Transformations of Flows by Discrete Random Measures

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0. Introduction. Let $\mathfrak{D} = (\Omega, \mathfrak{F}, P, \theta_t)$ be a dynamical system and $\alpha(dt, \omega)$ a homogeneous random measure on the real line $\mathbf{R}$, i.e. $\alpha(B, \omega)$ is measurable for each $B \in \mathfrak{B}$, is finite on compacts and homogeneous relative to $\mathfrak{D}$: for almost every $\omega \in \Omega$,

$$(1) \quad \alpha(B + t, \omega) = \alpha(B, \theta_t \omega), \quad t \in \mathbf{R}, \quad B \in \mathfrak{B},$$

with $\mathfrak{B}$ the Borel σ -field on $\mathbf{R}$. A random time $\tau : \Omega \rightarrow \mathbf{R}$ will be called a “translation time” if the transformation $\theta_\tau : \Omega \rightarrow \Omega$ given by $\theta_\tau(\omega) = \theta_{\tau(\omega)}(\omega)$ preserves $\hat{P}_\alpha$ -measure, $\hat{P}_\alpha$ being the Palm measure of α (all terminology will be explained momentarily). We deal in this paper with the question of when translation times exist.

Solutions to this problem are known in two special cases. The first is when α is a *counter*, i.e. for each ω , α places unit mass at time points $R_n(\omega)$ having no finite accumulation point, with $\cdots < R_{-1} < R_0 \leq 0 < R_1 < \cdots$; then each R_n , $n \in \mathbf{Z}$, is a translation time and $\hat{\theta}_n = \theta_{R_n}$ is a (discrete-time) flow on $\Omega_0 = \{\omega : R_0(\omega) = 0\}$ relative to $\hat{P}_\alpha$, [13], [14], [15]. This set-up is called a *stationary point process* and is the classical one of the subject. The case of a marked point process is included here since, if L is a subset of the space of marks, the customary Palm measure $\hat{P}_L$ is defined as the Palm measure of the counter ν_L which places unit mass at each time that a mark from L appears (see [6], [10]). The second case is when α is a *continuous* homogeneous measure, meaning that, a.s. α has no point masses. If, for $t \geq 0$,

$$\hat{\alpha}(t, \omega) = \inf \{s > 0 : \alpha((0, s], \omega) > t\}$$

(a similar definition is given for $t < 0$), then each $\hat{\alpha}(t)$ is a translation on time, and $\hat{\theta}_t = \theta_{\hat{\alpha}(t)}$ defines a flow $\Omega_0 = \{\omega : \hat{\alpha}(0, \omega) = 0\}$ relative to $\hat{P}_\alpha$ [9], [16]. In both cases there is an inversion formula retrieving P from $\hat{P}_\alpha$, and, if $\alpha(\mathbf{R}_+) = \infty$ a.s., the “dual” dynamical system $\hat{\mathfrak{D}} = (\Omega_0, \mathfrak{F}_0, \hat{P}_\alpha, \hat{\theta})$, where $\mathfrak{F}_0 = \mathfrak{F} \cap \Omega_0$, shares with $\mathfrak{D}$ the property of ergodicity. (These and related questions are discussed in [4].)

Coming to the general case, Logan, in his dissertation [8], constructed a family of operators L_t , $t \in \mathbb{R}$, taking the space of bounded random variables into itself, such that $\int L_t \xi d\hat{P}_\alpha = \int \xi d\hat{P}_\alpha$. However, Logan's transformations do not in general form a group of operators, and—more important from our point of view—are not induced by point transformations, in particular by θ_τ for a random time τ . Indeed, we shall exhibit situations in which no translation times exist.

Before describing our results, let us recall some definitions. A *dynamical system* $\mathfrak{D} = (\Omega, \mathcal{F}, P, \theta_t)$ consists of a probability space $(\Omega, \mathcal{F}, P)$ equipped with a (measurable) flow $\theta = (\theta_t)$, $t \in \mathbb{R}$, i.e., a one-parameter group of bi-measurable, measure-preserving transformations $\theta_t : \Omega \rightarrow \Omega$ such that $\theta_0 = \text{identity}$, and the mapping $(t, \omega) \rightarrow \theta_t(\omega)$ is $\mathcal{B} \times \mathcal{F}/\mathcal{F}$ -measurable. *Homogeneous random measures* α (we omit “random” hereafter) are described as at (1); we mention for purposes of reference that, defining $\alpha_t = \alpha(0, t]$, $t \geq 0$, $= -\alpha(t, 0]$, $t < 0$, we obtain an *additive functional*. If $\alpha(0, 1]$ is integrable, we call α *integrable*; in that case $E\alpha(B) = \lambda(B)E\alpha(0, 1]$ for $B \in \mathcal{B}$, where λ is Lebesgue measure. We can assume (1) holds for all ω ([4]).

Let $\Delta_0(\omega) = \alpha(\{0\}, \omega)$ be the mass on the point $t = 0$, so $\Delta_t = \Delta_0 \circ \theta_t$ gives the mass at t . Define $\hat{\Omega} = \{\omega : \Delta_0(\omega) > 0\}$. It is easy to see that $P(\hat{\Omega}) = 0$. The *Palm measure* of α is defined by

$$(2) \quad \hat{P}_\alpha(A) = E \int_0^1 I_A \circ \theta_t \alpha(dt), \quad A \in \mathcal{F}.$$

It is always σ -finite, and is finite if and only if α is integrable. A monotone class argument shows that for any non-negative random variable X , writing $\hat{E}_\alpha$ for $\hat{P}_\alpha$ -integration, we have

$$(2') \quad \hat{E}_\alpha(X) = E \int_0^1 X \circ \theta_t \alpha(dt).$$

The measure $\hat{P}_\alpha$ is always concentrated on $\{\omega : 0 \in \text{support of } \alpha(\cdot, \omega)\}$; if α is continuous ($\Delta_t \equiv 0$ a.s.), then $\hat{P}_\alpha$ is actually concentrated on $\{\omega : \alpha((0, t], \omega) > 0 \text{ for all } t > 0\}$. We shall be mostly concerned with α of *discrete type*, in which case, for each ω , $\alpha(\cdot, \omega)$ is purely atomic. $\hat{P}_\alpha$ is then concentrated on $\hat{\Omega}$ to which we may confine our attention for most purposes. Homogeneous measures of discrete type include counters and “multiple point processes” (point processes allowing arbitrary positive *integer* masses) as special cases, but it should be emphasized that we are *allowing* finite accumulation points for the mass locations and arbitrary non-negative mass sizes, both, of course, depending in general on ω .

