On the Classification of Reflective Submanifolds of Riemannian Symmetric Spaces

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Communicated by S. S. CHERN

Introduction. To generalize the classical Schwartz reflection invariant of minimal surfaces in E^3 to minimal submanifolds in a Riemannian symmetric space, the notion of a reflective submanifold was introduced in [6]. It is a connected component of the fixed point set of an involutive isometry of the ambient space and is therefore a geodesic submanifold. Using *ad hoc* methods, reflective submanifolds in spaces of constant sectional curvature and spaces of constant holomorphic sectional curvature were classified in [6]. It is the purpose of the present paper to give a classification of all the reflective submanifolds in an arbitrary simply connected Riemannian symmetric space. As a result of such classification, we have a list of geodesic submanifolds in every irreducible symmetric space. As far as the author knows, this is the first time such a comprehensive list of geodesic submanifolds in every irreducible symmetric space has been provided.

We would like to acknowledge our indebtedness to M. Berger, since our present method of classification relies strongly on his results on the classification of affine symmetric spaces [1]. The author also wishes to thank T. Nagano for many helpful conversations.

For general notations and terminologies in differential geometry as well as in symmetric spaces we will follow [5] closely. As in [5], we will denote Lie groups by capital Latin letters and their Lie algebras by the corresponding lower case German letters.

1. Elementary properties of reflective submanifolds. In this section we will recall some elementary properties of reflective submanifolds in symmetric spaces.

Let M be a complete Riemannian manifold. An imbedded submanifold B of M is said to be *reflective* if B is complete with respect to the induced metric and it is a connected component of the fixed point set of an involutive isometry of M . (To avoid difficulties with connectivity, we require B to be connected

in our present definition.) Using a well known fact about the fixed point set of isometries [5 II P. 61], we can conclude that *every reflective submanifold is totally geodesic*. Other geometric significance of reflective submanifolds can be found in [6].

For the remaining of this paper $M = G/H$ will denote a simply connected global Riemannian symmetric space. Here G is the largest connected group of isometries of M and H the isotropy subgroup of G at a point o (called the origin) of M . In this case, we have the following infinitesimal version of a reflective submanifold of M . Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be the orthogonal symmetric Lie algebra associated with $M = G/H$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be its canonical decomposition. Upon identifying $\mathfrak{m}$ with $T_o(M)$, there is an $ad(H)$ invariant inner product on $\mathfrak{m}$. A linear subspace $\mathfrak{b}$ of $\mathfrak{m}$ is said to be *reflective* if both $\mathfrak{b}$ and its orthogonal complement $\mathfrak{b}^\perp$ are Lie triple systems such that

$$[[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}] \subset \mathfrak{b}^\perp, \quad [[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}^\perp] \subset \mathfrak{b}.$$

It follows easily from the definition that $\mathfrak{b}^\perp$ is also a reflective subspace and that

$$[[\mathfrak{b}, \mathfrak{b}], \mathfrak{b}^\perp] \subset \mathfrak{b}^\perp, \quad [[\mathfrak{b}^\perp, \mathfrak{b}^\perp], \mathfrak{b}] \subset \mathfrak{b}.$$

$\{\mathfrak{b}, \mathfrak{b}^\perp\}$ is called a *complementary pair of reflective subspaces*.

Theorem 1.1. *Let $M = G/H$ be a simply connected Riemannian symmetric space. There is a one-to-one correspondence between the set of reflective linear subspaces $\mathfrak{b}$ of $\mathfrak{m}$ and the set of reflective submanifolds B through the origin o of M , the correspondence being given by $\mathfrak{b} = T_o(B) \subset T_o(M)$ under the identification $\mathfrak{m} = T_o(M)$.*

One can find a proof of this theorem in [6] (Theorem 3). For convenience of references, we will however note the following corollaries of the proof of Theorem 1.1. We will use the same notations as above.

Corollary 1.2. *Associated with every reflective subspace $\mathfrak{b}$ of $\mathfrak{m}$, there is an involutive automorphism ρ of the orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ such that $\mathfrak{b}$ is the eigenspace for $+1$ of ρ restricted to $\mathfrak{m}$. Conversely if ρ is an involutive isomorphism of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, the eigenspace for $+1$ (in fact, also for -1) of ρ restricted to $\mathfrak{m}$ is a reflective subspace of $\mathfrak{m}$. Furthermore if $\rho^\perp$ is the involutive automorphism associated with $\mathfrak{b}^\perp$, then we have*

$$\rho^\perp \rho = \rho \rho^\perp = \sigma.$$

Corollary 1.3. *If $(\mathfrak{g}, \mathfrak{h}, \sigma) = \sum_k (\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$ is a direct sum decomposition as orthogonal symmetric Lie algebras, where $(\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$ is either Euclidean or irreducible and $\mathfrak{g}_k = \mathfrak{h}_k + \mathfrak{m}_k$ is the canonical decomposition of $(\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$. Then $\mathfrak{b}_k = \mathfrak{b} \cap \mathfrak{m}_k$ is a reflective subspace of $\mathfrak{m}_k$.*

We will also call the pair of reflective submanifolds through o defined by a complementary pair of reflective subspaces a *complementary pair of reflective submanifolds*.

2. Preliminaries on the classification. In this section we will prove some

lemmas which we will need in carrying out the classification of reflective submanifolds.

By Theorem 1.1 and Corollary 1.3, it is clear that to classify all the complete reflective submanifolds of an arbitrary simply connected Riemannian symmetric space, it will be sufficient to do this for Euclidean spaces and for simply connected irreducible Riemannian symmetric spaces. However, every complete geodesic submanifold in an n -dimensional Euclidean space, in fact, in any simply connected space of constant curvature is reflective by Theorem 4 in [6]. For the purpose of classification in an arbitrary irreducible simply connected Riemannian symmetric space, we will introduce the notion of reflective subalgebras.

Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an effective orthogonal symmetric Lie algebra. An effective orthogonal subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is said to be a *reflective subalgebra*, if there exists an involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ with fixed point set $\mathfrak{g}^\rho$ such that $(\mathfrak{k}, \mathfrak{q}, \sigma)$ is the maximal effective orthogonal subalgebra contained in $\mathfrak{g}^\rho$.

