The Spherical Representations of Groups Transitive on S^n

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The purpose of this paper is an explicit description of the decomposition of the spherical harmonics into irreducible submodules under the action of groups transitive on the sphere, together with applications to harmonic mappings. Thus, as is explained in Observations 3.1–3.4, we have solved a subset of the following general problem:

Problem: Classify the mappings $S^m \rightarrow S^m$ which are defined on Euclidean space by homogeneous harmonic polynomials.

The significance of this problem for us is as follows: firstly, each such map of spheres is harmonic in the intrinsic sense of Eells and Sampson ([4], [15]); secondly, if such a map is topologically non-trivial, it can perhaps be joined to other such maps to produce harmonic representatives of various homotopy groups of spheres ([15], [16]). This of course bears directly on the fundamental problem of harmonic mappings [4]: does every homotopy class of maps between compact Riemannian manifolds admit a harmonic representative?

In this paper our results in the homotopy direction are negative (see Remark 3.12), as expected from the relatively large dimensions of the representations. The applications instead are of a geometric nature, particularly in their relation to minimal embeddings.

Our own contributions occur mainly in the study of the symplectic groups and Spin (9); the orthogonal and unitary cases have long been well-understood (for example, see Berger [1, p. 160] on $SO(n)$, Ikeda [6] on $U(n)$). For completeness, however, these results are summarized; furthermore, the relations between the irreducibles for the symplectic groups and those for $U(2n)$ lend themselves to a rather pretty (and hopefully useful) schematic presentation (*c.f.* Appendix).

The key case is that of $Sp(n) \times Sp(1)$, and it is understood through use of the following basic theorem of Kostant:

Theorem 0.1 ([13]). *Let $\mathcal{H}$ be the space of complex valued harmonic polynomials on the tangent space to the rank one symmetric space G/K . If the dimension of the*

2-root space $\mathfrak{g}^{2\alpha}$ is >1 , K decomposes $\mathcal{H}$ under the adjoint representation as a 2-parameter family of irreducible representations, each occurring with multiplicity one. More precisely,

$$\mathcal{H} = \bigoplus_{i,j \geq 0} W_{ij}$$

and the subspace of homogeneity k decomposes as

$$\mathcal{H}_k = \bigoplus_{2i+j=k} W_{ij}.$$

It should be noted that the above theorem also applies to the exceptional action of $\text{Spin}(9)$ on S^{15} . (Kostant's general theory in the case $\dim \mathfrak{g}^{2\alpha} = 0$ or 1 gives the cases $SO(n)$ and $U(n)$).

Our purpose then is to give explicit characterizations of the above spaces (Theorem 2.4) and determine more of their structure.

The statement of this problem and some examples can be found in [16]. Nolan Wallach has informed me that alternate characterizations of the irreducibles for $Sp(n) \times Sp(1)$ and the exceptional case of $\text{Spin}(9)$ have been given by K. Johnson and Wallach [9], [10] and Johnson [11], respectively. Their approach involves considering invariant differential operators other than the Laplacian. Many other results parallel to those in this paper have been obtained by Wallach (unpublished). In particular, he was kind enough to point out to me why the embedded images of these maps should all be minimal (Observation 3.3).

It is also a pleasure to acknowledge several helpful conversations with Bertram Kostant and Sigurdur Helgason. Finally, I owe much thanks to James Eells for his formulation of the problem.

1. The orthogonal and unitary cases. The following characterization of the eigenspaces of the Laplacian on S^n (standard metric) is well-known: let p be a harmonic polynomial, homogeneous of degree k in $(x_1, \dots, x_{n+1})$. Then $\tilde{p} = p|S^n$ satisfies $\Delta \tilde{p} = k(k+n-1)\tilde{p}$, and all eigenspaces arise this way. It is also well known that $SO(n+1)$ acts irreducibly on each such eigenspace $\mathcal{H}_k$ of spherical harmonics [1, p. 164]. The same applies equally well to the real module $\mathcal{H}_k^{\mathbf{R}}$ of real-valued harmonics. (Spaces of polynomials will be assumed complex. When necessary, we will denote spaces of real polynomials by the generic form $V^{\mathbf{R}}$, whereas the realification of a complex representation V will be denoted $\mathbf{R}V$, etc. Thus in the above case we have $\mathcal{H}_k \cong \mathbf{C}\mathcal{H}_k^{\mathbf{R}}$).

In the unitary case the results are also known (cf. Ikeda [6] or Koorwinder [12]), and are easily summarized. Consider the complex polynomials on $\mathbf{C}^n$ which are homogeneous of degree α in z_i and degree β in $\bar{z}_i$. Let $X_{\alpha,\beta}$ be the space of such polynomials which are harmonic; i.e., polynomials p such that $\sum \partial^2 p / \partial z_i \partial \bar{z}_i = 0$. (This is equivalent to the condition that the real and imaginary parts of p are harmonic).

Theorem 1.1 ([6], [12], [13]). *As representations of $U(n)$, $n \geq 1$, the $X_{\alpha,\beta}$ are irreducible and mutually non-isomorphic. Hence the entire space of harmonics has the unique decomposition*

$$\mathcal{H} = \bigoplus_{\alpha,\beta} X_{\alpha,\beta}.$$

Note that the uniqueness is equivalent by Frobenius reciprocity to the fact that the $U(n-1)$ -invariant zonal function in each $X_{\alpha,\beta}$ is unique. These zonals are constructed inductively in [12]. In [10] and [11] the zonal functions for $U(n)$, $Sp(n) \times Sp(1)$, and $Spin(9)$ are characterized as solutions to hypergeometric equations (compare Propositions 2.8 and 2.12). Furthermore ([6], [12]) we have

$$(1.2) \quad \dim X_{\alpha,\beta} = \frac{(\alpha+n-2)! (\beta+n-2)! (\alpha+\beta+n-1)!}{\alpha! \beta! (n-1)! (n-2)!}.$$

For studying spaces of real polynomials, the following lemma is useful

Lemma 1.3. *Let V be an irreducible complex representation of a compact group G by complex-valued functions on G/K . Define a representation over the reals by*

$$V^{\mathbf{R}} = \{\operatorname{Re}(f) : f \in V\}$$

Then a) If V contains a non-trivial real valued function, V is real, $V^{\mathbf{R}}$ is irreducible, and $\mathbf{C}V^{\mathbf{R}} \cong V$.

b) If V contains no non-trivial real-valued functions, $V^{\mathbf{R}} \cong \mathbf{R}V$. If V is real and $V \cong \mathbf{C}V_1$, then $V^{\mathbf{R}} \cong 2V_1$. Otherwise $V^{\mathbf{R}}$ is irreducible.

Now put $X^{\mathbf{R}}_{\alpha,\beta} = \{\operatorname{Re}(f) : f \in X_{\alpha,\beta}\}$. Then $X_{\alpha,\alpha} \cong \mathbf{C}X^{\mathbf{R}}_{\alpha,\alpha}$ and $X^{\mathbf{R}}_{\alpha,\beta} \cong \mathbf{R}X_{\alpha,\beta}$ if $\alpha \neq \beta$. Thus $X^{\mathbf{R}}_{\alpha,\alpha}$ is irreducible, and since $X_{\alpha,\beta}$ for $\alpha \neq \beta$ is not self conjugate (as can be seen from the highest weights, for example), $X^{\mathbf{R}}_{\alpha,\beta}$ is also irreducible. We also note that these spaces must all be non-isomorphic for $\alpha \geq \beta$, hence the previous result may be rephrased as follows: the decomposition of $L_2(S^{2n-1})$ under $U(n)$ is unique and is given by

$$L_2(S^{2n-1}) = \bigoplus_{\alpha \geq \beta} X^{\mathbf{R}}_{\alpha,\beta}.$$

The results in the case of $SU(n)$ for $n \geq 3$ are the same as above. To see that $X_{\alpha,\beta}$ is irreducible, it suffices that the $SU(n-1)$ -invariant function in this space is unique. As noted above, however, there is a unique $U(n-1)$ -invariant element. But since $SU(n-1)$ is transitive on S^{2n-3} , an $SU(n-1)$ -invariant function must be $U(n-1)$ -invariant. Hence $X_{\alpha,\beta}$ is irreducible and the rest follows.

