal{H}, \mathcal{H}' \oplus \mathcal{H}_\epsilon)$ such that

$$\|A - U_\epsilon^*(F \oplus R_\epsilon)U_\epsilon\| < \epsilon.$$

Here we also allow the possibility of R_ϵ being absent in which case we require that $U_\epsilon \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and that $\|A - U_\epsilon^*FU_\epsilon\| < \epsilon$. If for all $\epsilon > 0$ there is an unitary U_ϵ with $\|A - U_\epsilon^*FU_\epsilon\| < \epsilon$, we say that F is in the closure $\bar{\mathcal{U}}_A$ of the unitary equivalence class of A . We note that $F \in \mathcal{B}(\mathcal{H}')$ is a subspace approximant for A if and only if for all $\epsilon > 0$ there is an isometry $V_\epsilon : \mathcal{H}' \rightarrow \mathcal{H}$ such that $\|V_\epsilon^*AV_\epsilon - F\| < \epsilon$ and $\|V_\epsilon V_\epsilon^*A - AV_\epsilon V_\epsilon^*\| < \epsilon$.

It is natural to suppose that the set of subspace approximants is intimately related to the structure of the operator A . We will present several results to support this supposition. We remark that similar concepts are mentioned in [8, Chapter 2, §4] and in the announcement [13], both of which we received after obtaining our results. The condition (*) in Theorem 2 of [13] is equivalent to our definition of subspace approximant. A few general remarks can be made about subspace approximants. If F is a subspace approximant of A , then so is any operator unitarily equivalent to F . If G is a subspace approximant of F and F is a subspace approximant of A , then G is a subspace approximant of A . Also if F is a subspace approximant of A , then F^* is a subspace approximant of A^* .

Proposition 1. *If F is a subspace approximant of A , then the mapping $\pi(A) = F$ induces a *-representation of $C^*(A)$.*

Proof. Let F be a subspace approximant of A and let w be a word of length n in two noncommuting variables. Then using the inequality

$$||w(B, B^*) - w(C, C^*)|| \leq ||B - C|| \sum_{k=0}^{n-1} ||B||^k ||C||^{n-1-k},$$

we have that

$$||w(A, A^*) - U_\epsilon^*(w(F, F^*) \oplus w(R_\epsilon, R_\epsilon^*))U_\epsilon||$$

goes to zero as ϵ goes to zero. Hence the same is also true for any polynomial p in two noncommuting variables, and we have

$$||p(F, F^*)|| \leq ||p(A, A^*)||.$$

Thus the mapping $\pi(A) = F$ extends from the algebra of polynomials in A and A^* to a *-representation of $C^*(A)$.

Trivial examples show that direct sums of subspace approximants need not be subspace approximants. However we think the following conjecture is natural.

Conjecture. If π is an irreducible representation of $C^*(A)$, then $\pi(A)$ is a subspace approximant of A .

We are able to establish this conjecture for certain special cases. We will need the following lemma which is essentially Lemma 3.5.7 in [7].

Lemma. *Let $\mathfrak{A}$ be a C^* -subalgebra of $\mathfrak{B}(\mathfrak{H})$, π_0 a nonzero irreducible representation of $\mathfrak{A}$ on a closed subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. Let $a_1, \dots, a_p \in \mathfrak{A}$ and $x_1, \dots, x_n \in \mathfrak{H}_0$, $\epsilon > 0$. Then there exists a representation π of $\mathfrak{A}$ on $\mathfrak{H}$ such that π is unitarily equivalent to the identity representation of $\mathfrak{A}$ and*

$$||\pi(a_i)x_i - \pi_0(a_i)x_i|| \leq \epsilon$$

for all i , $1 \leq i \leq n$, and all j , $1 \leq j \leq p$.

Proof. We may assume that $x_1 \neq 0$, that $\mathfrak{A}$ contains I , $a_1 = I$, and $\|a_i\| \leq 1$ for all j and $\|x_i\| = 1$ all i .