Theorem 2.1. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an irreducible orthogonal symmetric Lie algebra with canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. There is a one-to-one correspondence between the reflective subspaces of $\mathfrak{m}$ and the reflective subalgebras of $(\mathfrak{g}, \mathfrak{h}, \sigma)$.*

Proof. Let $(\mathfrak{k}, \mathfrak{q}, \sigma)$ be a reflective subalgebra of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ defined by an involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ and with canonical decomposition $\mathfrak{k} = \mathfrak{q} + \mathfrak{b}$. Then obviously $\mathfrak{b}$ is the fixed point set of ρ restricted to $\mathfrak{m}$. Therefore $\mathfrak{b}$ is reflective. On the other hand, let $\mathfrak{b}$ be a reflective subspace of $\mathfrak{m}$ defined by an involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. As usual, we denote by $\mathfrak{b}^\perp$ the orthogonal complement of $\mathfrak{b}$ in $\mathfrak{m}$ and $\mathfrak{g}^\rho$ the fixed point set of ρ . Note that $\rho = -id$ on $\mathfrak{b}^\perp$. The irreducibility of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ implies that

$$\mathfrak{g}^\rho = [\mathfrak{b}^\perp, \mathfrak{b}^\perp] + [\mathfrak{b}, \mathfrak{b}] + \mathfrak{b}.$$

Let α be the kernel of the adjoint representation of $[\mathfrak{b}^\perp, \mathfrak{b}^\perp] + [\mathfrak{b}, \mathfrak{b}]$ on $\mathfrak{b}$. Then we have $[\mathfrak{b}^\perp, \mathfrak{b}^\perp] + [\mathfrak{b}, \mathfrak{b}] = \mathfrak{q} \oplus \alpha$ (direct sum decomposition as Lie algebras). Now if we put $\mathfrak{k} = \mathfrak{q} + \mathfrak{b}$, it is easy to see that $(\mathfrak{k}, \mathfrak{q}, \sigma)$ is a reflective subalgebra of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. **Q.E.D.**

Remark 2.2. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be the symmetric Lie algebra of a simply connected irreducible Riemannian symmetric space M . It is clear from the above proof that the reflective subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ which is associated with a reflective subspace $\mathfrak{b}$ is precisely the orthogonal symmetric Lie algebra of the geodesic submanifold B of M corresponding to $\mathfrak{b}$. In fact, it is by using $(\mathfrak{k}, \mathfrak{q}, \sigma)$ that we will be able to determine the local structure of B as a symmetric space.*

Remark 2.3. *In general the reflective subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ is not irreducible. We have $(\mathfrak{k}, \mathfrak{q}, \sigma) = (\mathfrak{k}_0, \mathfrak{q}_0, \sigma) + (\mathfrak{k}_1, \mathfrak{q}_1, \sigma)$ where $(\mathfrak{k}_0, \mathfrak{q}_0, \sigma)$ is of Euclidean type and $(\mathfrak{k}_1, \mathfrak{q}_1, \sigma)$ (not irreducible in general) is of compact type or non compact type depending on whether $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of compact type or non-compact type.*

Remark 2.4. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be as above and $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$ be its dual. Here $\mathfrak{g}^* = \mathfrak{h} + i\mathfrak{m} \subset \mathfrak{g}^c$. Obviously $\mathfrak{b} \subset \mathfrak{m}$ is a reflective subspace if and only if $i\mathfrak{b} \subset i\mathfrak{m}$*

is a reflective subspace of $\mathfrak{im}$. Using the proof of Theorem 2.1, it is easy to see that there is a 1 - 1 correspondence between the reflective subalgebras of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ and $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$ such that a reflective subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is mapped onto the reflective subalgebra $(\mathfrak{k}^*, \mathfrak{q}, \sigma^*)$ (dual of $(\mathfrak{k}, \mathfrak{q}, \sigma)$) of $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$.

Let $M = G/H$ be a simply connected symmetric space as in §1 but with M irreducible. It is clear that by classifying all the reflective subalgebras of the orthogonal symmetric algebras of M , we will have a local classification of the reflective submanifolds of M . When M is non-compact, every reflective submanifold B of M is also globally symmetric and simply connected, since its exponential map is a diffeomorphism. It follows that B is in general the product of a Euclidean space and a globally symmetric space of the non-compact type. Hence the local informations of B are sufficient to determine its global structure. However, when M is of compact type, the global structure of a reflective submanifold B depends on the topology of the largest connected subgroup of G which acts transitively and almost effectively on B . We collect in the rest of this section some facts which will be useful in determining the global structure of B in general.

Notations. Let G be a Lie group and σ be an involutive automorphism of G . In the remaining of this paper, G_0 will denote the component of identity in G and G^σ will denote the fixed point set of σ .

Let $M = G/H$ be a simply connected irreducible Riemannian symmetric space as above. Note that this implies that $H = H_0$, that is, H is connected. Let B be a reflective submanifold of M through the origin o of M which is contained in the fixed point set of an involutive isometry ρ of M . ρ induces an involutive automorphism on G , which will again be denoted by ρ , through conjugation with ρ . Let K denote the largest subgroup of G which acts transitively and almost effectively on B . We have the following corollary of the proof of Theorem 2.1.

Corollary 2.5. K_0 is contained in $(G^\rho)_0$.

In all the cases we will consider, $(G^\rho)_0$ will be known. Using Corollary 2.5 and the Lie algebra of K , we will be able to determine the global structure of K_0 . For the remainder of this section we will also assume M to be compact hence all the associated Lie groups are compact.

Lemma 2.6. We have $B = K_0/Q$ where $Q = H_0 \cap (G^\rho)_0$ and Q is connected.

Proof. Since B is connected, K_0 acts transitively on B . Therefore $B = K_0/Q$, where Q is the closed subgroup of K_0 leaving the origin fixed. It follows that $Q = H_0 \cap (G^\rho)_0$. H_0 and $(G^\rho)_0$ being compact, the exponential map is onto in each case. For $k \in H_0 \cap (G^\rho)_0$, $k = \exp X$ for some $X \in \mathfrak{h} \cap \mathfrak{g}^\rho$. It follows that $\exp tX \in H_0 \cap (G^\rho)_0$ for $0 \leq t \leq 1$. Therefore Q is connected. **Q.E.D.**

Lemma 2.7. Let $\bar{F}$ be a connected Lie group and N a finite subgroup in the center of $\bar{F}$. Let $F = \bar{F}/N$ and denote by $\pi: \bar{F} \rightarrow F$ the natural projection map. Let σ and $\bar{\sigma}$ be involutive automorphisms of F and $\bar{F}$ respectively such that $\pi \circ \bar{\sigma} =$

$\sigma \circ \pi$. We put $\Gamma = N/((\bar{F}^\sigma)_0 \cap N)$ and let Γ act on $\bar{F}/(\bar{F}^\sigma)_0$ by left multiplication. Then we have $(\bar{F}/(\bar{F}^\sigma)_0)/\Gamma$ and $F/(F^\sigma)_0$ are diffeomorphic.