The case of $SU(2) = Sp(1)$ will be treated later. The spaces $X^{\mathbf{R}}_{\alpha,\alpha}$ and $X_{\alpha,\alpha}$ are still irreducible however, and since in general these spaces are the eigenspaces of the Laplacian on $\mathbf{C}P^{n-1}$ we see that $SU(n)$ acts irreducibly on these eigenspaces in each case (as is well-known).

2. The symplectic cases. We first treat the case of $Sp(n) \times Sp(1)$. This group acts on quaternionic n -space $\mathbf{H}^n$ by left multiplication of matrices in $Sp(n)$ and right multiplication by scalars in $Sp(1)$. This is essentially the adjoint action in the rank one symmetric space $\mathbf{HP}^n = Sp(n+1)/Sp(n) \times Sp(1)$, so that we can apply Theorem 0.1 of the Introduction. (In the case at hand, $\dim \mathfrak{g}^{2\alpha} = 3$ if $n \geq 2$; henceforth we will assume $n \geq 2$ since we already know $Sp(1) \times Sp(1) = \text{Spin}(4)$ acts irreducibly on each $\mathcal{H}_k$). Our goal is now to give an explicit description of the spaces W_{ii} of Theorem 0.1 in terms of quaternionic polynomials.

Notation. Given any quaternion q , write $q = z + jw$ for complex z, w , and let $(q)_n$ denote z or w as n is 1 or 2. Variables in $\mathbf{H}^n$ will be labelled q_α, q_β , etc.

Definition 2.1. Let $V_{1,0}$ be the space of quadratic polynomials spanned by the monomials $(q_\alpha \bar{q}_\beta)_n$ and their complex conjugates. Let $V_{0,1}$ be the space spanned by monomials of the following 3 types: i) the $(q_\alpha i \bar{q}_\beta)_n$ and their conjugates, ii) holomorphic complex quadratics (type $(2, 0)$), iii) anti-holomorphic complex quadratics (type $(0, 2)$).

We remark that $V_{1,0}$ and the polynomials of type (i) above comprise the complex quadratics of type $(1, 1)$. Further, the span of types (ii) and (iii) above is exactly that of the monomials $(q_\alpha j \bar{q}_\beta)_n, (q_\alpha k \bar{q}_\beta)_n$ and their conjugates, as can be seen by an explicit calculation.

Note that all elements of $V_{1,0}$ are invariant with respect to the right action of $Sp(1)$, and that $V_{1,0}$ is itself an $Sp(n) \times Sp(1)$ invariant space. To see that $V_{0,1}$ is also invariant, one uses the above remark together with the fact that if $\xi \in \mathbf{H}$ has real part 0, then $\lambda \xi \bar{\lambda}$ has real part 0 for any λ .

Definition 2.2. Let $V_{r,s}$ be the span of monomials which are a product of r factors from $V_{1,0}$ and s factors from $V_{0,1}$.

Let $\tilde{V}_{0,0}$ be the space of arbitrary real-linear forms, and let $\tilde{V}_{r,s}$ be the span of products $p\eta$ where $p \in V_{r,s}$ and $\eta \in \tilde{V}_{0,0}$.

Note that the above spaces are all $Sp(n) \times Sp(1)$ invariant. They are not, however, disjoint, but are nested for homogeneity > 3 in a complicated manner [17]. This fact, whose proof need not concern us, necessitates the following

Definition 2.3. A complex harmonic polynomial p defined on $\mathbf{H}^n$ is said to be of *pure type* (r, s) if $p \in V_{r,s}$ and p cannot be written as a non-trivial sum $p = p_1 + p_2$ of harmonics p_1, p_2 with either p_1 or p_2 lying in the span of $V_{r+k, s-k}$ ($k \geq 1$). Similarly for (r, s) .

Theorem 2.4. Each space $V_{r,s}$ contains a distinguished $Sp(n) \times Sp(1)$ irreducible subspace of harmonics $H_{r,s}$, which is characterized as those harmonics of pure type (r, s) . Similarly for $\tilde{V}_{r,s}$.

The spaces $H_{r,s}$ are the spaces $W_{r,2s}$ of Theorem 0.1. $\tilde{H}_{r,s}$ is the space $W_{r,2s+1}$.

Proof. The simplest method is a direct study of the highest weights. Consider the standard maximal torus T in $Sp(n)$, namely the set of all diagonal matrices $(\lambda_1, \dots, \lambda_n)$, where the λ_i are complex scalars of norm 1. Thus $\lambda_\alpha q_\alpha = \lambda_\alpha z_\alpha + j\bar{\lambda}_\alpha w_\alpha$. Our torus in $Sp(1)$ consists of complex scalars $\lambda_{n+1} \equiv \lambda$, acting on the right; hence $q_\alpha \lambda = \lambda z_\alpha + j\bar{\lambda} w_\alpha$. Let $\lambda_i = e^{2\pi i x_i}$ and order the weights lexicographically via $x_n > x_{n-1} > \dots > x_1$. Then with the notation $[(m_1, \dots, m_n), m_{n+1}]$ for a weight, it follows that

$$u = \overline{(q_n \bar{q}_{n-1})_2} = z_{n-1} \bar{w}_n - z_n \bar{w}_{n-1}$$

and

$$v = z_n^2$$

are the unique highest weight vectors in $V_{1,0}$ and $V_{0,1}$, respectively, with weights $[(0, \dots, 1, 1), 0]$ and $[(0, \dots, 0, 2), 2]$. Hence u and v generate irreducible spaces of harmonics $H_{1,0}$ and $H_{0,1}$, respectively, which must therefore be the 2 spaces of quadratics predicted by Theorem 0.1.

It follows inductively that $u^r v^s$ is the unique polynomial in $V_{r,s}$ with (highest) weight $[(0, \dots, 0, 2s+r), 2s]$, and therefore generates a distinguished space of harmonics $H_{r,s}$. One also sees by inspection of this polynomial that $H_{r,s}$ does not lie in the span of $V_{r+k,s-k}$ for $k \geq 1$; hence each of these spaces is distinct. For a fixed homogeneity $2(r+s)$, we have thus exhibited the number of irreducibles predicted by Theorem 0.1.

In the case of odd homogeneity, we have similarly that $u^r v^s z_n$ is the highest weight vector in $\tilde{H}_{r,s}$, with weight $[(0, \dots, 0, r, 2s+r+1), 2s+1]$.

It follows from the Weyl dimension formula (7, p. 259) that

$$(2.5) \quad \dim H_{r,s} = \frac{[(2n+2s+r+2) \cdots (2s+r+2)][(2n+r-3) \cdots (r+1)](2s+1)^2(2n+2s+2r-1)}{(2n-1)!(2n-3)!}.$$

For $\dim \tilde{H}_{r,s}$, replace $2s$ by $2s+1$ everywhere.