1) First suppose that $n = 1$. Let $\epsilon_1 > 0$ be given. Then the mapping a to $(\pi_0(a)x_1, x_1)$ is a pure state on $\mathfrak{A}$ so by [7, 3.4.2 (ii)] there exists an $x \in \mathfrak{H}$ such that

$$|(a_i^* a_i x, x) - (\pi_0(a_i^* a_i)x_1, x_1)| \leq \epsilon_1$$

for all i, j . Then

$$|(a_j x, a_i x) - (\pi_0(a_j)x_1, \pi_0(a_i)x_1)| \leq \epsilon_1.$$

But if ϵ_1 is chosen as in [7, 3.5.6], then there exists a unitary U on $\mathfrak{H}$ such that

$$\|U a_j x - \pi_0(a_j)x_1\| < \epsilon/2$$

for all j . In particular, if $j = 1$ then $\|U x - x_1\| < \epsilon/2$ and, as in Dixmier, one verifies that

$$\|U a_i U^* x_1 - \pi_0(a_i)x_1\| < \epsilon$$

for all j . We may then take π to be the representation $\pi(a) = U a U^*$.

2) Suppose n is arbitrary. As in Dixmier, there exist $b_1, \dots, b_n \in \mathfrak{A}$ such that $\|x_i - \pi_0(b_i)x_1\| < \epsilon/4$. By 1) there exists a representation π , unitarily equivalent to the identity representation, such that

$$\|\pi(a_i b_i)x_1 - \pi_0(a_i b_i)x_1\| \leq \epsilon/4$$

and $\|\pi(b_i)x_1 - \pi_0(b_i)x_1\| \leq \epsilon/4$. An easy computation then shows that $\|\pi(a_i)x_i - \pi_0(a_i)x_i\| < \epsilon$ for all i, j .

Proposition 2. *If π_0 is a finite dimensional irreducible representation of $C^*(A)$, then $\pi_0(A) = F$ is a subspace approximant of A .*

Proof. Consider $\pi_0(A) = F$ as represented on a finite dimensional subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. Let $a_1 = A$, $a_2 = A^*$, and $x_1, x_2, \dots, x_n$ be an orthonormal basis for $\mathfrak{H}_0$. Then by the Lemma for every $\epsilon > 0$ there exists a unitary operator $U_\epsilon \in \mathfrak{B}(\mathfrak{H})$ such that

$$\|U_\epsilon a_i U_\epsilon^* x_i - \pi_0(a_i)x_i\| \leq \epsilon/2n^{1/2}$$

for all i, j , or equivalently that

$$\|U_\epsilon A U_\epsilon^* x_i - F x_i\| \leq \epsilon/2n^{1/2}$$

$$\|U_\epsilon A^* U_\epsilon^* x_i - F^* x_i\| \leq \epsilon/2n^{1/2}.$$

Hence, if $P_\mathfrak{L}$ denotes the projection on a closed subspace $\mathfrak{L}$ of $\mathfrak{H}$, we have

$$\|(U_\epsilon A U_\epsilon^* - F)P_{\mathfrak{H}_0}\| \leq \epsilon/2$$

$$\|(U_\epsilon A^* U_\epsilon^* - F^*)P_{\mathfrak{H}_0}\| \leq \epsilon/2.$$

We then have

$$\begin{aligned}
& ||U_\epsilon A U_\epsilon^* - (F \oplus P_{\mathcal{H}_0 \perp} U_\epsilon A U_\epsilon^*|_{\mathcal{H}_0 \perp})|| \\
& \leq ||(U_\epsilon A U_\epsilon^* - F \oplus P_{\mathcal{H}_0 \perp} U_\epsilon A U_\epsilon^*|_{\mathcal{H}_0 \perp})P_{\mathcal{H}_0}|| \\
& + ||P_{\mathcal{H}_0}(U_\epsilon A U_\epsilon^* - F \oplus P_{\mathcal{H}_0 \perp} U_\epsilon A U_\epsilon^*|_{\mathcal{H}_0 \perp})P_{\mathcal{H}_0 \perp}|| \\
& \leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Hence $||A - U_\epsilon^*(F \oplus P_{\mathcal{H}_0 \perp} U_\epsilon A U_\epsilon^*|_{\mathcal{H}_0 \perp})U_\epsilon|| \leq \epsilon$ and F is a subspace approximant of A .

Thus the conjecture is true for operators all of whose irreducible representations are finite dimensional; *e.g.*, normals, binormals, or n -normals. We remark that the special case of one-dimensional irreducible representations (characters) has received a great deal of attention lately [16]. The conjecture can be shown to be valid in many cases where the irreducible representation lives on an infinite dimensional space. We include some results.