Our results can now be stated roughly as follows. In §1 we distinguish two cases: (1) the number of masses in finite intervals is finite (α “counter-based”), (2) the number may be infinite. If α is counter-based and the *mean* number of masses per unit time is finite, we will show there are no translation times unless, with probability 1, the mass sizes repeat infinitely often, and then $\tau = \inf \{t > 0 : \Delta_t = \Delta_0\}$ is the (essentially unique) translation time.

In case (2), or when the mean number of masses per unit time is infinite, we decompose α into a sum of counter-based measures α_n . Under a "local integrability" condition, we show that τ above is a translation time whenever it is *a.s.* finite. Moreover, there are no "compatible" translation times unless repetition occurs, in which case τ is again essentially unique. The compatibility condition, explained in §1.2, is clearly undesirable, but we do not as yet know whether "incompatible" translation times exist even in various simple cases. When Δ_0 is discrete (*e.g.* α is a "multiple point process"), τ is a (finite) translation time regardless of local integrability. Section 1 closes with a new inversion formula for retrieving P from $\hat{P}_\alpha$, several applications of it, and a short discussion of the existence of translation times for homogeneous measures of mixed type.

Ergodic properties are taken up in §2. As we shall see, if the masses repeat, ergodicity does *not* carry over to the $\hat{P}_\alpha$ -preserving transformation θ_τ , unless $\Delta_0 = \text{constant}$ $\hat{P}_\alpha$ -*a.s.*, returning us to the case when α is continuous or (essentially) a counter. In fact, if Δ_0 is discrete under $\hat{P}_\alpha$, and P is ergodic, the σ -field of θ_τ -invariant sets in $\hat{\Omega}$ coincides up to null sets with that generated by Δ_0 . Finally we establish a lemma about return times in ergodic systems (which may be of interest in its own right) and use it to find conditions for the existence of a finite set G (countable set G_∞) such that $\Delta_t(\omega) \in G$ for all t ($\in G_\infty$ for all t) P -*a.s.* For α counter-based, P ergodic, and the mean number of masses per unit time finite, the condition for a finite set G is that $M^*(R_1 - R_0)^{-1}$ be integrable where M^* is the number of atoms passed until repetition of the first mass prior to the origin, and $R_1 - R_0$ is the length of the largest "atomless" interval about the origin. For α locally integrable, but not counter-based, a similar condition is sufficient for a countable set G_∞ .

1. Existence of translation times. Let $\alpha(dt, \omega)$ be a homogeneous measure of discrete type on $\mathfrak{D} = (\Omega, \mathfrak{F}, P, \theta_t)$. A point t such that $\Delta_t(\omega) > 0$ is an *atom* of $\alpha(dt, \omega)$. For each ω , there are countably many atoms. The measure $\nu(dt, \omega) = \Delta_t^{-1}(\omega)\alpha(dt, \omega)$, which puts unit mass at each atom of α , is the *base measure* of α . It has the homogeneity property (1), but need not be finite on compacts. If ν is finite on compacts, it will be a counter, and we call α *counter-based*. We will treat this case first, but before doing so, let us note

(3) Lemma. Suppose a random variable $\tau : \Omega \rightarrow \mathbf{R}$ is such that θ_τ preserves $\hat{P}_\alpha$ -measure. Then, $\hat{P}_\alpha$ -*a.s.*, τ is an atom of α .

Such a τ is a *translation time*. To prove (3), recall that $\hat{P}_\alpha$ is concentrated on $\hat{\Omega} = \{\Delta_0 > 0\}$, hence $\hat{P}_\alpha\{\Delta_\tau = 0\} = \hat{P}_\alpha\{\Delta_0 = 0\} = 0$. Notice that, if τ is a (putative) translation time, its values off $\hat{\Omega}$ are immaterial, so we shall take $\tau = 0$ off $\hat{\Omega}$.

Note. Set $L_\alpha = \{\alpha(\mathbf{R}_+, \omega) = \alpha(-\mathbf{R}_+, \omega) = \infty\}$, $\mathbf{R}_+ = [0, \infty)$. Then L_α is θ -invariant, $\omega \in L_\alpha$ $\hat{P}_\alpha$ -*a.s.*, and there will be no harm in §1 in assuming $L_\alpha = \Omega$. (See [4] and §2).

1.1 α Counter-based. In this case we enumerate the atoms of α (= atoms of ν) as $R_i, j \in \mathbf{Z}; \dots < R_{-1} < R_0 \leq 0 < R_1 < \dots$. Put $\hat{\Delta}_i = \Delta_{R_i}$ ($= \Delta_0 \circ \theta_{R_i}$) and $\hat{\theta}_i = \theta_{R_i}$. Since ν is a counter, the standard Palm theory [4], [14] applies; in particular, $\hat{\theta}_i$ preserves $\hat{P}_\nu$ for each $j \in \mathbf{Z}$. In addition,

$$(4) \quad d\hat{P}_\alpha = \Delta_0 d\hat{P}_\nu$$

and

$$(5) \quad E\xi = \hat{E}_\nu \int_0^{R_1} \xi \circ \theta_t dt$$

for any non-negative random variable ξ . Equation (4), which is immediate from (2) since $\alpha(dt) = \Delta_t \nu(dt)$, implies that $\hat{P}_\alpha$ and $\hat{P}_\nu$ are equivalent, *i.e.* have the same null sets. Equation (5) is standard [14] and follows from

$$(6) \quad E \int_{\mathbf{R}} u(s, \omega) \beta(ds, \omega) = \hat{E}_\beta \int_{\mathbf{R}} u(s, \theta_{-s} \omega) ds,$$

valid for any homogeneous measure $\beta(dt, \omega)$ and non-negative $\mathfrak{B} \times \mathfrak{F}$ -measurable function $u(s, \omega)$ [4], by choosing $\beta = \nu$ and, after replacing s by $-s$ in the right member, $u(s, \omega) = \xi(\omega) \times I_{(R_{-1}(\omega), 0)}(s)$. From (5) we obtain $E(R_1 - R_0)^{-1} = E_\nu 1 = E_\nu(0, 1] \leq \infty$ since $0 < s < R_1$ implies $R_1 \circ \theta_s = R_1 - s$ and $R_0 \circ \theta_s = -s$. Thus the “intensity” (= mean number of atoms per unit time) of ν equals the mean reciprocal of the length of the largest atomless interval about $t = 0$. (Curiously, however, $ER_1^{-1} = \infty$ always.)