Proposition 2.6. *Each space $\tilde{H}_{r,s}$, $H_{r,s}$ has a basis of real polynomials.*

Proof. Complex conjugation $\tau: \mathcal{H} \rightarrow \mathcal{H}$ is an $Sp(n) \times Sp(1)$ -isomorphism. Furthermore any (complex irreducible) representation of a symplectic group is self-conjugate, so for the spaces of Theorem 0.1 we have $\tau(W_{ij}) \cong W_{ij}$. On the other hand, each representation occurs with multiplicity one, hence $\tau(W_{ij}) = W_{ij}$.

Let us further define $H^{\mathbf{R}}_{r,s} = \{\text{Re } (f): f \in H_{r,s}\}$ and similarly for $\tilde{H}^{\mathbf{R}}_{r,s}$. Then by Lemma 1.3 we have $H_{r,s} \cong \mathbf{C}H^{\mathbf{R}}_{r,s}$, $\tilde{H}_{r,s} \cong \mathbf{C}\tilde{H}^{\mathbf{R}}_{r,s}$, and an entirely analogous 2-parameter decomposition of the harmonics is valid in the real case.

The content of the above spaces for $r=0$ is clarified by

Lemma 2.7. *Let p be a harmonic polynomial which involves only the coordinates of a single quaternionic variable q_α . If $\deg p = 2s$, $p \in H_{0,s}$; if $\deg p = 2s + 1$, $p \in \tilde{H}_{0,s}$.*

Proof. Suppose $\deg p = 2s$. By the proof of Theorem 2.4, $H_{0,s}$ contains z_n^{2s} ; hence by invariance it also contains z_α^{2s} , which is a polynomial of the same description as p . However, the diagonal elements in $Sp(n) \times Sp(1)$ act irreducibly on such polynomials (since $Sp(1) \times Sp(1) = \text{Spin}(4)$), so again by invariance $p \in H_{0,s}$. Similarly for odd degree.

Since these spaces (real or complex) are all mutually non-isomorphic, they each contain unique zonal functions; the above lemma will allow us to write them down. Let $(q_1, \dots, q_n)$ be our quaternionic variables, and let $R^2 = |q_1|^2 + \dots + |q_n|^2$. Let (x_1, x_2, x_3, x_4) be the real coordinates of q_1 , and let $L_k(x_1, r^2)$ denote the k^{th} Legendre polynomial on $\mathbf{R}^4$ (harmonic, $SO(3)$ -invariant of degree k), where $r^2 = x_2^2 + x_3^2 + x_4^2$.

Proposition 2.8. *The unique zonal in $H_{r,s}$ (resp., $\tilde{H}_{r,s}$) has the form $P(|q_1|^2, R^2)L_{2s}(x_1, r^2)$ (resp., PL_{2s+1}), where*

$$P = |q_1|^{2r} - a_{r-1} |q_1|^{2(r-1)} R^2 + \dots + (-)^r a_0 R^{2r}$$

and a_i are positive constants depending on the space in question.

Proof. The isotropy group for the action of $Sp(n) \times Sp(1)$ on the sphere consists of pairs $\{\binom{0}{A}, \bar{\lambda}\}$, where $\lambda \in Sp(1)$ and $A \in Sp(n-1)$. Hence this group acts as $SO(3)$ on (x_2, x_3, x_4) . Clearly, then, a polynomial of the above form is invariant. Further, if it is harmonic, it must lie in the space claimed (by the preceding lemma). However,

$$\Delta(PL_k) = (\Delta P)L_k + 2 \sum_{x_i} \frac{\partial P}{\partial x_i} \frac{\partial L_k}{\partial x_i} = P_1 L_k + P_2 \sum x_i \frac{\partial L_k}{\partial x_i}$$

where P_1, P_2 are polynomials in $|q_1|^2, R^2$. By homogeneity, $\sum x_i (\partial L_k) / (\partial x_i) = kL_k$, and now an explicit calculation of P_1, P_2 shows that there is a unique choice of a_i as claimed such that $P_1 + kP_2 = 0$.

We now turn to the case of $Sp(n)$. It is well-known that a finite dimensional complex irreducible representation V of a compact product group $G = G_1 \times G_2$ must have the form $V_1 \otimes V_2$, where V_i is an irreducible representation of G_i (e.g., look at the highest weights). In fact, the highest weight for V_i is the restriction of the highest weight for V . Hence there are irreducible representations $I_{r,s}$ and J_s of $Sp(n)$ and $Sp(1)$ respectively such that

$$H_{r,s} \cong I_{r,s} \otimes J_s$$

and similarly for $\tilde{H}_{r,s}$. From the weights we see $\dim J_s = 2s + 1$ and $\dim J_s = 2s + 2$, so the dimensions of $I_{r,s}$ and $\tilde{I}_{r,s}$ can be read from (2.5). Note also that these representations are all mutually non-isomorphic (look at the weights).

To get a more explicit picture of the structures here, we introduce some $Sp(n)$ -invariant spaces of complex polynomials. For each multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, with non-negative integer entries, let ψ_γ be the space spanned by "monomials" which are products of γ_1 factors from $V_{1,0}$, γ_2 factors of type $(q_\alpha i \bar{q}_\beta)_n$, and a complex polynomial of type $(\alpha, \beta) = (2\gamma_3, 2\gamma_4)$. Hence the following observations:

- i) (a) ψ_γ lies in that $V_{r,s}$ with $r = \gamma_1$, $s = \gamma_2 + \gamma_3 + \gamma_4$.
- (b) ψ_γ is of complex type (α, β) , with $\alpha = \gamma_1 + \gamma_2 + 2\gamma_3$, $\beta = \gamma_1 + \gamma_2 + 2\gamma_4$.

ii) ψ_γ is $Sp(n)$ -invariant.

iii) The highest weight vector in ψ_γ (cf. Thm. 2.4) is

$$p_\gamma = (z_{n-1}\bar{w}_n - z_n\bar{w}_{n-1})^{\gamma_1} (z_n\bar{w}_n)^{\gamma_2} (z_n)^{2\gamma_3} (\bar{w}_n)^{2\gamma_4}$$

and p_γ is the unique weight vector in ψ_γ with weight $= (0, \dots, 0, r, 2s + r)$.

iv) p_γ generates a unique irreducible subspace Φ_γ in ψ_γ , namely $\psi_\gamma \cap H_{r,s}$, and $\Phi_\gamma \cong I_{r,s}$.

We also define spaces ψ_γ' and ψ_γ'' by considering products of elements of ψ_γ with complex linear (resp., anti-linear) forms. Then in the same pattern as above we have $p_{\gamma'} = p_\gamma z_n$, $p_{\gamma''} = p_\gamma \bar{w}_n$; they generate spaces $\Phi_{\gamma'}$, $\Phi_{\gamma''}$ respectively which are both isomorphic to $I_{r,s}$.

Let us take for the moment the case of polynomials of even degree. It is clear that $\Phi_\gamma = \Phi_\mu \Leftrightarrow p_\gamma = p_\mu \Leftrightarrow$

$$\gamma_1 = \mu_1$$

*)

$$\gamma_2 + 2\gamma_3 = \mu_2 + 2\mu_3$$

$$\gamma_2 + 2\mu_4 = \mu_2 + \mu_4.$$

Hence from (*) and (i) above it follows:

Lemma 2.9. $\Phi_\gamma, \Phi_\mu \subset H_{r,s} \cap X_{\alpha,\beta} \Rightarrow \Phi_\gamma = \Phi_\mu$.