Proposition 3. *If A is quasinormal and π is an irreducible representation of $C^*(A)$, then $\pi(A) = F$ is a subspace approximant of A .*

Proof. If A is quasinormal (that is, A commutes with A^*A), then A is unitarily equivalent to an operator of the form $N \oplus \hat{P}$ where N is normal, P is a positive operator on $\mathcal{H}'$ and $\hat{P}$ acts on the Hilbert space of square summable sequences with entries from $\mathcal{H}'$ by $\hat{P}(x_0, x_1, x_2, \dots) = (0, Px_0, Px_1, \dots)$. The only irreducible representations of $C^*(A)$ are characters and representations determined by $\pi(A) = \lambda U_+$, where $\lambda \in \sigma(P)$ and U_+ is the unilateral shift [5]. Thus we need only establish that the latter are subspace approximants. But this follows easily from the one-dimensional case of Proposition 2. Indeed, if $\lambda \in \sigma(P)$ then $\pi(P) = \lambda$ induces a character on $C^*(P)$ so that for every $\epsilon > 0$ there is an operator $R_\epsilon \in \mathcal{B}(\mathcal{H}_\epsilon)$ and a unitary operator $U_\epsilon \in \mathcal{B}(\mathcal{H}', \mathbf{C} \oplus \mathcal{H}_\epsilon)$ such that

$$||P - V_\epsilon^*(\lambda \oplus R_\epsilon)V_\epsilon|| < \epsilon.$$

Letting $U_\epsilon = I \oplus \tilde{V}_\epsilon$, where $\tilde{B}$ denotes the inflation of B defined by $\tilde{B}(x_0, x_1, \dots) = (Bx_0, Bx_1, \dots)$, and rearranging $\sum_{n=1}^\infty \oplus (\mathbf{C} \oplus \mathcal{H}_\epsilon)$ as $(\sum_{n=1}^\infty \oplus \mathbf{C}) \oplus (\sum_{n=1}^\infty \oplus \mathcal{H}_\epsilon)$ (this rearrangement also changes the unitary U_ϵ , but we keep the same notation), we obtain

$$||(N \oplus P) - U_\epsilon^*(N \oplus (\lambda U_+ \oplus \hat{R}_\epsilon))U_\epsilon|| < \epsilon$$

or

$$||A - U_\epsilon^*(\lambda U_+ \oplus (N \oplus \hat{R}_\epsilon))U_\epsilon|| < \epsilon.$$

The conjecture is also true for operators normal modulo compacts and for power partial isometries (that is, operators all of whose powers are partial isometries). The first statement can be deduced from Proposition 2 and from

[2, Theorem 1] or [9, Theorems 1, 3]. Indeed, if A is normal modulo compacts, then A is the direct sum of a normal operator and operators A_n irreducible and normal modulo compacts. It easily follows that the only irreducible representations of $C^*(A)$ are characters and those induced by the A_n . The case of characters is covered by Proposition 2, while each A_n is a direct summand and hence a subspace approximant. The statement for power partial isometries follows from the recent characterization of the closure of the unitary equivalence classes for such operators by D. Hadwin. We include two special results. We first need a lemma.

Lemma 4. *Let $A \in \mathfrak{B}(\mathfrak{H})$ be compact, $F \in \mathfrak{B}(\mathfrak{H}')$ and suppose for all $\epsilon > 0$ there is an unitary $U_\epsilon \in \mathfrak{B}(\mathfrak{H}', \mathfrak{H})$ such that $\|A - U_\epsilon F U_\epsilon^*\| < \epsilon$. Then if $0 \neq \lambda \in \sigma(A)$, we have*

$$\dim(N(A - \lambda)) \geq \dim(N(F - \lambda)),$$

where $N(A - \lambda) = \{x: (A - \lambda)x = 0\}$.

Proof. Let $e_1, e_2, \dots, e_n$ be an orthonormal basis for $N(F - \lambda)$. Then

$$\|(A - \lambda)U_\epsilon e_i\| < \epsilon.$$

Let P be the projection onto $N(A - \lambda)$. Then since $A - \lambda$ has closed range, $A - \lambda$ is bounded below on $(I - P)(\mathfrak{H})$, so that there exists a $\beta > 0$ with

$$\beta \|(I - P)U_\epsilon e_i\| \leq \|(A - \lambda)(I - P)U_\epsilon e_i\| = \|(A - \lambda)U_\epsilon e_i\| < \epsilon.$$

Hence for each j , $(I - P)U_{1/n}e_i$ converges to zero in norm. But some subsequence of $PU_{1/n}e_i$ converges in norm; hence some subsequence of $U_{1/n}e_i$ converges in norm to a unit vector f_i . Then we have $(A - \lambda)f_i = 0$. Also $\delta_{ii} = (e_i, e_i) = (U_{1/n}e_i, U_{1/n}e_i)$; hence $(f_i, f_i) = \delta_{ii}$ and $\dim(N(A - \lambda)) \geq \dim(N(F - \lambda))$.