Proof. By considering suitable differentiable action of $\bar{F}$ on $F/(F^\sigma)_0$ analogous to that in the proof of the third isomorphism theorem in the theory of finite groups (c.f. p. 27, [8]) one can see that $\bar{F}/\pi^{-1}((F^\sigma)_0)$ and $F/(F^\sigma)_0$ are diffeomorphic. $(\bar{F}^\sigma)_0$ is the identity component of the subgroup $\pi^{-1}(F^\sigma)$ of $\bar{F}$. It is easy to see that we have $\pi^{-1}((F^\sigma)_0) = N(\bar{F}^\sigma)_0$. Let N act on $\bar{F}/(\bar{F}^\sigma)_0$ by left multiplication. Then we have that $(\bar{F}/(\bar{F}^\sigma)_0)/\Gamma$ is diffeomorphic to $\bar{F}/N(\bar{F}^\sigma)_0$.

Q.E.D.

Remarks 2.8. If $(\bar{F}^\sigma)_0$ is a compact Lie group, it follows from Theorem 8.3.11 in [9] that Γ can actually be realized as a discrete subgroup of the full group of isometries of the symmetric space $\bar{F}/(\bar{F}^\sigma)_0$.

3. Classification in symmetric spaces of type II and IV. Although the classification in symmetric spaces of type II and IV can also be carried out by the more general procedure to be described in the next section, however, we will do it in the present section by a more direct and elementary method. This method also enables us to pair off complementary reflective subspaces immediately. We begin with the following proposition.

Proposition 3.1. Let $\mathfrak{g}$ be a simple Lie algebra and σ be the involutive automorphism of $\mathfrak{g} \times \mathfrak{g}$ defined by $\sigma(x, y) = (y, x)$. If ρ is an involutive automorphism of $\mathfrak{g} \times \mathfrak{g}$ which commutes with σ , then there exists an involutive automorphism μ of $\mathfrak{g}$ such that either

$$\rho(x, y) = (\mu(x), \mu(y)) \text{ or } \rho(x, y) = (\mu(y), \mu(x))$$

for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Proof. It is well known that any involutive automorphism ρ of $\mathfrak{g} \times \mathfrak{g}$ is one of the following:

- (i) $\rho(x, y) = (\mu_1(x), \mu_2(y))$ or
- (ii) $\rho(x, y) = (\mu_1(y), \mu_2(x))$

for some involutive automorphisms μ_1 and μ_2 of $\mathfrak{g}$ (see for example [4 II p. 230]). If ρ commutes with σ , then we must have $\mu_1 = \mu_2$.

Q.E.D.

For the classical groups we will use the notations as in [4]. We will also use $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$ and $\mathbf{G}_2$ to denote the simply connected compact real Lie groups whose Lie algebras are of type E_6 , E_7 , E_8 , F_4 and G_2 respectively.

Lemma 3.2. For $n > 4$ the fixed point set of an involutive automorphism $\bar{\sigma}$ of $\mathbf{Spin}(n)$ is isomorphic to one of the following: $\mathbf{SU}(n) \times \mathbf{T}^1 (= \mathbf{U}(n))$, $\{\mathbf{Spin}(n-p) \times \mathbf{Spin}(p)\}/\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2)$, $p \geq 2$, and $\mathbf{Spin}(n-1)$.

Proof. First of all let us note that $\mathbf{Spin}(n)^\bar{\sigma}$ is connected by Theorem 3.4 in [2]. $\mathbf{Spin}(n)$ is the universal covering group of $\mathbf{SO}(n)$. Denote its covering map by π . Using the well-known fact that local homomorphism from a simply

connected Lie group to an arbitrary Lie group can be extended to a unique globally defined homomorphism [7, Theorem 63], the following fact is easy to see. Every involutive automorphism σ of $\mathbf{SO}(n)$ is covered by a unique involutive automorphism $\tilde{\sigma}$ of $\mathbf{Spin}(n)$, that is, $\sigma \circ \pi = \pi \circ \tilde{\sigma}$. Since every involutive automorphism of $\mathfrak{so}(n)$ can also be extended to an involutive automorphism of $\mathbf{SO}(n)$, the converse of the previous statement also follows easily. By Lemma 2.6 we have that $\mathbf{Spin}(n)/\pi^{-1}((\mathbf{SO}(n)^\sigma)_0)$ is a covering of $\mathbf{SO}(n)/(\mathbf{SO}(n)^\sigma)_0$. However, the latter space is simply connected. Therefore $\pi^{-1}((\mathbf{SO}(n)^\sigma)_0) = \mathbf{Spin}(n)^\tilde{\sigma}$. We have also $\mathbf{Spin}(n)/\mathbf{Spin}(n)^\tilde{\sigma} = \mathbf{SO}(n)/(\mathbf{SO}(n)^\sigma)_0$. Restricting π , we have therefore $\pi: \mathbf{Spin}(n)^\tilde{\sigma} \rightarrow \mathbf{SO}(n)^\sigma$ as a covering homomorphism with kernel $\mathbf{Z}_2$. It is well known that up to isomorphisms $((\mathbf{SO}(n)^\sigma)_0)$ is either $\mathbf{U}(n) = (\mathbf{SU}(n) \times \mathbf{T}^1)$ or $\mathbf{SO}(n-p) \times \mathbf{SO}(p)$.