Thus $H_{r,s}$ is the direct sum of those distinct Φ_γ lying in $H_{r,s}$. For distinct Φ_γ in $H_{r,s}$ are of distinct complex type, hence independent. On the other hand, one easily sees (e.g., from the formula for p_γ) that the complex types achieved by Φ_γ in $H_{r,s}$ are those which lie between $(\alpha, \beta) = (r, 2s + r)$ and $(\alpha, \beta) = (r + 2s, r)$ inclusive. As there are exactly $2s + 1 = \dim J_s$ of these subrepresentations, each isomorphic to $I_{r,s}$, they also span.

It also follows that $X_{\alpha,\beta}$ is the direct sum of those distinct Φ_γ which it contains. For again, by Lemma 2.9, distinct Φ_γ in $X_{\alpha,\beta}$ fall in distinct $H_{r,s}$, hence are independent. On the other hand, by the above we see that the Φ_γ span the totality of harmonics, hence those of type (α, β) will span $X_{\alpha,\beta}$.

At this point it is convenient to change our notation slightly and put

$$\Phi_{r,s,\alpha,\beta} = H_{r,s} \cap X_{\alpha,\beta}$$

$$\tilde{\Phi}_{r,s,\alpha,\beta} = \tilde{H}_{r,s} \cap X_{\alpha,\beta}.$$

Diagrams A and B of the Appendix now follow from:

Proposition 2.10. *i) $\Phi_{r,s,\alpha,\beta} \neq 0 \Leftrightarrow r \leq \alpha \leq r + 2s$ and $\alpha + \beta = 2(r + s)$, in which case $\Phi_{r,s,\alpha,\beta} \cong I_{r,s}$.
ii) $\tilde{\Phi}_{r,s,\alpha,\beta} \neq 0 \Leftrightarrow r \leq \alpha \leq r + 2s + 1$ and $\alpha + \beta = 2(r + s) + 1$, in which case $\tilde{\Phi}_{r,s,\alpha,\beta} \cong \tilde{I}_{r,s}$.*

On the other hand, each of the above spaces is also invariant under the larger group $Sp(n) \times U(1)$, which is the third group we wish to study.

Proposition 2.11. *As representations of $Sp(n) \times U(1)$ the non-zero spaces $\Phi_{r,s,\alpha,\beta}$ and $\tilde{\Phi}_{r,s,\alpha,\beta}$ are all irreducible and mutually non-isomorphic. Hence they give the unique decomposition of the harmonics under this group.*

Proof. Irreducibility is immediate. As a maximal torus for this group we may take the one previously considered for $Sp(n) \times Sp(1)$; the polynomials given previously for $Sp(n)$ are still highest weight vectors, with weights $[(0, \dots, 0, r, 2s + r), \alpha - \beta]$ and $[(0, \dots, 0, r, 2s + r + 1), \alpha - \beta]$ respectively. Hence the proposition.

Hence each such space contains a unique zonal, which again we can write down. Let the notation be as before Proposition 2.8, but now let (z_1, w_1) be the complex coordinates of q_1 , and let $Q_{\alpha,\beta}(z_1, |w_1|^2)$ denote the unique $U(1)$ -invariant harmonic of type (α, β) in (z_1, w_1) . A similar proof gives

Proposition 2.12. *The unique zonal in $\Phi_{r,s,\alpha,\beta}$ (resp., $\tilde{\Phi}_{r,s,\alpha,\beta}$) for the group $Sp(n) \times U(1)$ has the form $P(|q_1|^2, R^2)Q_{\alpha-r,\beta-r}(z_1, |w_1|^2)$, where P has the same form as in Proposition (2.8).*

The structure of the corresponding real spaces follows from

Proposition 2.13. *For each (r, s) , $I_{r,s}$ is real and $\tilde{I}_{r,s}$ is quaternionic. Similarly for J_s and $\tilde{J}_s$.*

Proof. First we consider the spaces $X_{\alpha,\beta}$ on $\mathbb{C}^{2n}$ as representations of $Sp(n)$. There is an anti-linear $Sp(n)$ -map $\xi: X_{\alpha,\beta} \rightarrow X_{\alpha,\beta}$ defined as follows:

$$(\xi f)(x) = \bar{f}(xj).$$

Notice that since $(z + jw)j = -\bar{w} + j\bar{z}$, ξ preserves complex type, hence $X_{\alpha,\beta} \cdot \xi$ clearly commutes with $Sp(n)$, and is anti-linear. Note also that $\xi^2 = (-1)^{\alpha+\beta}$.

Now consider the decomposition of $X_{\alpha,\beta}$ into irreducibles under $Sp(n)$ given in Diagrams A and B (Appendix). The components are mutually non-isomorphic, so since each is self-conjugate, each must be preserved by ξ (compare Proposition 2.6). Therefore ξ is the required structure map on $I_{r,s}$ and $\tilde{I}_{r,s}$.

For the spaces J_s and $\tilde{J}_s$, one may proceed in two ways. The first is to observe that since $H_{r,s}$ and $I_{r,s}$ are real, and $H_{r,s} \cong I_{r,s} \otimes J_s$, then J_s must be real, etc. The second is to note that J_s (resp. $\tilde{J}_s$) is isomorphic to the representation of $SU(2)$ on complex harmonics of type (α, β) for any (α, β) with $\alpha + \beta =$

$2s$ (resp. $2s + 1$) (this is immediate from the highest weight). Now apply the above structure map to this irreducible $SU(2)$ -module.

It follows that there are irreducible representations over $\mathbf{R}$ defined by: $\mathbf{C}I_{r,s}^{\mathbf{R}} \cong I_{r,s}$, $\mathbf{C}J_s^{\mathbf{R}} \cong J_s$, $\tilde{I}_{r,s}^{\mathbf{R}} \cong \mathbf{R}\tilde{I}_{r,s}$, $\tilde{J}_s^{\mathbf{R}} \cong \mathbf{R}\tilde{J}_s$.

Proposition 2.14. *As representations of $Sp(n) \times Sp(1)$, we have*

$$H_{r,s}^{\mathbf{R}} \cong I_{r,s}^{\mathbf{R}} \otimes J_s^{\mathbf{R}}, \quad 4\tilde{H}_{r,s}^{\mathbf{R}} \cong \tilde{I}_{r,s}^{\mathbf{R}} \otimes \tilde{J}_s^{\mathbf{R}}.$$

As representations of $Sp(n)$,

$$H_{r,s}^{\mathbf{R}} \cong (2s + 1)I_{r,s}$$

$$\tilde{H}_{r,s}^{\mathbf{R}} \cong (s + 1)\tilde{I}_{r,s}.$$

Proof. The first isomorphism follows by considering the complexification. The second is a direct consequence of the (undoubtedly standard)

Lemma 2.15. *Let V and W be (finite-dimensional) complex self-conjugate representations of the group G ; then*

$$\mathbf{R}V \otimes_{\mathbf{R}} \mathbf{R}W \cong 2\mathbf{R}(V \otimes_{\mathbf{C}} W).$$

If we form the real spaces $\Phi_{r,s,\alpha,\beta}^{\mathbf{R}}$, $\tilde{\Phi}_{r,s,\alpha,\beta}^{\mathbf{R}}$ in the usual manner, then the following situation obtains: for each such space which is non-zero, then as representations of $Sp(n)$ we have (cf. Lemma 1.3)

$$\Phi_{r,s,\alpha,\beta}^{\mathbf{R}} \cong \begin{cases} I_{r,s}^{\mathbf{R}} & \text{if } \alpha = \beta \\ 2I_{r,s}^{\mathbf{R}} & \text{if } \alpha \neq \beta \end{cases}$$

$$\tilde{\Phi}_{r,s,\alpha,\beta}^{\mathbf{R}} \cong \tilde{I}_{r,s}^{\mathbf{R}}.$$

As representations of $Sp(n) \times U(1)$, the above spaces are irreducible and mutually non-isomorphic.