Let τ be a translation time. Because of (3), τ is of the form R_N for some (random) integer N , $\hat{P}_\alpha$ -a.s. This leads us to define a random variable $N : \hat{\Omega} \rightarrow \mathbf{Z}$ to be a *translation index* if $\hat{\theta}_N = \theta_{R_N}$ preserves $\hat{P}_\alpha$ -measure, *i.e.* if R_N is a translation time. By an earlier remark, we may extend the definition of R_N to all of Ω by letting it be zero off $\hat{\Omega}$; however, in the counter-based situation, $R_0 + R_N \circ \theta_{R_0}$ gives an alternate method (which we shall not pursue further). In what follows, $\hat{\theta}_i$ will denote the restriction of $\hat{\theta}_i$ to $\hat{\Omega}$ and the underlying measure space will be $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}_\alpha)$ with $\hat{\mathfrak{F}} = \mathfrak{F} \cap \hat{\Omega}$. Here and in §1.2 we allow translation indices to take the value $+\infty$ on a set of measure zero.

(7) Theorem. *A random variable $N : \hat{\Omega} \rightarrow \mathbf{Z}$ is a translation index iff*

$$(8) \quad \hat{\Delta}_0 = \sum_{k \in \mathbf{Z}} \hat{\Delta}_{-k} I_{\{N(\hat{\theta}_{-k})=k\}} \quad \hat{P}_\alpha - a.s. .$$

Applying $\hat{\theta}_i$ to both sides of (8), this is the same as

$$\hat{\Delta}_i = \sum_{k \in \mathbf{Z}} \hat{\Delta}_{i-k} I_{\{N(\hat{\theta}_{i-k})=k\}} \quad \text{for all } j \in \mathbf{Z} \quad \hat{P}_\alpha - a.s. .$$

For the proof of (7), let $0 \leq \xi$ be $\mathfrak{F}$ -measurable. Then,

$$\begin{aligned}
\hat{E}_\alpha \xi \circ \hat{\theta}_N &= \sum_{k \in \mathbf{Z}} \hat{E}_\alpha (\xi \circ \hat{\theta}_k I_{\{N=k\}}) \\
&= \sum_{k \in \mathbf{Z}} \hat{E}_\nu (\hat{\Delta}_0 \xi \circ \hat{\theta}_k I_{\{N=k\}}) \quad \text{--- (4)} \\
&= \sum_{k \in \mathbf{Z}} \hat{E}_\nu (\hat{\Delta}_{-k} \xi I_{\{N \circ \hat{\theta}_{-k} = k\}}) \\
&\equiv \hat{E}_\nu \xi \tilde{Z},
\end{aligned}$$

where $\tilde{Z} = \sum_{k \in \mathbf{Z}} \hat{\Delta}_{-k} I_{\{N \circ \hat{\theta}_{-k} = k\}}$.

Now $\hat{\theta}_N$ preserves $\hat{P}_\alpha$ iff $\hat{E}_\nu \xi \tilde{Z} = \hat{E}_\alpha \xi = \hat{E}_\nu \xi \Delta_0$ for all ξ as described, hence iff $\tilde{Z} = \Delta_0$.

Applying (7) directly to any counter ν , we obtain a special case of a result due to Neveu [12]:

(9) Corollary. *Let $N : \hat{\Omega} \rightarrow \mathbf{Z}$. Then $\hat{\theta}_N$ preserves $\hat{P}_\nu$ -measure iff*

$$1 = \sum_{k \in \mathbf{Z}} I_{\{N \circ \hat{\theta}_{-k} = k\}} \quad a.s. .$$

As an application of (7), suppose the range of Δ_0 is in a fixed countable set $B_0 = \{b_1, b_2, \dots\}$, $b_k > 0$, $\hat{P}_\alpha$ -a.s.. (If $b_n = n$ we have a "multiple point process".) Put $\nu_k(dt) = b_k^{-1} I_{[\Delta_t = b_k]} \alpha(dt)$: this is the number of atoms with mass b_k during time dt . It is well-known [4], that either $\nu_k(\mathbf{R}_+) = \nu_k(-\mathbf{R}_+) = \infty$ or $\nu_k(\mathbf{R}_+) = \nu_k(-\mathbf{R}_+) = 0$ except on a P -null strictly invariant set A_0 . Consequently $\hat{P}_\alpha(A_0) = 0$. Let

$$D_k = \{\nu_k(\mathbf{R}_+) = \nu_k(\mathbf{R}_-) = \infty\}$$

and

$$\begin{aligned}
\tau(\omega) &= \inf \{t > 0 : \Delta_t(\omega) = \Delta_0(\omega)\} \\
&= \infty \quad (\text{if the indicated set is empty}).
\end{aligned}$$

Because of the above dichotomy (or simply because $L_\alpha = \Omega$), $\hat{\Omega} \subset \cup D_k \hat{P}_\alpha$ -a.s. so $0 < \tau < \infty$ $\hat{P}_\alpha$ -a.s., and it is immediate that (8) holds where N is the index for which $\tau = R_N$ (cf. (3)). Thus, "first mass repetition time" is a translation time. Note, in comparison with Theorem (10) below, that no integrability assumptions are made in this case.

From now on we consider only translation indices with values in $\mathbf{N}$, the set of positive integers. Define a function $M : \hat{\Omega} \rightarrow \mathbf{N} \cup \{\infty\}$ by

$$\begin{aligned}
M &= \min \{n \in \mathbf{N} : \hat{\Delta}_n = \Delta_0\} \\
&= \infty \quad \text{if the above set is empty.}
\end{aligned}$$

Let $\beta = \hat{P}_\alpha\{M = \infty\}$. The main result of this section is:

(10) Theorem. *Assume ν is integrable. Then: (a) if $\beta > 0$, there are no translation indices. (b) If $\beta = 0$, then M is the unique translation index except for random iterations.*

Note. The meaning of (b) is that, if N is a translation index, then $\hat{\theta}_N = (\hat{\theta}_M)^k$ for some $k = k(\omega)$. Of course, a given random iteration $(\hat{\theta}_M)^k$, $k = k(\omega)$, may or may not preserve measure.

Proof. (a) Let $N : \hat{\Omega} \rightarrow \mathbf{N}$ be a translation index. Since $\Delta_0 > 0$ a.s., we find from (8) that

$$(11) \quad \bigcup_{k \in \mathbf{N}} [N \circ \hat{\theta}_{-k} = k] = \hat{\Omega} \quad \text{a.s.}$$

(Recall from (4) that “a.s.” may refer to $\hat{P}_\alpha$ or $\hat{P}_\nu$ indifferently.)