If $(\mathbf{SO}(n)^\sigma)_0 = \mathbf{U}(n)$, then $\mathbf{Spin}(n)^\tilde{\sigma}$, being compact, must be $\mathbf{SU}(n) \times \mathbf{T}^1$. In fact π restricting to $\mathbf{SU}(n) \times \{1\}$ is the identity map and π restricting to $\{I_n\} \times \mathbf{T}^1$ is a 2-fold covering. Next we will treat the case when $(\mathbf{SO}(n)^\sigma)_0 = \mathbf{SO}(n-p) \times \mathbf{SO}(p)$, $n-p, p \geq 3$. Now since $\pi_1(\mathbf{SO}(n)/\mathbf{SO}(n-p)) = \{0\}$ for $p \geq 1$, the inclusion map $i_1: \mathbf{SO}(n-p) \rightarrow \mathbf{SO}(n)$ induces an isomorphism on the fundamental groups. Since $\pi_1(\mathbf{SO}(n-p) \times \mathbf{SO}(p)) = \mathbf{Z}_2 \times \mathbf{Z}_2$ and $\pi_1(\mathbf{SO}(n)) = \mathbf{Z}_2$, if $i_2: (\mathbf{SO}(n-p)) \times \mathbf{SO}(p) \hookrightarrow \mathbf{SO}(n)$ is the inclusion map, $(i_2)_*$ restricted to the diagonal of $\mathbf{Z}_2 \times \mathbf{Z}_2$ must be trivial. In fact, $\ker(i_2)_* = \Delta(\mathbf{Z}_2 \times \mathbf{Z}_2)$. Let $\pi_1(\mathbf{SO}(n-p) \times \mathbf{SO}(p))$ and $\pi_1(\mathbf{SO}(n))$ acts on $\mathbf{Spin}(n)^\tilde{\sigma}$ and $\mathbf{Spin}(n)$ as usual. Let H be the subgroup of $\pi_1(\mathbf{SO}(n-p) \times \mathbf{SO}(p))$ which fixes the base point of $\mathbf{Spin}(n)^\tilde{\sigma}$. The images of H in $\pi_1(\mathbf{SO}(n))$ must also fix the base point of $\mathbf{Spin}(n)$. Hence $H \subset \ker(i_2)_*$. H is not trivial so it must be equal to $\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2)$. This determines $\mathbf{Spin}(n)^\tilde{\sigma}$ as a covering group of $\mathbf{SO}(n-p) \times \mathbf{SO}(p)$. By a moment of inspection, one can conclude that

$$\mathbf{Spin}(n)^\tilde{\sigma} = (\mathbf{Spin}(n-p) \times \mathbf{Spin}(p))/\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2).$$

When $(\mathbf{SO}(n)^\sigma)_0 = \mathbf{SO}(n-2) \times \mathbf{SO}(2)$, we have $\pi_1(\mathbf{SO}(n-2) \times \mathbf{SO}(2)) = \mathbf{Z}_2 \times \mathbf{Z}$. Let $i: \mathbf{SO}(n-2) \times \mathbf{SO}(2) \rightarrow \mathbf{SO}(n)$ be the inclusion map. Then we have $i_*: \mathbf{Z}_2 \times \{0\} \rightarrow \mathbf{Z}_2$ is an isomorphism. Since $\pi_1(\mathbf{SO}(3)/\mathbf{SO}(2)) = \{0\}$, we also have that $i_*: \{0\} \times \mathbf{Z} \rightarrow \mathbf{Z}_2$ is a surjective homomorphism. It follows readily that $\ker i_* = \{(0, 2m), (1, 2m+1); m \in \mathbf{Z}\}$. Let H be the subgroup of $\pi_1(\mathbf{SO}(n-2) \times \mathbf{SO}(2))$ which fixes the base point of $\mathbf{Spin}(n)^\tilde{\sigma}$. $i_*(H)$ must also fix the base point of $\mathbf{Spin}(n)$. Hence $H \subset \ker i_*$. Using the facts that $H = \pi_1(\mathbf{Spin}(n)^\sigma)$ and $\mathbf{Spin}(n)^\tilde{\sigma}$ is a 2-fold covering space of $\mathbf{SO}(n-2) \times \mathbf{SO}(2)$, we can readily conclude that $H = \ker i_*$ and $\mathbf{Spin}(n)^\tilde{\sigma} = \{\mathbf{Spin}(n-2) \times \mathbf{Spin}(2)\}/\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

Finally when $(\mathbf{SO}(n)^\sigma)_0 = \mathbf{SO}(n-1)$ obviously we must have $\mathbf{Spin}(n)^\tilde{\sigma} = \mathbf{Spin}(n-1)$. Q.E.D.

We will only give the details of the classifications of reflective submanifolds for symmetric spaces of type II. Using Remark 2.4 and the fact that reflective submanifolds of an irreducible symmetric space of type IV are simply connected, it is then straightforward to write down the reflective submanifolds

in irreducible symmetric spaces of type IV. First of all we will remark on some notations to be used. Let G_1 and G_2 be two compact Lie groups and A a finite abelian group. Let i_1, i_2 be injective homomorphisms of A into the centers of G_1 and G_2 respectively and let $N = \{(i_1(a), i_2(a)) \mid a \in A\}$. The quotient group $(G_1 \times G_2)/N$ is called an *almost product (or local product)* of G_1 and G_2 . Note that G_1 and G_2 can be considered as subgroups of $(G_1 \times G_2)/N$ in a natural way such that $G_1 \cap G_2$ is isomorphic to A . When no confusion is likely, we will simply denote the almost product defined above by $(G_1 \times G_2)/A$.

Theorem 3.3. *The reflective submanifolds of irreducible simply connected Riemannian symmetric spaces of type II are given up to isometries as below.*