The most convenient way of representing the occurrence of non-zero Φ , $\tilde{\Phi}$, etc. is diagrammatically. Hence our final structure theorems are not explicitly stated, but are instead pictured in the appendix. However, a few comments are in order. The first is that we have depicted decomposition into irreducibles for $Sp(n)$ only. For the case of $Sp(n) \times U(1)$, one must modify diagram C by collapsing each pair occurring in an $H_{r,s}^{\mathbf{R}}$ column into a single irreducible. Furthermore, of course, the representations of $Sp(n) \times U(1)$ are all mutually distinct. Hence the decomposition of the complete space of harmonics is unique in this case, whereas this is evidently not so for $Sp(n)$.

Secondly, the $X_{k,k}^{\mathbf{R}}$ and $X_{k,k}$ rows correspond to the eigenspaces on $\mathbf{C}P^{2n-1}$. The decomposition of these spaces under $Sp(n)$ is read from the diagrams, and the complete decomposition of $\mathcal{L}_2(\mathbf{C}P^{2n-1})$ is in fact unique.

Thirdly, for the case of $Sp(1) = SU(2)$, we find that only the right-hand columns in each case are non-trivial (cf. Lemma 2.7), but the conclusions as depicted are still valid with this *proviso*.

3. Applications. The background and motivation for this study is contained in the ensuing sequence of observations.

In general, if V is a space of (complex) functions on G/K which is irreducible under G (compact), let v_0 be the unique norm 1 reproducing kernel for V (with respect to an appropriately normalized measure $d\mu$). For any $v \in V$,

$$v(gK) = \langle v_0, g^{-1}v \rangle = \int_{G/K} v_0(x) \overline{v(gx)} d\mu(x).$$

Then by fixing an isometry of V^m with $\mathbf{C}^m$ one sees that the orthonormal basis (v_i) for V^m so determined has this property: the associated map $G/K \rightarrow \mathbf{C}^m$ given by $gK \mapsto (v_i(gK))$ has image exactly equal to the orbit $G \circ v_0$ under the above identification. That is,

$$gv_0 = \sum v_i(gK)v_i$$

Hence we have the well-known

Observation 3.1. *The image of G/K under V is diffeomorphic to $G/I(v_0)$, where $I(v_0)$ is the isotropy subgroup of v_0 , and lies in the unit sphere in $\mathbf{C}^m$.*

Of course, v_0 is necessarily a zonal function, so $I(v_0) \supseteq K$. What makes life interesting is that $I(v_0)$ is often larger.

The importance of these observations for minimal submanifold theory is given in

Observation 3.2. (DoCarmo and Wallach [2]). *If V is an eigenspace of the Laplacian and the isotropy action of K is irreducible, the above map is a minimal isometric (up to a scalar) immersion of G/K into the sphere.*

Nolan Wallach also made the next crucial observation, and for completeness we have included a proof.

Observation 3.3. (Wallach). *Suppose that up to scalar multiples v_0 is the unique zonal in V , and further that $\lambda v_0 \notin G \circ v_0$ for all $\lambda \in S^1$. Then $G \circ v_0$ is minimal in the Euclidean sphere in V .*

Proof. It follows from [5] that $G \circ v_0$ is minimal if $G \circ v_0$ is an isolated orbit type. However, for nearby v , $G_v \supseteq gG_v g^{-1} = G_{gv}$ for some g . But if $G_{gv} = G_{v_0}$, $gv = \lambda v_0 \notin G \circ v_0$ by assumption. Hence $v \notin G \circ v_0$, so $G \circ v_0$ is isolated.

Finally, for harmonic maps we have

Observation 3.4. ([16]). *If V is an invariant irreducible subspace of an eigenspace of the Laplacian, the map of G/K to the unit sphere in V is harmonic.*

In this paper we consider cases in which the space G/K is itself a Euclidean sphere. The ways in which a sphere can be written as a homogeneous space were essentially classified by Montgomery-Samelson (completed by Borel), and the Lie algebras $(\mathfrak{g}, \mathfrak{k})$ can be listed (see Wang [19]).

We consider only the classical cases here. Of the exceptional cases, only $\text{Spin}(9)/\text{Spin}(7) = S^{15}$ has non-transitive isotropy action; hence, this is the only remaining case in which the eigenspaces of Δ will not be irreducible [1]. Results for $\text{Spin}(9)$ are presented in Section 4.

Let us now take up each case in turn. Our main object will be to determine the isotropy subgroups $I(v_0)$, and hence the diffeomorphism type of the image of each map. In the case of $SO(n)$ (covered by Observation 3.2; see also [3]), it is known that the image is S^{n-1} or $\mathbf{R}P^{n-1}$, depending on whether the degree of the harmonics is odd or even. Hence we begin with

Lemma 3.5. *Let f be an eigenfunction of Δ on S^{2n-1} which is invariant under $U(n-1)$ and also a unitary $T \notin U(1) \times U(n-1)$. Then if $n \geq 3$, f is constant; if $n = 2$, $f(z, w)$ is a function of $|z|$ and $|w|$ only, and is symmetrical in z and w .*

Proof. In general we have $f(z) = F(z_1)$ for $z \in S^{2n-1}$. Here F is some function on the unit disc in $\mathbf{C}$. Let $T(z)_1 = \alpha_1 z_1 + \cdots + \alpha_n z_n = \alpha(z)$, where we assume $\alpha_i \neq 0$ for some $j \neq 1$.

In the case $n \geq 3$ we restrict ourselves to the plane $z_1 = 0$. Hence $F(\alpha(z)) = F(0)$ for any z with $|z| = 1$ and $z_1 = 0$. We claim that if $\epsilon = |(\alpha_2, \dots, \alpha_n)|$, the image of the unit sphere in $\mathbf{C}^{n-1}$ under α is the ϵ -disc in $\mathbf{C}$. Certainly all vectors of length ϵ are obtained (let $(z_2, \dots, z_n) = (1/\epsilon)(\bar{\alpha}_2, \dots, \bar{\alpha}_n)$, etc.) and by considering combinations with non-trivial elements in $\ker(\alpha|_{\mathbf{C}^{n-1}})$ we get the rest of the disc. Therefore f is constant on an open set in S^{2n-1} , and being an eigenfunction, must be constant throughout.

In the case $n = 2$, let $M = \ker \alpha \cap S^3$ and let $\tilde{M} = \{(z_1, z_2\lambda) : (z_1, z_2) \in M, |\lambda| = 1\}$. Then f is constant on M , and in fact on $\tilde{M}$ by $U(1)$ invariance. However, for points in $\tilde{M}$ we have

$$\alpha(z_1, z_2\lambda) = \alpha_2 z_2(\lambda - 1).$$

As F is constant on $\alpha(\tilde{M})$ and $\alpha_2 \neq 0$ by assumption, we see F must be constant on an open set unless $\ker \alpha = \{z_2 = 0\}$. Hence f is constant unless $\alpha_1 = 0$. In the latter case T has the form $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$; hence $f(z_1, |z_2|^2) = f(\alpha z_2, |z_1|^2)$, f only depends on $|z_1|^2$, and is symmetrical in z_1, z_2 . Hence the Lemma.