Proposition 5. *If A is an irreducible operator which is normal modulo compacts (that is, $AA^* - A^*A$ is compact), then*

$$\bar{\mathfrak{U}}_A = \cup \{\mathfrak{U}_{A \oplus N} : N \text{ normal, } \sigma(N) \subseteq \sigma_e(A)\}$$

where $\sigma_e(A)$ denotes the essential spectrum of A , and the normal operator N need not appear.

Proof. That $\bar{\mathfrak{U}}_A \supseteq \mathfrak{U}_{A \oplus N}$ for N normal and $\sigma(N) \subseteq \sigma_e(A)$ is Corollary 4.9 of [16]. If $F \in \bar{\mathfrak{U}}_A$ then F is a subspace approximant of A and so by Proposition 1 $\pi(A) = F$ determines a representation of $C^*(A)$. Since A is irreducible and normal modulo compacts, the only irreducible representations of $C^*(A)$ are unitarily equivalent to the identity representation or are characters determined by points of $\sigma_e(A)$. (If $\pi(AA^* - A^*A) \neq 0$, then since $\mathfrak{K} \subseteq C^*(A)$, π must be unitarily equivalent to the identity representation [7, 4.1.10], while if $\pi(AA^* - A^*A) = 0$, then π is a character.) Also $C^*(A)$ is clearly GCR, so that we

can apply §8.6 of [7], where it is shown that representations are direct integrals of irreducible representations over the spectrum $C^*(A)^\wedge$. In our case, since the set of characters is closed in $C^*(A)^\wedge$, one concludes from [7, 8.6.8 (ii)] that F is unitarily equivalent to an operator of the form

$$\left(\sum_{k=0}^n \oplus A \right) \oplus N$$

where $0 \leq n \leq \aleph_0$, N is normal with $\sigma(N) \subset \sigma_e(A)$, and we do not yet know that N or A must occur. Since $\|A - U_\epsilon^* F U_\epsilon\| < \epsilon$ for some unitary U_ϵ , we have that

$$\|(AA^* - A^*A) - U_\epsilon^*(FF^* - F^*F)U_\epsilon\| < 2\epsilon(\|A\| + \|F\|).$$

Since $AA^* - A^*A$ and $FF^* - F^*F$ are nonzero compact operators, letting $0 \neq \lambda \in \sigma(A^*A - AA^*)$, we have by Lemma 4 that

$$\begin{aligned} 0 \neq \dim(N(A^*A - AA^* - \lambda)) &= \dim(N(F^*F - FF^* - \lambda)) \\ &= (n+1) \dim(N(A^*A - AA^* - \lambda)). \end{aligned}$$

Hence A occurs once, and we have our result.

Proposition 5 generalizes part of the main result in [10]. From Proposition 5 and its proof one easily determines all subspace approximants to A . Up to unitary equivalence they are A , $A \oplus N$, or N where N is normal and $\sigma(N) \subseteq \sigma_e(A)$.

We conclude this section by presenting some results for a particular power partial isometry. Let J_n denote the usual Jordan nilpotent block on $\mathbb{C}^n$ and let $J = \sum_{n=2}^\infty \oplus J_n$ on $\sum_{n=2}^\infty \oplus \mathbb{C}^n$. Then J is a power partial isometry (that is, J^k is a partial isometry for every $k \geq 0$). It follows from [11] that every power partial isometry generates a GCR C^* -algebra. In particular $C^*(J)$ was studied in [6], and it was shown that the irreducible representations of $C^*(J)$ are precisely those given by $\pi(J) = J_n$ for some n , U_+ , U_+^* or $e^{i\theta}$ for some $0 \leq \theta < 2\pi$.