Next, $\sum_{k \in \mathbf{N}} \hat{P}_\nu \{N \circ \hat{\theta}_{-k} = k\} = \hat{P}_\nu \{N < \infty\} = \hat{P}_\nu(\hat{\Omega}) < \infty$ (ν being integrable) implies that the events $\{N \circ \hat{\theta}_{-k} = k\}$ are “almost” pairwise disjoint, i.e. except for null sets. Hence $1 = \sum_{k \in \mathbf{N}} I_{[N \circ \hat{\theta}_{-k} = k]}$ a.s. and $\hat{\theta}_N$ preserves $\hat{P}_\nu$ by (9) (the replacement of $\mathbf{Z}$ by $\mathbf{N}$ causes no trouble). Since $\hat{P}_\alpha$ and $\hat{P}_\nu$ are both $\hat{\theta}_N$ -invariant, (4) shows that $\Delta_0 = \Delta_0 \circ \hat{\theta}_N = \hat{\Delta}_N$ a.s., so $M \leq N$ a.s. and $\beta = 0$.

(b) From the definition of M , one sees that the sets $\{M \circ \hat{\theta}_{-k} = k\}$ are pairwise disjoint. Now $\sum_{k \in \mathbf{N}} \hat{P}_\nu \{M \circ \hat{\theta}_{-k} = k\} = \hat{P}_\nu \{M < \infty\} = \hat{P}_\nu(\hat{\Omega})$ because $\beta = 0$ implies $\hat{P}_\nu(M = \infty) = 0$, so we conclude (11) holds for M , hence also (8) holds for M , and M is a translation index. If N is any translation index, then, arguing as in (a), $\Delta_0 = \Delta_0 \circ \hat{\theta}_N$ which implies the desired result.

(12) Corollary. $\hat{\theta}_M$ preserves $\hat{P}_\nu$ if $\beta = 0$.

Define $M_1 = M$ and $M_{n+1} = M_n + M \circ \hat{\theta}_{M_n}$ when $\beta = 0$. Then we have

(13) Corollary. (a) Each M_n is a translation index. (b) $\hat{P}_\alpha \{M_n = \infty \text{ for some } n\} = 0$.

Part (a) is trivial since $\hat{\theta}_{M_n} = (\hat{\theta}_{M_1})^n$. Proceeding by induction in (b) we have $\hat{P}_\alpha \{M_{n+1} = \infty\} = \hat{P}_\alpha \{M \circ \hat{\theta}_{M_n} = \infty\}$ (induction hypothesis) $= \beta = 0$, which yields the result.

Notice that M_n is the n^{th} time that the mass at zero is repeated; part (b) says that each mass size is repeated infinitely often. Finally we remark that $N \equiv 1$ is a translation index iff $\Delta_0 = \hat{\Delta}_j$ for every $j \in \mathbf{Z}$.

1.2 The General Discrete Case. In this section we first extend some of the results of §1.1 to an arbitrary homogeneous measure $\alpha(dt, \omega)$ of discrete type. Next, we give a new inversion formula for P in terms of $\hat{P}_\alpha$, and use it to investigate a special class of homogeneous measures including all multiple point processes. We conclude with several examples.

In the present case, the base measure $\nu(dt, \omega) = \Delta_t^{-1}(\omega) \alpha(dt, \omega)$ is no longer assumed to be a counter, i.e., finite on compacts. To enumerate the atoms of α , we introduce a partition $(B_n)_{n \in \mathbf{N}}$ of $(0, \infty)$, $B_n \in \mathfrak{B}$, each B_n bounded away from 0 and ∞ . Since α is finite on compacts, the set $\{t \in \mathbf{R} : \Delta_t(\omega) \in B_n\}$ is locally finite for each n so we can order its points: $\dots < R_{-1}^n(\omega) < R_0^n(\omega) \leq 0 < R_1^n(\omega) < \dots$. Define

$$(14) \quad \begin{cases} \nu_n(B, \omega) = \int_B (\Delta_t(\omega))^{-1} I_{B_n}(\Delta_t(\omega)) \alpha(dt, \omega) \\ \alpha_n(B, \omega) = \int_B \Delta_t(\omega) \nu_n dt, \omega, \quad B \in \mathfrak{B}. \end{cases}$$

Clearly, $\nu_n(B, \omega)$ counts the number of atoms in B of mass $s \in B_n$ and is finite on compacts. Moreover, $\nu = \sum \nu_n$, $\alpha = \sum \alpha_n$, and $\hat{P}_\alpha = \sum \hat{P}_{\alpha_n}$. It is easy to see that

(15) Lemma. α_n is integrable iff ν_n is integrable.

We will say that α is *locally integrable* if, for each $0 < a < b < \infty$, the homogeneous measure $I_{(a,b)}(\Delta_t(\omega)) \alpha(dt, \omega)$ is integrable.

Because of (3), we can again restrict our attention to translation indices: a measurable function $N : \hat{\Omega} \rightarrow \mathbf{N} \times \mathbf{Z}$ is a *translation index* if $\hat{\theta}_N$ preserves $\hat{P}_\alpha$, where we write $\hat{\theta}_N = \hat{\theta}_{(n,i)} = \theta_{R^{n,i}}$ when $N = (n, j)$. Put $\hat{\Omega}_n = \{\Delta_0 \in B_n\}$, noting for later use that $\hat{P}_{\alpha_n}$ is concentrated on $\hat{\Omega}_n$.

Denote by $\Delta^{(n)}_i$ the function $\Delta^{(n)}_0 \circ \theta_{R^{n,i}}$ where $\Delta^{(n)}_0 = \Delta_0 I_{\hat{\Omega}_n}$, and define $M : \hat{\Omega} \rightarrow (\mathbf{N} \times \mathbf{N}) \cup \{\infty\}$, the index of first mass repetition, by

$$M(\omega) = \begin{cases} (n, j) & \text{if } \omega \in \hat{\Omega}_n \text{ and } j = \min \{i \in \mathbf{N} : \Delta^{(n)}_i = \Delta^{(n)}_0\} \\ \infty & \text{if the above set is empty.} \end{cases}$$

Let $\beta = \hat{P}_\alpha(M = \infty)$. We have

(15) Theorem. If α is locally integrable and $\beta = 0$, then M is a translation index.

Proof. Since $\beta = 0$, we can define a *finite* random variable M_n on $\hat{\Omega}_n$ by $M_n = \inf \{j \geq 1 : \Delta^{(n)}_j = \Delta^{(n)}_0\}$. By Theorem (10), M_n is a translation index in $(\hat{\Omega}_n, \hat{P}_{\alpha_n})$ for each $n \in \mathbf{N}$. This easily implies that M is a translation index.