Corollary 3.6. *Let v_0 be the unique $U(n)$ zonal in $X_{\alpha, \beta}$. Then $I(v_0)$ has the form $H \times U(n-1)$, where H is a subgroup of $U(1)$, except possibly in the case $n = 2$, $\alpha = \beta$. In the latter case $I(v_0)$ is either $U(1) \times U(1)$ or the normalizer of this group.*

Proof. Essentially immediate from Lemma 3.5. In the case $n = 2$, if $f(z, w)$ is a function of $|z|$ and $|w|$ only, then $\alpha = \beta$ and $I(v_0)$ contains $U(1) \times U(1)$. In the symmetrical case, $I(v_0)$ must be the normalizer (that is, it must contain all matrices $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ in $U(2)$).

Remark. Here and throughout we will ignore the space of constants when no confusion is possible.

Let us denote by Z_p the p^{th} roots of unity embedded as scalars in $U(n)$. Then for the unitary case we have

Proposition 3.7. $X_{\alpha,\beta}(S^{2n-1})$ is diffeomorphic to a) $S^{2n-1}/Z_{|\alpha-\beta|}$ if $\alpha \neq \beta$; b) $\mathbf{CP}^{n-1}$ if $\alpha = \beta$ and $n \geq 3$ or α odd. c) $\mathbf{RP}^2$ if $n = 2$ and $\alpha = \beta$ is even. In each case the image is minimally embedded.

Proof. We simply verify that $I(v_0)$ is the same as the isotropy for the natural action of $U(n)$ on these spaces. In general this follows from Corollary 3.6 and the fact that if p is any polynomial of type (α, β) , then $p(\lambda z) = \lambda^{\alpha-\beta} p(z)$ for scalars λ of norm 1. In the problem case $n = 2$, $\alpha = \beta$ we observe that $X_{\alpha,\alpha}$ is an eigenspace for $\mathbf{CP}^1 = S^2$, and that even α correspond to even polynomials on S^2 (the Hopf map $S^3 \rightarrow S^2$ is itself a quadratic polynomial). Hence as is well known an embedding of $\mathbf{RP}^2$ is induced. Similarly, odd α must give an embedding of S^2 .

To check minimality, we use Observation 3.3 directly in case $\alpha \neq \beta$. For then $\lambda v_0 \in U(n) \circ v_0$ follows from $p(\lambda z) = \lambda^{\alpha-\beta} p(z)$. In case $\alpha = \beta$ one may proceed in two ways. The first is to observe (as above), that $X_{\alpha,\alpha}$ is a complete eigenspace of the Laplacian on $\mathbf{CP}^{n-1}$, so that by [2] the image is in fact an *isometric* (up to a scalar)-minimal immersion of $\mathbf{CP}^{n-1}$. Alternately, one can show that $X_{\alpha,\alpha}$ has a basis of real polynomials. Hence we can apply Observation 3.3 (in essence) to the corresponding real irreducible space and its unique zonal.

The following results are quite similar to Lemma 3.5 and Corollary 3.6.

Lemma 3.8. Let f be an eigenfunction of Δ on S^{4n-1} which is invariant under $Sp(n-1)$ and a symplectic $T \nmid Sp(1) \times Sp(n-1)$. Then if $n \geq 3$, f is constant. If $n = 2$, $f(q_1, q_2)$ is a symmetrical function of $|q_1|, |q_2|$.

Corollary 3.9. Let v_0 be the unique reproducing kernel in $\Phi_{(r,s,\alpha,\beta)}$ (resp. $\tilde{\Phi}_{(r,s,\alpha,\beta)}$). Then with respect to $Sp(n)$, $I(v_0)$ has the form $H \times Sp(n-1)$, where H is a subgroup of $Sp(1)$, except possibly in the case $n = 2$, $s = 0$, $\alpha = \beta$. In the latter case, $I(v_0)$ is either $Sp(1) \times Sp(1)$ or the normalizer of this group.

Let us denote by $\mathbf{CP}^{2n-1}/\sim$ the space gotten by identifying complex lines which are related under right multiplication by j . $Sp(n)$ acts naturally on this space, which is doubly covered by $\mathbf{CP}^{2n-1}$. Then for the symplectic case we have

Proposition 3.10. (i) $\tilde{\Phi}_{r,s,\alpha,\beta}(S^{4n-1})$ is diffeomorphic to $S^{4n-1}/Z_{|\alpha-\beta|}$.
(ii) If $n \geq 3$, $\Phi_{r,s,\alpha,\beta}(S^{4n-1})$ is diffeomorphic to
a) $S^{4n-1}/Z_{|\alpha-\beta|}$ if $\alpha \neq \beta$,
b) $\mathbf{CP}^{2n-1}$ if $\alpha = \beta$ and s odd,
c) $\mathbf{CP}^{2n-1}/\sim$ if $\alpha = \beta$ and s even $\neq 0$,
d) $\mathbf{HP}^{n-1}$ if $\alpha = \beta$ and $s = 0$.
(iii) If $n = 2$ the results are the same except in the case $\alpha = \beta$, $s = 0$, where the image is diffeomorphic to S^4 if α is odd and $\mathbf{RP}^4$ if α is even.

In each case the image is minimally embedded.

Proof. If $\alpha \neq \beta$ or $s \neq 0$, $I(v_0)$ has the form $H \times Sp(n-1)$ by Corollary 3.9. By Proposition 2.12, H must then be the isotropy of $Q_{\alpha-r, \beta-r} \equiv Q$ in $SU(2)$; that is, the intersection of $SU(2)$ with the isotropy of Q in $U(2)$. If $\alpha \neq \beta$, then $\alpha - r \neq \beta - r$ and the isotropy of Q in $U(2)$ consists of matrices $\text{diag}(\lambda, \mu)$ where λ is an $|\alpha - \beta|^{\text{th}}$ root of unity and μ is arbitrary. Hence elements in H have the form $\text{diag}(\lambda, \bar{\lambda})$ for such λ , as desired.

If $s \neq 0$ but $\alpha = \beta$, then $2(r+s) = \alpha + \beta = 2\alpha$, so $(\alpha - r, \beta - r) = (s, s)$. Hence the index for Q is odd or even as s is odd or even. From Proposition 3.7, we see that H is thus the torus $T = \{\text{diag}(\lambda, \bar{\lambda})\}$ or $N(T) = T \cup jT$, respectively. This takes care of cases (b) & (c).

If $s = 0$ but $n \geq 3$, the isotropy still has the form $H \times Sp(n-1)$. But as $(s, s) = (0, 0)$, Q is a constant, hence $H = Sp(1)$. This is (d). Finally, if $n = 2$ the functions involved are defined on $\mathbf{HP}^1 = S^4$. This is analogous to the special case in Proposition 3.7.

To check minimality, we use Observation 3.3 in case $\alpha \neq \beta$. Uniqueness of v_0 under $G = Sp(n) \times U(1)$ is established so we need $\lambda v_0 \notin G \circ v_0$. However, since $Sp(1) \times U(1) \rightarrow U(2)$ is surjective, it follows that any unitary scalar $\text{diag}_{2n}(\mu) \in G$. As noted before, this element acts as $\mu^{\alpha-\beta}$ on v_0 .

In case $\alpha = \beta$, our zonal is real valued, so Φ has a basis of real polynomials. Hence we may apply Observation 3.3 to the corresponding real irreducible subspace.

Remark. In case $\alpha = \beta$, $s = 0$, the space Φ is a complete eigenspace of the Laplacian on $\mathbf{HP}^{n-1}$. Hence the associated immersion is isometric, up to a scalar [2].

Finally, we consider the case $Sp(n) \times Sp(1)$.

Proposition 3.11.