Ambient space	Reflective submanifolds through the identity
$\mathbf{SU}(n)$ $n \geq 2$	$(1_r) \mathbf{S}(\mathbf{U}(r) \times \mathbf{U}(n-r)); (2_r) \mathbf{SU}(n)/\mathbf{S}((\mathbf{U}(r) \times \mathbf{U}(n-r)));$ $(3) \mathbf{SO}(n); (4) \mathbf{SU}(n)/\mathbf{SO}(n); ((5) \text{ and } (6) \text{ for } n \text{ even only})$ $(5) \mathbf{Sp}(n/2); (6) \mathbf{SU}(n)/\mathbf{Sp}(n/2).$ <i>Complementary pairs are $\{(1_r), (2_r)\}, \{(3), (4)\}$ and $\{(5), (6)\}$.</i>
$\mathbf{Spin}(n)$	$(1_1) \mathbf{Spin}(n-1);$ $(1_r) \{\mathbf{Spin}(r-r) \times \mathbf{Spin}(r)\}/\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2), r \geq 2;$ $(2_r) \mathbf{SO}(n)/\mathbf{SO}(n-r) \times \mathbf{SO}(r); ((3) \text{ and } (4) \text{ for } n \text{ even only})$ $(3) \mathbf{U}(n/2); (4) \mathbf{SO}(n)/\mathbf{U}(n/2).$ <i>Complementary pairs are $\{(1_r), (2_r)\}$ and $\{(3), (4)\}$.</i>
$\mathbf{E}_6$	$(1) \{[\mathbf{SU}(6)/\mathbf{Z}_3] \times \mathbf{SU}(2)\}/\mathbf{Z}_2; (2) \mathbf{E}_6/[(\mathbf{SU}(6)/\mathbf{Z}_3) \times \mathbf{SU}(2)]/\mathbf{Z}_2;$ $(3) \{\mathbf{SO}(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2; (4) \mathbf{E}_6/[\{\mathbf{SO}(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2];$ $(5) \mathbf{F}_4; (6) \mathbf{E}_6/\mathbf{F}_4; (7) \mathbf{Sp}(4)/\mathbf{Z}_2;$ $(8) \mathbf{E}_6/[\mathbf{Sp}(4)/\mathbf{Z}_2].$ <i>Complementary pairs are $\{(1), (2)\}, \{(3), (4)\}, \{(5), (6)\}$ and $\{(7), (8)\}$.</i>
$\mathbf{E}_7$	$(1) \{\mathbf{SO}(12) \times \mathbf{SU}(2)\}/\mathbf{Z}_2; (2) \mathbf{E}_7/[(\mathbf{SO}(12) \times \mathbf{SU}(2)]/\mathbf{Z}_2;$ $(3) \mathbf{SU}(8)/\mathbf{Z}_2; (4) \mathbf{E}_7/[\mathbf{SU}(8)/\mathbf{Z}_2];$ $(5) \{\mathbf{E}_6 \times \mathbf{SO}(2)\}/\mathbf{Z}_3; (6) \mathbf{E}_7/[(\mathbf{E}_6 \times \mathbf{SO}(2)]/\mathbf{Z}_3.$ <i>Complementary pairs are $\{(1), (2)\}, \{(3), (4)\}$ and $\{(5), (6)\}$.</i>
$\mathbf{E}_8$	$(1) \mathbf{SO}(16); (2) \mathbf{E}_8/\mathbf{SO}(16); (3) \{\mathbf{E}_7 \times \mathbf{SU}(2)\}/\mathbf{Z}_2;$ $(4) \mathbf{E}_8/[(\mathbf{E}_7 \times \mathbf{SU}(2)]/\mathbf{Z}_2.$ <i>Complementary pairs are $\{(1), (2)\}$ and $\{(3), (4)\}$.</i>
$\mathbf{F}_4$	$(1) \mathbf{Spin}(9); (2) \mathbf{F}_4/\mathbf{Spin}(9); (3) \{\mathbf{Sp}(3) \times \mathbf{Sp}(1)\}/\mathbf{Z}_2;$ $(4) \mathbf{F}_4/[\mathbf{Sp}(3) \times \mathbf{Sp}(1)]/\mathbf{Z}_2.$ <i>Complementary pairs are $\{(1), (2)\}$ and $\{(3), (4)\}$.</i>
$\mathbf{G}_2$	$(1) \mathbf{SO}(4); (2) \mathbf{G}_2/\mathbf{SO}(4).$ <i>Complementary pair is $\{(1), (2)\}$.</i>

Proof. Let G be a connected compact simple Lie group. As a Riemannian symmetric space its largest connected group of isometries is $G \times G$. The associated involutive automorphism is defined by the mapping $\sigma: (g_1, g_2) \rightarrow (g_2, g_1)$ (cf. [4], p. 188). Its canonical decomposition is $\mathfrak{g} \times \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \{(x, x) \mid x \in \mathfrak{g}\}$ and $\mathfrak{m} = \{(y, -y) \mid y \in \mathfrak{g}\}$. Let μ be an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{q}$ and $\mathfrak{b}$ denote the eigenspaces of μ for $+1$ and -1 respectively. Let $\rho = \mu \times \mu$ and $\bar{\rho} = (\mu \times \mu) \circ \sigma$. Then ρ and $\bar{\rho}$ are involutive isomorphisms of $\mathfrak{g} \times \mathfrak{g}$ which commute with σ . In fact, by Proposition 3.1, all involutive automorphisms of $\mathfrak{g} \times \mathfrak{g}$ which commute with σ can be obtained this way. Since $\rho\bar{\rho} = \sigma$ the reflective subspaces of $\mathfrak{m}$ defined by ρ and $\bar{\rho}$ are complementary to each other. They are respectively $\mathfrak{Q} = \{(x, -x) \mid x \in \mathfrak{q}\}$ and $\mathfrak{B} = \{(y, -y) \mid y \in \mathfrak{b}\}$. Now computing the reflective subalgebras corresponding to $\mathfrak{Q}$ and $\mathfrak{B}$ as described in Theorem 2.1, they are found to be $(\mathfrak{g} \times \mathfrak{g}, \Delta(\mathfrak{g} \times \mathfrak{g}), \sigma)$ and (isomorphic to) $(\mathfrak{g}, \mathfrak{q}, \mu)$ respectively. These give the local structure of the reflective submanifolds defined by $\mathfrak{Q}$ and $\mathfrak{B}$. Let Q be the fixed point set of the involutive automorphism μ of G . Then the fixed point set of the involutive automorphisms of ρ and $\bar{\rho}$ of $G \times G$ are respectively $Q \times Q$ and (isomorphic to) G . Using Lemma 2.6, the corresponding reflective submanifolds are Q_0 and G/Q_0 respectively. Theorem 3.3 now follows directly from the list of E. Cartan on compact symmetric spaces [9, p. 289] with the exception of $\mathbf{Spin}(n)$ and the almost products involved in the exceptional Lie groups. The required computations for $\mathbf{Spin}(n)$, have been taken care of by Lemma 3.2. The more detailed informations about the almost products also follow readily from a remark on p. 285 in [9]. Q.E.D.

The reflective submanifolds in Theorem 3.3 were already known to E. Cartan as totally geodesic submanifolds in [3].

4. Classification in symmetric spaces of type I and III. In this section we will describe a method which can be used to classify reflective submanifolds up to isometries in an arbitrary simply connected irreducible symmetric space and apply it to symmetric spaces of type I and III.

Let $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$ be an irreducible orthogonal symmetric Lie algebra of the compact type and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be its noncompact dual. Reflective subalgebras are preserved under duality. Therefore to classify the reflective subalgebras in $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$ and $(\mathfrak{g}, \mathfrak{h}, \sigma)$, it will be sufficient to classify such for any one of them. We will now describe a method to carry out the classification in the noncompact case.

The first step is to classify all involutive automorphisms $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Using Lemma 10.2 in [1], this is the same as to classify all involutive automorphism of the real noncompact simple Lie algebra $\mathfrak{g}$. It has also been proved in [1] that an involutive automorphism ρ of $\mathfrak{g}$ is determined up to automorphisms of $\mathfrak{g}$ by its fixed point set $\mathfrak{g}^\rho$. All the possibilities for $\mathfrak{g}^\rho$ for a given simple real Lie algebra $\mathfrak{g}$ were given in Table II in [1]. From the structure of $\mathfrak{g}^\rho$ one can then determine the structure of the reflective subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ determined by ρ .