Remarks. (i) Of course (15) is actually independent of the partition used, the conclusion being that θ_τ preserves $\hat{P}_\alpha$ where $\tau(\omega) = \inf \{t > 0 : \Delta_t(\omega) = \Delta_0(\omega)\}$.

(ii) We have been unable to determine if M is unique (up to random iterates) or if $\beta = 0$ is necessary for the existence of translation indices. However, if we restrict ourselves to *compatible* indices, meaning that $N(\hat{\Omega}_n) \subset \{n\} \times \mathbf{Z}$ a.s. for all $n \in \mathbf{N}$, then the rest of (10) carries over:

(16) Theorem. (a) A compatible $N : \hat{\Omega} \rightarrow \mathbf{N} \times \mathbf{Z}$ is a translation index iff

$$\Delta^{(n)}_0 = \sum_{i \in \mathbf{Z}} \Delta^{(n)}_0 \circ \hat{\theta}_{(n,-i)} I_{\{N \circ \hat{\theta}_{(n,-i)} = (n,i)\}} \quad \text{a.s.} \quad \hat{P}_\alpha.$$

Assuming α is locally integrable,

(b) If $\beta > 0$, there exist no compatible translation indices.

(c) If $\beta = 0$, M is unique except for random iterates.

Proof. (a) As we have indicated, N will be a translation index iff it is a translation index relative to $(\hat{\Omega}_n, \hat{P}_{\alpha_n})$ for all $n \in \mathbf{N}$. The result now follows from (7); we omit the details.

(b) Let N be a compatible translation index. Since α_n is counter-based we have $\hat{P}_{\alpha_n}(M_n = \infty) = 0$ by (10) and (15). Hence, $\beta = \hat{P}_\alpha(M = \infty) = \sum_{n \geq 1} \hat{P}_{\alpha_n}(M_n = \infty) = 0$.

(c) This obtained essentially as in (10).

Next, we turn to an inversion formula for P in the case when a translation time $\tau : \Omega \rightarrow [0, \infty)$ exists. Define the iterates of τ by $\tau_1 = \tau$, $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$; each τ_n is then a translation time.

(17) Theorem. *The measure $\tilde{P}$ on $(\Omega, \mathfrak{F})$ given by*

$$(18) \quad \tilde{P}(A) = \hat{E}_\alpha \int_0^\tau I_A \circ \theta_t dt, \quad A \in \mathfrak{F},$$

is θ -invariant, and finite iff $\hat{E}_\alpha(\tau) < \infty$. When σ -finite, $\tilde{P}$ is equivalent to P .

Notes. (i) Either of the following conditions is sufficient for σ -finiteness of $\tilde{P}$: (a) the set $\{s < 0 : s + \tau \circ \theta_s(\omega) > 0\}$ be bounded below a.s.; (b) if α is counter-based and $\tau = R_N$, where N is a translation index, then $\hat{E}_\alpha(N) < \infty$.

(ii) The method of (17) may be used to construct stationary measures for certain Markov processes; see [4].

Proof. We write $\tilde{E}$ for integration with $\tilde{P}$. Consider a $\tilde{P}$ -integrable random variable $\xi \geq 0$. We show first that $E\xi = n^{-1} \hat{E}_\alpha \int_0^{\tau_n} I_A \circ \theta_t dt$. When $n = 1$, this follows from (18) by a monotone class argument. It suffices to look at $n = 2$. We have

$$\begin{aligned} \hat{E}_\alpha \int_0^{\tau_2} \xi \circ \theta_t dt &= \hat{E}_\alpha \int_0^{\tau_1} \xi \circ \theta_t dt + \hat{E}_\alpha \int_0^{\tau \circ \theta_{\tau_1}} \xi \circ \theta_t \circ \theta_{\tau_1} dt \\ &= 2\tilde{E}\xi. \end{aligned}$$

For the next part of the proof we need the identity

$$(*) \quad E\alpha(B)\xi = \hat{E}_\alpha \int_B \xi \circ \theta_{-s} ds, \quad B \in \mathfrak{B},$$

which is true for any homogeneous measure $\alpha(dt, \omega)$ by selecting $u(s, \omega) = I_B(s)\xi(\omega)$ in (6) (see [4]).

It will be enough to show that $\tilde{E}\xi \circ \theta_t = \tilde{E}\xi (\leq \infty)$ for each t and each bounded $\xi \geq 0$ such that $E\xi\alpha(-2t, 0] < \infty$. For such $\xi : E\xi = \hat{E}_\alpha Y_1 + \hat{E}_\alpha Y_2$, where

$$Y_1 = I_{(\tau \leq t)} \int_0^\tau \xi \circ \theta_s ds, \quad Y_2 = I_{(\tau > t)} \int_0^\tau \xi \circ \theta_s ds,$$

and

$$\tilde{E}\xi \circ \theta_t = \hat{E}_\alpha X_1 + \hat{E}_\alpha X_2,$$

where

$$X_1 = I_{(\tau \leq t)} \int_t^{t+\tau} \xi \circ \theta_s ds, \quad X_2 = I_{(\tau > t)} \int_t^{t+\tau} \xi \circ \theta_s ds.$$

Observe that $\max(X_1, Y_1) \leq \int_0^{2t} \xi \circ \theta_s ds$ so that $\hat{E}_\alpha X_1 < \infty$ and $\hat{E}_\alpha Y_1 < \infty$ by choosing $B = (-2t, 0]$ in (*). A straightforward computation yields:

$$|X_2 - Y_2| \leq \int_0^t \xi \circ \theta_s \circ \theta_\tau ds + \int_0^t \xi \circ \theta_s ds,$$

and hence, since θ_τ preserves $\hat{P}_\alpha$,

$$\hat{E}_\alpha |X_2 - Y_2| \leq 2E\xi\alpha(-t, 0] < \infty.$$

This implies $\hat{E}_\alpha X_2 < \infty$ if and only if $\hat{E}_\alpha Y_2 < \infty$. Consequently $\tilde{E}\xi < \infty$ if and only if $\tilde{E}\xi \circ \theta_t < \infty$, in which case,

$$\begin{aligned} |\tilde{E}\xi - \tilde{E}\xi \circ \theta_t| &= \frac{1}{n} \left| \hat{E}_\alpha \left[\int_0^{\tau_n} \xi \circ \theta_s ds - \int_t^{t+\tau_n} \xi \circ \theta_s ds \right] \right| \\ &\leq \frac{1}{n} (2E\xi\alpha(-2t, 0] + 2E\xi\alpha(-t, 0]) \quad \text{for all } n \geq 1 \\ &= \frac{1}{n} \times \text{const.} \rightarrow 0 \quad (\text{as } n \rightarrow \infty), \end{aligned}$$

the inequality by an argument similar to that above.