- i) $\tilde{H}_{r,s}(S^{4n-1})$ is diffeomorphic to S^{4n-1} .
- ii) If $n \geq 3$, $H_{r,s}(S^{4n-1})$ is diffeomorphic to
 - a) $\mathbf{RP}^{4n-1}$ if $s > 0$,
 - b) $\mathbf{HP}^{n-1}$ if $s = 0$.
- iii) If $n = 2$, the results are the same except for $H_{r,0}$, whose image is S^4 if r is odd and $\mathbf{RP}^4$ if r is even.

In each case the image is minimally embedded.

Proof. As usual, we identify the isotropy group of the zonal (Prop. 2.8). In all cases except $H_{r,0}$, $n = 2$, this amounts to identifying the isotropy of L in $Sp(1) \times Sp(1)$. (These are basically the same arguments as in Lemma 3.8, and Corollary 3.9). But as is well known, for L_{2s+1} this is $\{(\lambda, \bar{\lambda}) : \lambda \in Sp(1)\}$. For L_{2s} and $s \neq 0$ it is $\{(\lambda, \pm \bar{\lambda})\}$, and for $s = 0$ it is $Sp(1) \times Sp(1)$. This takes care of everything but the exceptional case, which itself was treated in Proposition 3.10 (iii).

In each case the zonal is real, and minimality follows by considering the real irreducible subspace of real polynomials.

Remark 3.12. Each case considered may in fact be expanded to a *diagram* of harmonic mappings: *e.g.*, in Proposition 3.10, case (ii)b, we have

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{h} & S^{m-1} \\ \pi \downarrow & \nearrow \tilde{h} & \\ \mathbf{CP}^{2n-1} & & \end{array}$$

where π is a harmonic Riemannian fibration [4] and $\tilde{h}$ is harmonic because $\tilde{h} \circ \pi = h$ is harmonic [14] (this last is a special result applying only to harmonic fibrations). Here $m = \dim \Phi^{\mathbf{R}}_{r,s,\alpha,\alpha}$, since the polynomials may be assumed real.

Secondly, no new *essential* harmonic polynomial maps are to be found here. By looking at dimension formulae we see that the only essential examples in this study are as follows: the $U(2)$ -module $X_{1,1}$ defines the Hopf map $S^3 \rightarrow S^2$; the $Sp(2)$ module $H_{1,0}$ gives the Hopf map $S^7 \rightarrow S^4$. The remaining Hopf map $S^{15} \rightarrow S^8$ is associated to the action of $\text{Spin}(9)$.

4. The exceptional action of $\text{Spin}(9)$. Let C be the real division algebra of Cayley numbers. Elements of C may be represented as pairs of quaternions with multiplication

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma - \bar{\delta}\beta, \delta\alpha + \beta\bar{\gamma}).$$

Then S^{15} is the unit sphere in C^2 and there is the Hopf fibration $\sigma: S^{15} \rightarrow S^8$ given by homogeneous harmonic polynomials:

$$\sigma(c_1, c_2) = (c_1\bar{c}_1 - c_2\bar{c}_2, 2c_1\bar{c}_2).$$

The subgroup of $SO(16)$ which preserves the fibres of σ is the (real) spin representation of $\text{Spin}(9)$. The latter acts transitively on S^{15} with isotropy a non-standard $\text{Spin}(7)$. The $\text{Spin}(9)$ action itself is the isotropy action in the (non-compact) rank one symmetric space $F_4/\text{Spin}(9) = P^2C$, the latter being the hyperbolic Cayley plane. (The basic reference is Mostow [14, ch. 19]).

C is also intimately related to the principle of triality. Recall that there are three inequivalent irreducible eight-dimensional representations of $\text{Spin}(8)$. These are: the $SO(8)$ representation v , and the $\pm 1/2$ spin representations $\rho^\pm$. If we assume all representations take place on C , they may be related by several identities discovered by van der Blij and Springer. A convenient one is [18, p. 161]

$$(4.1) \quad v(g)(x\bar{y}) = \rho^+(g)(x)\overline{\rho^-(g)(y)}$$

for all $g \in \text{Spin}(8)$, $x, y \in C$. This will be useful for us since $\text{Spin}(8)$ is the stabilizer of the fibre of σ through the point $(1, 0)$, and acts on the Cayley lines through $(1, 0)$ and $(0, 1)$ via ρ^+ and ρ^- respectively.

Finally, we need some facts about G_2 , the automorphism group of C . Let B be any subalgebra of C which is isomorphic to the quaternions, and let $B^\perp$ be

its ortho-complement. Then the reflection τ which is $+1$ on B and -1 on $B^\perp$ is an automorphism of C , and any element of G_2 is a product of such reflections (see Jacobson [8, pp. 66–67]). Furthermore, any two such reflections are conjugate in G_2 [8, p. 69].

We now define some $\text{Spin}(9)$ -invariant spaces of polynomials. All spaces will be assumed complex. They will be generated from certain C -valued polynomials by taking the eight real components of each. Since the resulting structure has similarities to that obtained for $Sp(n) \times Sp(1)$, we will use some of the same notation (although the spaces involved are of course quite different).

Definition 4.2. Let $V_{1,0}$ be generated by the quadratics $c_i \bar{c}_i$. Let $V_{0,1}$ be generated by the quadratics $c_i \tau(c_i)$, where τ is the reflection in a quaternionic subalgebra.

Proposition 4.3. $\text{Spin}(9)$ stabilizes $V_{1,0}$ and $V_{0,1}$. Furthermore, $V_{0,1}$ consists entirely of harmonics.

Proof. First note that $V_{1,0} = R^2 \oplus H_{1,0}$, where $H_{1,0}$ is the space of harmonics generated by the components of σ . But σ is exactly the map

$$\text{Spin}(9)/\text{Spin}(7) \rightarrow \text{Spin}(9)/\text{Spin}(8),$$

hence $H_{1,0}$ is *de facto* $\text{Spin}(9)$ -invariant.

To show that $V_{0,1}$ consists of harmonics it suffices to consider polynomials $c \tau(c)$. Verification can be done explicitly in the case

$$p_0(c) = \overline{c \tau_0(c)}$$

where

$$\tau_0(\alpha, \beta) = (\alpha, -\beta).$$

However, for any other $\tau = g \tau_0 g^{-1}$, with $g \in G_2$, we have

$$\overline{c \tau(c)} = g(p_0(g^{-1}(c))).$$

Hence our new polynomial is also harmonic.

We now claim that $V_{0,1} = (H_{1,0})^\perp$ in L_2 (within the space of quadratic harmonics). Elementary considerations show that polynomials $c_i \tau(c_i)$ are perpendicular to $H_{1,0}$. For the remaining elements in $V_{0,1}$, one verifies first that $\bar{p}_0 = c_1 \tau_0(c_2)$ is perpendicular and proceeds as before. Hence it suffices to show that $V_{0,1} \oplus H_{1,0}$ span the harmonic quadratics. To see this, we write down some reflections.

Let ϵ be one of the quaternions (i, j, k) and let δ be taken from the complement. If we define

$$\tau^\pm(\alpha, \beta) = (\epsilon \alpha \bar{\epsilon}, \pm \epsilon \beta \bar{\epsilon}),$$

then τ^+ is the reflection in the algebra generated by $(\epsilon, 0)$ and $(0, \epsilon)$ and τ^- is the reflection in the algebra generated by $(\epsilon, 0)$ and $(0, \delta)$. Now write down

the polynomials determined by the $\tau^\pm$ and τ_0 . The classification of the quadratics in terms of quaternionic polynomials given in Definition 2.1 and remarks thereafter shows that $V_{0,1} \oplus H_{1,0}$ does indeed span. This concludes the proof.