This depends on the fact proved in [1] that $\mathfrak{g}^\rho$ is either semisimple or it has a center of dimension one. In all cases, we have $\mathfrak{g}^\rho = \mathfrak{k} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{h}$. If $\mathfrak{g}^\rho$ is semisimple, the reflective subalgebra $(\mathfrak{k}, \mathfrak{q}, \sigma)$ must be noncompact by Remark 2.3. In fact $\mathfrak{k}$ is the direct sum of all the noncompact components of $\mathfrak{g}^\rho$. When $\mathfrak{g}^\rho$ is not semisimple, it has also been determined in [1] whether the center $\mathfrak{z}$ of $\mathfrak{g}^\rho$ lies in $\mathfrak{a}$ or $\mathfrak{k}$ in each case. These facts are also indicated in Table II of [1]. (See also p. 115 in [1] for notations.) From this we can conclude that $\mathfrak{k}$ is either $\mathfrak{k}_-$ or $\mathfrak{k}_- \oplus \mathfrak{z}$ respectively, where $\mathfrak{k}_-$ is the direct sum of the noncompact components of $\mathfrak{g}^\rho$.

It is well-known that the noncompact orthogonal symmetric Lie algebra $(\mathfrak{k}_-, \mathfrak{q}, \sigma)$ is completely determined once $\mathfrak{k}_-$ is known. Therefore by referring to Table II in [1], we can easily obtain a complete classification of all reflective subalgebras of every noncompact orthogonal symmetric Lie algebra. Using the fact that every reflective submanifold of a noncompact Riemannian (global) symmetric space is also simply connected, we also have a classification of all the reflective submanifolds of Riemannian symmetric spaces of type III.

By duality, we can also obtain a classification of all the reflective subalgebras of every compact irreducible orthogonal symmetric Lie algebra. However the structures of the reflective submanifolds defined by them do not follow immediately. Let $M = G/H$ be a simply connected irreducible Riemannian symmetric space, with G the largest connected group of isometries of M . Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be the orthogonal symmetric Lie algebra associated with M . If $(\mathfrak{k}, \mathfrak{q}, \sigma)$ is a reflective subalgebra of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, it determines a reflective submanifold B of M . Let ρ denote the involutive isometry of M which determines B as well as the involutive automorphisms it induces on G and $\mathfrak{g}$. Using the structure of $\mathfrak{g}^\rho$ and consulting the table in Theorem 3.3, if necessary, we can determine $(G^\rho)_0$. Let K be the subgroup of G leaving B invariant. Then using the structure of $\mathfrak{k}$ and Corollary 2.5, we can determine K_0 . If K_0 is a direct product of Lie groups, using Lemma 2.6 we can determine B as a compact Riemannian symmetric space. We still need to consider the case when $K_0 = (G_1 \times G_2)/A$ is an almost product of two compact Lie groups. Using the fact that the exponential map is onto for K_0 , one can conclude readily that G_1 and G_2 are also connected. σ thus induces involutive automorphisms on K_0 , G_1 and G_2 denoted by σ , σ_1 and σ_2 respectively. We put $Q_1 = G_1 \cap H$, $Q_2 = G_2 \cap H$. Since $(\mathfrak{k}, \mathfrak{q}, \sigma) = (\mathfrak{k}_1, \mathfrak{q}_1, \sigma) \oplus (\mathfrak{k}_2, \mathfrak{q}_2, \sigma)$, G_1 and G_2 determine respectively the totally geodesic submanifolds $M_1 = G_1/Q_1$ and $M_2 = G_2/Q_2$ of M . Since H is connected, by arguments analogous to that in the proof of Lemma 2.6, we can also conclude that Q_1 and Q_2 are the identity components of $(G_1)^{\sigma_1}$ and $(G_2)^{\sigma_2}$ respectively. We can now apply Lemma 2.7 and Remark 2.8 to conclude that $B = (M_1 \times M_2)/\Gamma$ for some suitable finite subgroup Γ of the full group of isometries of $M_1 \times M_2$. Note that neither M_1 nor M_2 collapses under the action of Γ .

Unlike the method discussed in §3, the present method does not enable us to pair off complementary reflective submanifolds immediately. However, this can be achieved by the following simple observation. We will use notations as above.

Lemma 4.1. *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two reflective subspaces, then they are complementary pairs only if the Lie subalgebra $\mathfrak{q} \oplus \mathfrak{a}$ as constructed in Theorem 2.1 are the same for $\mathfrak{h}_1$ and $\mathfrak{h}_2$.*

Proof. If they are complementary pair, a glance at the proof of Theorem 2.1 shows that they are both equal to $[\mathfrak{h}_1, \mathfrak{h}_1] + [\mathfrak{h}_2, \mathfrak{h}_2]$. **Q.E.D.**

In every case, the structure of $\mathfrak{q} \oplus \mathfrak{a}$ can easily be determined from the reflective subalgebra $(\mathfrak{f}, \mathfrak{q}, \sigma)$ and $\mathfrak{g}''$.

We will call a reflective submanifold B *self-complementary* if its complementary reflective submanifolds are isometric to B . Obviously B can only be self-complementary if its dimension is half that of the ambient space. By Lemma 3.1 and the fact that there always exists a unique complementary reflective submanifolds, one can easily pair off complementary ones or decide if a given reflective submanifold is self-complementary. As a result of the classification, it follows that *all reflective submanifolds whose dimension is half of that of the ambient space is self-complementary*.

For convenience of references, we will give below a complete list of all the reflective submanifolds in simple connected irreducible Riemannian symmetric spaces of type I. From this list one can easily obtain the corresponding list for irreducible symmetric spaces of type III.