Finally changing s to $-s$ on the right side of (6) and choosing $u(s, \omega) = \xi(\omega)I_{(-\tau(\theta_s \omega), 0)}(s)$; we obtain

$$\begin{aligned} \tilde{E}\xi &= E\xi \int_{\mathbf{R}} I_{(-\tau(\theta_s \omega), 0)}(s) \alpha(ds, \omega) \\ &= E\xi \eta, \end{aligned}$$

with $\eta(\omega) = \int_{\mathbf{R}} I_{(-\tau(\theta_s \omega), 0)}(s) \alpha(ds, \omega)$. Thus $\tilde{P} \ll P$, and σ -finiteness is equivalent to the $a.s.$ finiteness of η , which is implied by the condition stated in Note (i) above. Suppose $\tilde{P}$ is a σ -finite. Then η must be "almost" invariant: $\eta \circ \theta_t = \eta$ $a.s.$ for each t . It is easy to find a strictly invariant random variable $a.s.$ equal to η , which we again denote by η , so $\tilde{E}\xi = E\xi\eta$, η strictly invariant. Then $\tilde{P}\{\eta = 0\} = E(\eta; \eta = 0) = 0$, hence $\hat{E}_\alpha \int_0^\tau I_{\{\eta=0\}} \circ \theta_t dt = 0 = \hat{E}_\alpha(\tau; \eta = 0)$. Since $\tau > 0$ $\hat{P}_\alpha$ - $a.s.$, $\hat{P}_\alpha\{\eta = 0\} = 0$, and it then follows [4] that $P\{\eta = 0\} = 0$, which implies P and $\tilde{P}$ are equivalent.

The following inversion formula is now immediate.

(19) Theorem. If $\tilde{P}$ is σ -finite, $E\xi = \hat{E}_\alpha(\eta^{-1} \int_0^\tau \xi \circ \theta_t dt)$ for each random variable $\xi \geq 0$ on $(\Omega, \mathcal{F})$.

Remark. Both the "classical" inversion formulae [4], [14] are special cases of (19). Notice that the discreteness of α is not required.

Consider again the special case, discussed in §1.1 following (9), in which Δ_0 is atomic, but now we do *not* assume α is counter-based. Defining τ as before we still have $0 < \tau < \infty$ $\hat{P}_\alpha$ -a.s. and τ is a translation time. Denote by r^n_j the time of the j^{th} occurrence of the mass b_n , so $\cdots < r^n_{-1} < r^n_0 \leq 0 < r^n_1 < \cdots$. Going back to the random variable η above we find

$$\begin{aligned}\eta &= \sum_{n,j} b_n I_{(-\tau(\theta_{r^n_j}), 0)}(r^n_j) \\ &= \sum_n b_n I_{D_n},\end{aligned}$$

since the only non-zero terms in the double sum are for $j = 0$.

If $\hat{P}$ is σ -finite, then $\eta < \infty$ a.s., and we conclude: if the set B_0 is bounded away from zero, $P(D_n \text{ i.o.}) = 0$. (Thus, a.s. only a finite number of distinct masses appear.)

Finally, let us mention two examples. The first is simply an example of a measure α which is not counter-based, while the second shows the possibility of part (a) in Theorem (16) (or (10)).

Example 1. Let ν_n , $n \in \mathbf{N}$, be a family of independent Poisson processes, all of intensity 1, and $\nu = \sum \nu_n$. Also let $a_n > 0$ be such that $\sum a_n < \infty$. Then $\alpha = \sum a_n \nu_n$ is an integrable homogeneous random measure with base measure ν . A simple Borel–Cantelli argument shows that ν is not a counter.

Example 2. Let $U = [0, 1)$ and $T : U \rightarrow U$ the (Lebesgue) measure-preserving transformation $Tx = (x + \gamma) \pmod{1}$, where γ is fixed. We take $\Omega = U_1 \times U_2$, where U_i are copies of U . The σ -field $\mathfrak{F}$ is the Borel σ -field on Ω , with P being Lebesgue measure on $\mathfrak{F}$. Finally, we take θ_t to be the special flow [16] on Ω with ceiling function $\equiv 1$, and $(U_1, \mathfrak{G}_1, \lambda, T)$ as the “base space”, where $\mathfrak{G}_1$ is the Borel σ -field in U_1 , and λ is Lebesgue measure. A particle starting at $(x, y) \in \Omega$ moves with the flow θ_t according to the following intuitive description: first it moves vertically along $(x, y + t)$ until it touches the line $y = 1$, when it falls down onto U_1 , is moved over by T , and then travels vertically along (Tx, t) , etc. We define a counter ν by

$$\nu((0, t], (x, y)) = [t + y], \quad t \geq 0,$$

where $[\ ]$ means “greatest integer in”, and a similar definition is given for $t < 0$, so that $\nu(B)$ counts the number of falls suffered by the particle during the time set B . The Palm measure $\hat{P}_\nu$ in this case is just Lebesgue measure λ on U_1 ; more precisely, for $A \in \mathfrak{F}$, $\hat{P}_\nu(A) = \lambda(A \cap U_1)$. Let $\Delta_0(x)$ be a function defined on U_1 which is > 0 . We extend Δ_0 to all of Ω by $\Delta_0(x, y) = \Delta_0(x)$, and define $\alpha(B) = \int_B \Delta_0 \circ \theta_t \nu(dt)$. Then $d\hat{P}_\alpha = \Delta_0 d\hat{P}_\nu$, or, $\hat{P}_\alpha(A) = \int_{A \cap U_1} \Delta_0(x) dx$.

The successive masses of α are $\Delta_0(x)$, $\Delta_0(Tx)$, $\Delta_0(T^2x)$, $\cdots$. Thus, if γ is an irrational number and $\Delta_0(x)$ is one-one, there is no mass repetition, since T is then ergodic. (For a treatment of point processes and special flows, see [5].)