Proposition 6. *The operators U_+ and U_+^* are subspace approximants of J .*

Proof. In [6, p. 269] it was shown that J is unitarily equivalent to the operator with I , U_+^* , U_+^* , $\dots$ on the first diagonal below the main diagonal and zeroes elsewhere. Since U_+ is irreducible and normal modulo compacts, Proposition 5 implies that for every $\epsilon > 0$ there exist unitaries $V_\epsilon \in \mathcal{B}(\ell^2, \ell^2 \oplus \mathbb{C})$ such that

$$\|U_+^* - V_\epsilon^*(U_+^* \oplus 1)V_\epsilon\| < \epsilon.$$

Using the inflation $\tilde{V}_\epsilon \in \mathcal{B}(\sum_{1=1}^\infty \oplus \ell^2, \sum_{1=1}^\infty \oplus (\ell^2 \oplus \mathbb{C}))$ we conclude

$$\|J - U_\epsilon^*(J \oplus U_+)U_\epsilon\| < \epsilon,$$

where U_* is the unitary obtained from $\tilde{V}_*$ by rearranging $\sum_{i=1}^{\infty} (\ell^2 \oplus \mathbf{C})$ as $(\sum_{i=1}^{\infty} \ell^2) \oplus (\sum_{i=1}^{\infty} \mathbf{C})$. The fact that U_+^* is also a subspace approximant follows since J is unitarily equivalent to J^* . It also follows, by breaking off an identity on a big space in the above proof, that unilateral shifts of arbitrary multiplicity are subspace approximants of J .

We remark that it is not difficult using the techniques in the proof of Proposition 6 to show that

$$\bar{\mathfrak{U}}_J = \bigcup \{ \mathfrak{U}_{J \oplus (\sum_{i=1}^n \oplus_{U_+}) \oplus (\sum_{i=1}^m \oplus_{U_+^*}) \oplus U} : U_+, U_+^*, \\ U \text{ need not appear, } n, m \geq 0, U \text{ unitary} \}.$$

We can again determine all subspace approximants of J .

§3. GCR operators. In [2] H. Behnke asked whether the set $\mathfrak{G}$ of GCR operators is closed or dense. It was shown in [6] that $\mathfrak{G}$ is not closed. We now show that $\mathfrak{G}$ is not dense. In fact we will prove more. We recall some notation from [17]. Let $\mathfrak{R}_n$ denote operators having n -dimensional reducing subspaces and let $\mathfrak{F}$ denote the set of “finite operators”; that is, operators A such that $0 \in \text{closure } W(AX - XA)$ for all $X \in \mathfrak{B}(\mathfrak{H})$, where W denotes the numerical range.

Proposition 7. *The set $\mathfrak{G}$ of all GCR operators is not dense; in fact, $\mathfrak{G} \subset \bigcup_{n=1}^{\infty} \bar{\mathfrak{R}}_n \subset \mathfrak{F}$.*

Proof. Suppose that $A \in \mathfrak{G}$. Then by Corollary 2 in [4], $C^*(A)$ has nonzero finite dimensional $*$ -representations. It then follows from Proposition 2 that $A \in \bar{\mathfrak{R}}_{n_0}$ for some $n_0 < \infty$. Thus $\mathfrak{G} \subset \bigcup_{n=1}^{\infty} \bar{\mathfrak{R}}_n \subset \mathfrak{F}$. To see that $\mathfrak{G}$ is not dense we use the fact that $\mathfrak{F}$ is closed and proper [17]. It follows, in fact, that $\mathfrak{G}$ is of first category from [13].

In [1] it is shown that every GCR C^* -algebra $\mathfrak{A}$ has exactly one composition series $\{\mathfrak{g}_\rho\}_{0 \leq \rho \leq \alpha}$ with the property that $\mathfrak{g}_{\rho+1}/\mathfrak{g}_\rho$ is the largest CCR ideal in $\mathfrak{A}/\mathfrak{g}_\rho$ for every $0 \leq \rho < \alpha$. It is of interest to study the composition series for singly generated GCR C^* -algebras $C^*(A)$. Since $C^*(A)$ is a separable C^* -algebra, the composition series is countable. We will show that there exist singly generated GCR C^* -algebras whose composition series has length n for every $0 < n \leq \aleph_0$. An example of length 4 was given in [6]. The question of which CCR algebras can appear in the composition series of singly generated GCR C^* -algebras seems open.