Theorem 4.2. *The reflective submanifolds of simply connected irreducible Riemannian symmetric spaces of type I are as follows.*

Ambient space	Reflective submanifolds through the origin
AI $\mathrm{SU}(n)/\mathrm{SO}(n)$	$(1_r) [\mathrm{SU}(r)/\mathrm{SO}(r)] \times [\mathrm{SU}(n-r)/\mathrm{SO}(n-r)] \times S^1$; $(2_r) \mathrm{SO}(n)/\mathrm{SO}(r) \times \mathrm{SO}(n-r)$; $((3) \text{ and } (4) \text{ for even } n \text{ only})$ $(3) \mathrm{Sp}(n/2)/\mathrm{U}(n/2)$; $(4) \mathrm{SU}(n/2)$. Complementary pairs: $\{(1_r), (2_r)\}, \{(3), (4)\}$.
AII $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ $n > 1$	$(1_r) [\mathrm{SU}(2r)/\mathrm{Sp}(r)]$ $\times [\mathrm{SU}(2n-2r)/\mathrm{Sp}(n-r)] \times S^1$; $(2_r) \mathrm{Sp}(n)/\mathrm{Sp}(r) \times \mathrm{Sp}(n-r)$; $(3) \mathrm{SO}(2n)/\mathrm{U}(n)$; $(4) \mathrm{SU}(n)$. Complementary pairs: $\{(1_r), (2_r)\}, \{(3), (4)\}$.
AIII $\mathrm{SU}(p+q)/$ $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$(1_{r,s}) [\mathrm{SU}(r+s)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(s))]$ $\times [\mathrm{SU}(p+q-r-s)/\mathrm{S}(\mathrm{U}(p-r)$ $\times \mathrm{U}(q-s))]$; $(2) \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$; $((3) \text{ for } p, q \text{ even only})$ $(3) \mathrm{Sp}(p)/\mathrm{U}(p)$; $((4), (5), (6) \text{ when } p = q \text{ only})$ $(4) \mathrm{SO}(2p)/\mathrm{U}(p)$; $(5) \mathrm{Sp}(p)/\mathrm{U}(p)$; $(6) \mathrm{SU}(p) \times S^1$. Complementary pairs: $\{(1_{r,s}), (1_{r,q-s})\}, \{(4), (5)\}$. Self-complementary spaces: $(2), (3), (6)$.

<i>Ambient space</i>	<i>Reflective submanifolds through the origin</i>
BDI $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$	$(1_{r,s}) [\mathrm{SO}(r+s)/\mathrm{SO}(r) \times \mathrm{SO}(s)]$ $\times [\mathrm{SO}(p+q-r-s)/\mathrm{SO}(p-r) \times \mathrm{SO}(q-s)];$ (2) (for p, q even only) $\mathrm{SU}((p+q)/2)/\mathrm{S}(\mathrm{U}(p/2) \times \mathrm{U}(q/2));$ $((3)$ and (4) when $p = q$ only) $(3) [\mathrm{SU}(p)/\mathrm{SO}(p)] \times S^1;$ $(4) \mathrm{SO}(p).$ <i>Complementary pairs:</i> $\{(1_{r,s}), (1_{p-r,s})\}, \{(3), (4)\}.$ <i>Self-complementary space:</i> $(2).$
DIII $\mathrm{SO}(2n)/\mathrm{U}(n)$	$(1_r) [\mathrm{SO}(2r)/\mathrm{U}(r)] \times [\mathrm{SO}(2n-2r)/\mathrm{U}(n-r)];$ $(2_r) \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r)); (3) \mathrm{SO}(n);$ (4) (for n even only) $[\mathrm{SU}(n)/\mathrm{Sp}(n/2)] \times S^1.$ <i>Complementary pairs:</i> $\{(1_r), (2_r)\}$ <i>Self-complementary spaces:</i> $(3), (4).$
CI $\mathrm{Sp}(n)/\mathrm{U}(n)$	$(1_r) [\mathrm{Sp}(r)/\mathrm{U}(r)] \times [\mathrm{Sp}(n-r)/\mathrm{U}(n-r)];$ $(2_r) \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r));$ $((3)$ and (4) for n even only) $(3) \mathrm{SU}(n)/\mathrm{Sp}(n/2);$ $(4) \mathrm{Sp}(n/2).$ <i>Complementary pairs:</i> $\{(1_r), (2_r)\}.$ <i>Self-complementary spaces:</i> $(3), (4).$
CII $\mathrm{Sp}(p+q)/\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$(1_{r,s}) [\mathrm{Sp}(r+s)/\mathrm{Sp}(r) \times \mathrm{Sp}(s)] \times$ $[\mathrm{Sp}(p+q-r-s)/\mathrm{Sp}(p-r) \times \mathrm{Sp}(q-s)];$ $(2) \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)); ((3)$ and (4) when $p = q$ only) $(3) [\mathrm{SU}(2p)/\mathrm{Sp}(p)] \times S^1; (4) \mathrm{Sp}(p).$ <i>Complementary pairs:</i> $\{(1_{r,s}), (1_{p-r,s})\}, \{(3), (4)\}.$ <i>Self-complementary space:</i> $(2).$
EI $\mathrm{E}_6/\{\mathrm{Sp}(4)/\mathrm{Z}_2\}$	$(1) F_4/\{[\mathrm{Sp}(4) \times \mathrm{Sp}(1)]/\mathrm{Z}_2\}; (2) [\mathrm{SU}(6)/\mathrm{Sp}(3)]/\mathrm{Z}_3 ;$ $(3) \{[\mathrm{SO}(10)/\mathrm{SO}(5) \times \mathrm{SO}(5)] \times S^1\}/\mathrm{Z}_2 ;$ $(4) \mathrm{Sp}(4)/\mathrm{Sp}(2) \times \mathrm{Sp}(2);$ $(5) \mathrm{Sp}(4)/\mathrm{U}(4); (6) \{[\mathrm{SU}(6)/\mathrm{SO}(6)]/\mathrm{Z}_3\} \times$ $[\mathrm{SU}(2)/\mathrm{SO}(2)].$ <i>Complementary pairs:</i> $\{(1), (2)\}, \{(3), (4)\},$ $\{(5), (6)\}.$