Let us put $H_{0,1} = V_{0,1}$ and find the highest weights $\wedge_{1,0}$ and $\wedge_{0,1}$ for the spaces $H_{1,0}$ and $H_{0,1}$. Let $T = \{(\lambda_1, \dots, \lambda_4)\}$ with $\lambda_i = e^{2\pi i x_i}$ be the standard maximal torus in $SO(8)$. Then $\tilde{T} = v^{-1}(T)$ is the standard maximal torus for $\text{Spin}(8)$. If weights are ordered by $x_1 > \dots > x_4$, then $\wedge_+ = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$ is the highest weight for $\rho^\pm$, and $\wedge_0 = (1, 0, 0, 0)$ is that for v . Considering $\tilde{T}$ as a torus in $\text{Spin}(9)$ we have

Lemma 4.4. $\wedge_{1,0} = \wedge_0$; $\wedge_{0,1} = 2\wedge_+$.

Proof. The first is essentially immediate from the triality identity (4.1) and remarks following. The highest weight vector in this case is the first complex component of $c_1 \bar{c}_2$. The second is a consequence of the fact that $\wedge_+$ is the highest weight for the spin representation of $\text{Spin}(9)$. Since this representation is real, the dual linear form on $\mathbf{R}^{10}$ corresponding to the highest weight vector also has (highest) weight $\wedge_+$. Thus the highest weight in the quadratic polynomials is $2\wedge_+$. The highest weight vector in this case is a harmonic polynomial in the c_1 variable alone.

The analogue of Theorem 2.4 now follows in the $\text{Spin}(9)$ case, with spaces of harmonics $H_{r,s}$ and $\tilde{H}_{r,s}$ arising as in Definitions 2.2, 2.3. The highest weights are $\wedge_{r,s} = (r+s, s, s, s)$ and $\tilde{\wedge}_{r,s} = \wedge_{r,s} + \wedge_+$. From the Weyl formula we get

$$(4.5) \quad \dim H_{r,s} = \frac{(2r+2s+7)(2s+5)(2s+3)(2s+1)}{7 \cdot 5^2 \cdot 4^2 \cdot 3} \cdot \binom{r+3}{3} \binom{r+2s+6}{3} \binom{2s+4}{3}.$$

For $\dim \tilde{H}_{r,s}$ replace $2s$ by $2s+1$ everywhere.

To determine the zonal functions in these spaces, we let $L_k(c_1)$ denote the Legendre polynomial of degree k on $\mathbf{R}^8 = C$. It follows that $L_{2s} \in H_{0,s}$ and $L_{2s+1} \in \tilde{H}_{0,s}$. The reason is that the highest weight polynomial for either of these spaces is a harmonic in the coordinates of c_1 (cf. Lemma 4.4), and $\text{Spin}(8)$ acts irreducibly on such spaces. On the other hand, since the isotropy $\text{Spin}(7) \subset \text{Spin}(8)$ acting on the first Cayley coordinate clearly leaves each L_k invariant, we have

Proposition 4.6. *The unique zonal in $H_{r,s}$ (resp. $\tilde{H}_{r,s}$) has the form $P(|c_1|^2, |c_2|^2)L_{2s}$ (resp. PL_{2s+1}), where P has the same form as in Prop. 2.8. In particular, this zonal is real-valued.*

Hence by Lemma 1.3 we have

Corollary 4.7. *The representations $H_{r,s}$ and $\tilde{H}_{r,s}$ are real.*

Finally, arguments similar to those in Section 3 will demonstrate

- Proposition 4.8.** *i) $\tilde{H}_{r,s}$ yields a harmonic minimal embedding of S^{15} .
 ii) $H_{r,s}$ yields a harmonic map of S^{15} whose image is a minimally embedded*
- a) $\mathbf{RP}^{15}$ if $s > 0$,
 - b) S^8 if $s = 0$, r odd,
 - c) $\mathbf{RP}^8$ if $s = 0$, r even.

APPENDIX

The Spherical Representations of $Sp(n)$.

Diagram A. Complex harmonics of degree $2k$. Each dash in an $\tilde{H}_{r,s}$ column represents an $Sp(n)$ -module isomorphic to $I_{r,s}$.

Diagram B. Complex harmonics of degree $2k + 1$. Each dash in an $\tilde{H}_{r,s}$ column represents an $Sp(n)$ -module isomorphic to $\tilde{I}_{r,s}$.

Diagram C. Real harmonics of degree $2k$. Each dash in an $H^{\mathbf{R}}_{r,s}$ column represents an $Sp(n)$ module isomorphic to $I_{r,s}$.

$$\begin{array}{ccccccc}
 & & & & & \overline{\hspace{1.5cm}} & \left. \begin{array}{c} \overline{\hspace{1.5cm}} \\ \overline{\hspace{1.5cm}} \end{array} \right\} X^{\mathbf{R}}_{2k,0} = X^{\mathbf{R}}_{0,2k} \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & \left. \begin{array}{c} \overline{\hspace{1.5cm}} \\ \overline{\hspace{1.5cm}} \end{array} \right\} X^{\mathbf{R}}_{k+2,k-2} = X^{\mathbf{R}}_{k-2,k+2} \\
 & & & & & \overline{\hspace{1.5cm}} & \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & \left. \begin{array}{c} \overline{\hspace{1.5cm}} \\ \overline{\hspace{1.5cm}} \end{array} \right\} X^{\mathbf{R}}_{k+1,k-1} = X^{\mathbf{R}}_{k-1,k+1} \\
 & & & & & \overline{\hspace{1.5cm}} & \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & X^{\mathbf{R}}_{k,k} \\
 \overline{\hspace{1.5cm}} & \overline{\hspace{1.5cm}} & \overline{\hspace{1.5cm}} & \dots & \overline{\hspace{1.5cm}} & & \\
 H^{\mathbf{R}}_{k,0} & H^{\mathbf{R}}_{k-1,1} & H^{\mathbf{R}}_{k-2,2} & & H^{\mathbf{R}}_{0,k} & &
 \end{array}$$

Diagram D. Real harmonics of degree $2k + 1$. Each dash in an $\tilde{H}^{\mathbf{R}}_{r,s}$ column represents an $Sp(n)$ module isomorphic to $\tilde{I}_{r,s}$.

$$\begin{array}{ccccccc}
 & & & & & \overline{\hspace{1.5cm}} & X^{\mathbf{R}}_{2k+1,0} = X^{\mathbf{R}}_{0,2k+1} \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & \left. \begin{array}{c} \overline{\hspace{1.5cm}} \\ \overline{\hspace{1.5cm}} \end{array} \right\} X^{\mathbf{R}}_{k+2,k-1} = X^{\mathbf{R}}_{k-1,k+2} \\
 & & & & & \overline{\hspace{1.5cm}} & \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & X^{\mathbf{R}}_{k+1,k} = X^{\mathbf{R}}_{k,k+1} \\
 & & & & & \overline{\hspace{1.5cm}} & \\
 & & & & & \vdots & \\
 & & & & & \overline{\hspace{1.5cm}} & \\
 \overline{\hspace{1.5cm}} & \overline{\hspace{1.5cm}} & \overline{\hspace{1.5cm}} & \dots & \overline{\hspace{1.5cm}} & & \\
 \tilde{H}^{\mathbf{R}}_{k,0} & \tilde{H}^{\mathbf{R}}_{k-1,1} & \tilde{H}^{\mathbf{R}}_{0,k} & & & &
 \end{array}$$

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