1.3 The General case. Suppose α is an arbitrary homogeneous measure. We indicate briefly how to obtain translation times under certain circumstances. Define $S_\alpha(\omega)$ to be the support of the measure $\alpha(\cdot, \omega)$. Then it is easy to see that $\hat{P}_\alpha$ is concentrated on $\Omega_0 = \{\omega : 0 \in S_\alpha(\omega)\}$. Denoting the mass at $t = 0$ by $\Delta_0(\omega)$, and $\hat{\Omega} = \{\Delta_0 > 0\}$ as before, we have $\hat{\Omega} \subset \Omega_0$, the inclusion being strict in general. For each ω , the measure $\alpha(\cdot, \omega)$ decomposes into a continuous measure $\alpha_c(\cdot, \omega)$ plus a discrete measure $\alpha_d(\cdot, \omega)$, both of which are homogeneous. Define $\hat{\alpha}_c(t) = \inf \{s : \alpha_c(s) > t\}$, and write $\hat{P}_c, \hat{P}_d$ for the Palm measures of α_c, α_d respectively. Then [16], for each t , $\hat{\alpha}_c(t)$ is a translation time for $\hat{P}_c$. Notice that $\hat{P}_\alpha = \hat{P}_c + \hat{P}_d$.

(20) Theorem. Define

$$\tau(\omega) = \begin{cases} \alpha_c(1, \omega) & \omega \in \Omega_0 - \hat{\Omega} \\ \inf \{t > 0 : \Delta_t(\omega) = \Delta_0(\omega)\} & \omega \in \hat{\Omega}. \end{cases}$$

Then, if α is integrable and $\hat{P}_\alpha(\tau = \infty) = 0$, τ is a translation time.

Note. Integrability may be relaxed somewhat; we omit the details.

Proof. Clearly $\hat{P}_d(\hat{\Omega}^c) = 0$. We show also that $\hat{P}_c(\hat{\Omega}) = 0$:

$$\begin{aligned} \hat{P}_c(\hat{\Omega}) &= E \int_0^1 I_{\hat{\Omega} \circ \theta, \alpha_c}(dt) \\ &= E \int_0^1 I_{\{\Delta_t > 0\}} \alpha_c(dt) \\ &\leq E \alpha_c\{t : \Delta_t > 0\} = 0 \end{aligned}$$

since α_c is a continuous measure and, for each ω , $\Delta_t > 0$ for at most countably many t .

Now, for $A \in \mathcal{F}$, and, putting $\tau_0 = \inf \{t > 0; \Delta_t = \Delta_0\}$,

$$\begin{aligned} \hat{P}_\alpha(\theta_\tau^{-1}A) &= \hat{P}_c(\theta_{\hat{\alpha}(1)}^{-1}A) + \hat{P}_d(\theta_{\tau_0}^{-1}A) \\ &= \hat{P}_c(A) + \hat{P}_d(A) \\ &= \hat{P}_\alpha(A), \end{aligned}$$

since $\hat{P}_c$ is concentrated on $\Omega_0 - \hat{\Omega}$, and similarly for $\hat{P}_d$.

Finally, we remark that $\Delta_0 = \Delta_0 \circ \theta_\tau = \Delta_\tau$ $\hat{P}_\alpha$ -a.s. under the conditions of (20), i.e. τ does not "mix" the supports of α_c and α_d . To see this, consider

$$\hat{P}_\alpha(\Delta_0 \neq \Delta_\tau) = \hat{P}_c(\Delta_0 \neq \Delta_\tau) + \hat{P}_d(\Delta_0 \neq \Delta_\tau).$$

The second term obviously vanishes, while the first term equals

$$\begin{aligned} \hat{P}_c(\Delta_0 \circ \theta_\tau > 0) &= \hat{P}_c(\Delta_0 \circ \theta_{\hat{\alpha}(1)} > 0) \\ &= \hat{P}_c(\Delta_0 > 0) \\ &= 0. \end{aligned}$$

2. Ergodic properties. Consider again $\alpha(dt, \omega)$ of discrete type and suppose

$$\tau(\omega) = \inf \{t > 0 : \Delta_t(\omega) = \Delta_0(\omega)\}$$

is a (finite) translation time. Recall that this will be true if Δ_0 takes all its values in a countable set $B_0 = \{b_1, b_2, \dots\}$ $\hat{P}_\alpha$ -a.s., or when τ is finite $\hat{P}_\alpha$ -a.s. and either the base measure ν is integrable (Theorem (10)) or α is locally integrable (Theorem (15)). If $\beta(dt, \omega)$ is any homogeneous measure, let $L_\beta = \{\omega : \beta(\mathbf{R}_+, \omega) = \beta(-\mathbf{R}_+, \omega) = \infty\}$, as before. Ruling out the trivial case $P(L_\beta) = 0$, we remark for later use that $\hat{P}_\beta$ and $P(\cdot | L_\beta)$ (= ordinary conditional probability given L_β) are equivalent on the σ -field $\mathcal{Q}$ of (θ_t) -invariant events in Ω [4].

Let $\mathcal{Q}_\tau$ be the σ -field of θ_τ -invariant events A in $\hat{\Omega} : \theta_\tau^{-1}A = A$. When α is itself a counter and $P(L_\alpha) = 1$, it is well known [14] that the dynamical systems $\mathfrak{D} = (\Omega, \mathfrak{F}, P, \theta_t)$ and $\hat{\mathfrak{D}} = (\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}_\alpha, \theta_\tau)$ are simultaneously ergodic (i.e. the σ -fields $\mathcal{Q}$ and $\mathcal{Q}_\tau$ are trivial up to null sets) or non-ergodic (similarly for α continuous). But in the present case Δ_0 is clearly a θ_τ -invariant random variable, so $\mathcal{Q}_\tau$ is never trivial unless Δ_0 is constant, which brings us back to the counter case. Notice, however, that $\hat{\mathfrak{D}}$ ergodic implies $\mathfrak{D}$ ergodic. (One consequence of these remarks is that it is generally impossible to represent $\mathfrak{D}$ as a special flow (cf. [5], [16]) with θ_τ as the base automorphism.)

Denote by $\sigma(\Delta_0)$ the σ -field generated by Δ_0 , so $\sigma(\Delta_0) \subseteq \bar{\mathcal{Q}}_\tau$, where the bar signifies completion by all $\hat{P}_\alpha$ -null sets.