<i>Ambient space</i>	<i>Reflective submanifolds through the origin</i>
EII $\mathbf{E}_6/\{[(\mathbf{SU}(6)/\mathbf{Z}_3) \times \mathbf{SU}(2)]/\mathbf{Z}_2\}$	(1) $\mathbf{SO}(10)/\mathbf{U}(5)$; (2) $\mathbf{SO}(10)/\mathbf{SO}(4) \times \mathbf{SO}(6)$; (3) $\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(2) \times \mathbf{U}(4))$; (4) $[\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$; (5) $\mathbf{Sp}(4)/\mathbf{Sp}(1) \times \mathbf{Sp}(3)$; (6) $\mathbf{F}_4/\{[\mathbf{Sp}(3) \times \mathbf{SU}(2)]/\mathbf{Z}_2\}$; (7) $\mathbf{Sp}(4)/\mathbf{U}(4)$. <i>Complementary pairs:</i> $\{(2), (3)\}, \{(5), (6)\}$. <i>Self-complementary spaces:</i> (1), (4), (7).
EIII $\mathbf{E}_6\{[\mathbf{SO}(10) \times \mathbf{SO}(2)]/\mathbf{Z}_2\}$	(1) $\mathbf{F}_4/\mathbf{Spin}(9)$; (2) $\mathbf{SO}(10)/\mathbf{SO}(2) \times \mathbf{SO}(8)$; (3) $\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(2) \times \mathbf{U}(4))$; (4) $\mathbf{Sp}(4)/\mathbf{Sp}(2) \times \mathbf{Sp}(2)$; (5) $[\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(5))] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$; (6) $\mathbf{SO}(10)/\mathbf{U}(5)$. <i>Complementary pair:</i> $\{(5), (6)\}$. <i>Self-complementary spaces:</i> (1), (2), (3), (4).
EIV $\mathbf{E}_6/\mathbf{F}_4$	(1) $[\mathbf{SU}(6)/\mathbf{Sp}(3)]/\mathbf{Z}_3$; (2) $\mathbf{Sp}(4)/\mathbf{Sp}(1) \times \mathbf{Sp}(3)$; (3) $\{[\mathbf{SO}(10)/\mathbf{SO}(9)] \times S^1\}/\mathbf{Z}_2$, (4) $\mathbf{F}_4/\mathbf{Spin}(9)$. <i>Complementary pairs:</i> $\{(1), (2)\}, \{(3), (4)\}$.
EV $\mathbf{E}_7\{\mathbf{SU}(8)/\mathbf{Z}_2\}$	(1) $\mathbf{SO}(12)/\mathbf{U}(6)$; (2) $\mathbf{E}_6/\{[\mathbf{SU}(2)/\mathbf{Z}_3] \times \mathbf{SU}(2)]/\mathbf{Z}_2\}$; (3) $[\mathbf{SO}(12)/\mathbf{SO}(6) \times \mathbf{SO}(6)] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$; (4) $\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}(4) \times \mathbf{U}(4))$; (5) $\mathbf{SU}(8)/\mathbf{Sp}(4)$; (6) $\{[\mathbf{E}_6/(\mathbf{Sp}(4)/\mathbf{Z}_2)] \times S^1\}/\mathbf{Z}_3$; <i>Complementary pairs:</i> $\{(1), (2)\}, \{(3), (4)\}, \{(5), (6)\}$.
EVI $\mathbf{E}_7/\{\mathbf{E}_6 \times \mathbf{SO}(2)]/\mathbf{Z}_3\}$	(1) $\mathbf{E}_6/\{[\mathbf{SO}(10) \times \mathbf{SO}(2)]/\mathbf{Z}_2\}$; (2) $\mathbf{SO}(12)/\mathbf{SO}(8) \times \mathbf{SO}(4)$; (3) $\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}(4) \times \mathbf{U}(4))$; (4) $\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}(2) \times \mathbf{U}(6))$; (5) $\mathbf{E}_6/\{[(\mathbf{SU}(6)/\mathbf{Z}_3) \times \mathbf{SU}(2)]/\mathbf{Z}_2\}$; (6) $[\mathbf{SO}(12)/\mathbf{SU}(6)] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$. <i>Complementary pair:</i> $\{(4), (5)\}$. <i>Self-complementary spaces:</i> (1), (2), (3), (6).
EVII $\mathbf{E}_7/\{[\mathbf{E}_6 \times \mathbf{SO}(2)]/\mathbf{Z}_3\}$	(1) $\mathbf{SU}(8)/\mathbf{Sp}(4)$; (2) $[\mathbf{SO}(12)/\mathbf{SO}(2) \times \mathbf{SO}(10)] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$; (3) $\mathbf{E}_6/\{[\mathbf{SO}(10) \times \mathbf{SO}(2)]/\mathbf{Z}_2\}$; (4) $\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}(2) \times \mathbf{U}(6))$; (5) $\mathbf{SO}(12)/\mathbf{U}(6)$; (6) $\{[\mathbf{E}_6/(\mathbf{F}_4/\mathbf{Z}_2)] \times S^1\}/\mathbf{Z}_3$. <i>Complementary pairs:</i> $\{(2), (3)\}, \{(4), (5)\}$. <i>Self-complementary spaces:</i> (1), (6).

<i>Ambient space</i>	<i>Reflective submanifolds through the origin</i>
EVIII $\mathbf{E}_8/\mathbf{SO}(16)$	(1) $E_7/[\mathbf{SO}(12) \times \mathbf{SU}(2)]/\mathbf{Z}_2$; (2) $\mathbf{SO}(16)/\mathbf{SO}(8) \times \mathbf{SO}(8)$; (3) $\mathbf{SO}(16)/\mathbf{U}(8)$ (4) $[\mathbf{E}_7/[\mathbf{SU}(8)/\mathbf{Z}_2]] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$. <i>Complementary pairs:</i> $\{(3), (4)\}$. <i>Self-complementary spaces:</i> (1), (2).
EIX $\mathbf{E}_8/[\{\mathbf{E}_7 \times \mathbf{SO}(2)]/\mathbf{Z}_2$	(1) $\mathbf{SO}(16)/\mathbf{U}(8)$; (2) $\mathbf{SO}(16)/\mathbf{SO}(4) \times \mathbf{SO}(12)$; (3) $\mathbf{E}_7/[\mathbf{SO}(12) \times \mathbf{SU}(2)]/\mathbf{Z}_2$; (4) $[\mathbf{E}_7/[(E_6 \times \mathbf{SO}(2))/\mathbf{Z}_3]] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$. <i>Complementary pair:</i> $\{(2), (3)\}$. <i>Self-complementary spaces:</i> (1), (4).
FI $\mathbf{F}_4/[\{\mathbf{Sp}(3) \times \mathbf{Sp}(1)]\mathbf{Z}_2$	(1) $\mathbf{SO}(9)/\mathbf{SO}(4) \times \mathbf{SO}(5)$; (2) $\mathbf{Sp}(3)/\mathbf{Sp}(1) \times \mathbf{Sp}(2)$; (3) $[\mathbf{Sp}(3)/\mathbf{U}(3)] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$. <i>Complementary pair:</i> $\{(1), (2)\}$. <i>Self-complementary space:</i> (3).
FII $\mathbf{F}_4/\mathbf{Spin}(9)$	(1) $\mathbf{SO}(9)/\mathbf{SO}(8)$; (2) $\mathbf{Sp}(3)/\mathbf{Sp}(1) \times \mathbf{Sp}(2)$. <i>Self-complementary spaces:</i> (1), (2).
G $\mathbf{G}_2/\mathbf{SO}(4)$	(1) $[\mathbf{SU}(2)/\mathbf{SO}(2)] \times [\mathbf{SU}(2)/\mathbf{SO}(2)]$ <i>Self-complementary space:</i> (1).

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Date communicated: DECEMBER 31, 1973