Proposition 8. *For each $1 \leq N < +\infty$ there exists an operator A_N such that $C^*(A_N)$ is a GCR C^* -algebra with composition series $\{\mathfrak{g}_k\}_{0 \leq k \leq N}$ with $\mathfrak{g}_0 = \{0\}$, $\mathfrak{g}_N = C^*(A_N)$ and $\mathfrak{g}_{k+1}/\mathfrak{g}_k \cong \mathfrak{K}(\mathfrak{H})$ for all $k = 0, 1, 2, \dots, N-1$.*

Proof. Fix the Hilbert space $\mathfrak{H} = \ell^2$ with the usual orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Let P_n denote the projection onto the span of e_n , and let A_1 be defined

by $A_1 e_n = (1/(n+1))e_{n+1}$. Then the desired operators are A_1 , $A_2 = A_1 \otimes P_0 + I \otimes A_1$, $A_3 = A_2 \otimes P_0 + I \otimes A_1$, $\dots$, $A_N = A_{N-1} \otimes P_0 + I \otimes A_1$ on $\mathcal{H}$, ${}^{(2)}\mathcal{H}$, ${}^{(3)}\mathcal{H}$, $\dots$, ${}^{(N)}\mathcal{H}$, where I is the identity on the appropriate space, and ${}^{(i)}\mathcal{H}$ denotes the tensor product of $\mathcal{H}$ i -times.

For $1 \leq k \leq N$, let $\mathcal{K}_k = I \otimes \mathcal{K} (\otimes_{k \leq i \leq N} \mathcal{H})$ on ${}^{(N)}\mathcal{H}$. Define $\mathcal{G}_0 = (0)$, $\mathcal{G}_k = C^*(\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_k)$. Then $\{\mathcal{G}_k\}_{0 \leq k \leq N}$ is an increasing family of closed two-sided ideals in $\mathcal{G}_N$ satisfying $\mathcal{G}_{k+1}/\mathcal{G}_k \cong \mathcal{K}(\mathcal{H})$. We need only show that $\mathcal{G}_N = C^*(A_N)$. Clearly $C^*(A_N) \subseteq \mathcal{G}_N$ since $A_{N-1} \otimes P_0 \in \mathcal{G}_{N-1}$ and $I \otimes A_1 \in \mathcal{K}_N$. So that we need only establish that $\mathcal{K}_k \subseteq C^*(A_N)$ for $1 \leq k \leq N$. To do this we need two facts: (1) if A is irreducible and $T = A \otimes P_0 + I \otimes A_1$, then T is irreducible, (2) if $\|A^*A\|$ is an isolated eigenvalue of multiplicity one for A^*A , then $\|T^*T\|$ is an isolated eigenvalue of multiplicity one for T^*T . For then since A_1 is irreducible and $\|A_1^*A_1\|$ is an isolated eigenvalue of multiplicity one, it follows by induction that A_N is irreducible and $\|A_N^*A_N\|$ is an isolated eigenvalue of multiplicity one for $A_N^*A_N$. Then $C^*(A_N)$ contains the projection onto the associated eigenvector. Thus $C^*(A_N)$ is irreducible and contains a nonzero compact operator, and so $C^*(A_N) \supseteq \mathcal{K}_1 = \mathcal{K}({}^{(N)}\mathcal{H})$ by [7, 4.1.10]. Hence $C^*(A_N)$ also contains $A_N - A_1 \otimes P_0 \otimes \dots \otimes P_0 = I \otimes A_{N-1}$ and $C^*(A_N)$ contains $\mathcal{K}_2 = I \otimes \mathcal{K}({}^{(N-1)}\mathcal{H})$. Continuing, we see that $C^*(A_N)$ contains every $\mathcal{K}_k$ so that $C^*(A_N) = \mathcal{G}_N$. In order to prove fact (1), suppose Q is a projection that commutes with T . Then Q commutes with $T^*T = (A^*A + I) \otimes P_0 + \sum_{i=1}^{\infty} (1/(n+1))^2 I \otimes P_n$. Since the spectra of any two of the operators $(A^*A + I)$, $(1/n^2)I$, $n = 2, 3, \dots$, are disjoint, we conclude that $Q = \sum (Q_n \otimes P_n)$ with Q_n projections [14]. Consideration of the equation $QT = TQ$ then yields $AQ_0 = Q_0A$ and $Q_0 = Q_1 = Q_2 = \dots$. If A is irreducible, then $Q_0 = 0$ or I , so that $Q = 0$ or I and T is irreducible. Fact (2) is obvious.

Proposition 9. *There exists a singly generated GCR C^* -algebra with an infinite composition series.*

Proof. Let $\mathfrak{A}$ be the C^* -algebra obtained by adjoining the identity to the $C^*(\infty)$ -sum of the C^* -algebras $C^*(A_N)$, $N = 1, 2, \dots$ constructed in Proposition 8. Then $\mathfrak{A}$ is a singly generated GCR C^* -algebra [12], and must have an infinite composition series.

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