(21) Theorem. *If $\mathfrak{D}$ is ergodic, and Δ_0 takes all its values in B_0 , then $\overline{\sigma(\Delta_0)} = \bar{\mathcal{Q}}_\tau$.*

Proof. Define the counters ν_k , $k \in \mathbf{N}$, and atoms r^k_i , $k \in \mathbf{N}$, $j \in \mathbf{Z}$, as in §§1.1, 1.2, and put $\Omega_k = \{\Delta_0 = b_k\}$. Since $\mathfrak{D}$ is ergodic, so is each $(\Omega_k, \hat{P}_{\nu_k}, \theta_{\tau_{k_1}})$. Let ξ be an $\mathcal{Q}_\tau$ -measurable function; then $\xi_k = \xi|_{\Omega_k}$ is $\theta_{\tau_{k_1}}$ -invariant for all k , hence constant $\hat{P}_{\nu_k}$ -a.s., say $\xi_k = c_k$ a.s.. It follows easily that $\xi = \sum c_k I_{\Omega_k}$ $\hat{P}_\alpha$ -a.s. so ξ is $\sigma(\Delta_0)$ -measurable.

Remarks. (i) We do not know whether (21) remains true for Δ_0 non-discrete; on the other hand, we have not been able to give an example of an ergodic $\mathfrak{D}$ and a non-discrete Δ_0 (partially because of (26)–(29) below).

(ii) Using the notation of §1.3 we sketch the result of (21) in the general case, namely: if $\mathfrak{D}$ is ergodic and Δ_0 has a discrete distribution, then $\bar{\mathcal{Q}}_\tau = \overline{\sigma(\Delta_0)}$ with τ given by (20). It was shown already (end of §1.3) that Δ_0 is “almost” θ_τ -invariant, so $\sigma(\Delta_0) \subseteq \bar{\mathcal{Q}}_\tau$. Let B_0 be as above, $b_0 = 0$, and $\Omega_k = \{\Delta_0 = b_k\}$, $k \geq 0$. If ξ is $\mathcal{Q}_\tau$ -measurable, then, as before, $\xi_k = \xi|_{\Omega_k}$ is constant $\hat{P}_{\nu_k}$ -a.s. and $\hat{P}_c$ -a.s. for all k , so $\xi = \sum \xi_k I_{\Omega_k}$ is $\sigma(\Delta_0)$ -measurable.

We next prove a lemma which may be construed as a refutation of the ergodic hypothesis in its strict form [7, pp. 52–54], then apply it to find conditions under which all masses Δ_t , $t \in \mathbf{R}$, fall in some fixed countable set P -a.s.

Let $(W, \mathfrak{W}, \mu)$ be a probability space with an automorphism $T : W \rightarrow W$.

For a random variable X on W , put $X_n = X \circ T^n$ ($T^0 = \text{identity}$) and define the "repetition time"

$$N = \min \{n \geq 1 : X_n = X_0\} \\ = \infty \text{ if the indicated set is empty.}$$

Finally, let $F(dx)$ be the distribution of X , and, if $N < \infty$ a.s., $\mathfrak{A}_N$ the σ -field of T^N -invariant events A : $(T^N)^{-1}A = A$ a.s.

(22) Lemma. *If $(W, \mathfrak{W}, \mu, T)$ is ergodic and $\mu\{N < \infty\} = 1$, then*

$$F(\{X\}) = \frac{1}{E_\mu(N | X)} \quad \text{a.s. .}$$

Notes. (i) $F(\{x\})$ is the mass on the point x . (ii) The conditional expectation does not require integrability of N . (iii) Choosing $X = I_B(\mu(B) > 0)$ yields Kac's recurrence formula: $E_\mu(N; B) = 1$, where N is the return time to B .

(23) Corollary. *If $E_\mu(N | X) < \infty$ a.s., F is purely atomic and $E_\mu(N; X \in \Gamma) =$ the number of atoms of F in Γ for any $\Gamma \in \mathfrak{B}$.*

Before proving (22), it is instructive to consider the case where $\{X_n\}$ is an i.i.d. sequence with common distribution F . Obviously $N < \infty$ a.s. iff F is purely atomic. Assuming this, let the atoms be $\{e_i\}$ with masses $\{p_i\}$, $i \leq R \leq \infty$. Then $P\{N > n\} = P\{X_i \neq X_0, 1 \leq i \leq n\} = \sum_{1 \leq i \leq R} (1 - p_i)^n p_i$ (valid also for $n = 0$), hence $EN = \sum_{0 \leq n} P\{N > n\} = R$.

Proof of (22). Define (random) probability measures on $\mathbf{R}$ by

$$(24) \quad F_n(w, B) = \frac{1}{n} \sum_{k=0}^{n-1} I_B(X_k(w)), \quad B \in \mathfrak{B}.$$

By the ergodic theorem, $F_n(w, B) \rightarrow F(B)$ μ -a.s. for each $B \in \mathfrak{B}$. Thus, there is a single set $\Gamma \in \mathfrak{W}$ with $\mu(\Gamma) = 1$, on which

$$F_n(w, \{d\}) \rightarrow F(\{d\}) \quad \text{for each atom } d \text{ of } F$$

and

$$F_n(w, r) \rightarrow F(r) \quad \text{for each rational,}$$

where $F_n(w, r) = F_n(w, (-\infty, r])$. It follows (see e.g. [2, p. 124]) that

$$(25) \quad F_n(w, \{x\}) \rightarrow F(\{x\}) \quad \text{for all } x \in \mathbf{R}$$

whenever $w \in \Gamma$.

Define the successive repetition times $N_1, N_2, \dots$ by

$$N_1 = N, \quad N_{n+1} = N_n + N \circ T^{N_n}.$$

Then $N_1 < \infty$ a.s. by hypothesis, and $\mu\{N_{n+1} = \infty\} = \mu\{N \circ T^{N_n} = \infty\}$.

But each N_n has the property that T^{N_n} preserves μ -measure [12], hence the last probability is zero, *i.e.* $N_n < \infty$ for all n *a.s.* Thus there is a set $K \in \mathfrak{W}$, $\mu(K) = 1$, such that $N_n < \infty$ for all n and (25) holds for every $w \in K$. It follows that

$$F_n(w, \{X_0(w)\}) \rightarrow F(\{X_0(w)\}) \quad \text{for all } w \in K.$$

But $F_n(w, \{X_0(w)\}) = n^{-1}(1 + \#\{N_m(w) : N_m(w) \leq n - 1\})$, where $\#$ indicates "cardinality of", so we are left with evaluating

$$\lim_{n \rightarrow \infty} \frac{1}{n} \# \{N_m(w) : N_m(w) \leq n\}$$

(the replacement of $n - 1$ by n is harmless). By the ergodic theorem in the from which it is easy to evaluate the above limit as $1/E_\mu(N \mid \mathcal{G}_N)$; a further conditioning by X (which is $\mathcal{G}_N$ -measurable) yields the result.

Note. Lemma (22) can no doubt also be derived using the material in [1, §6.8]. It would be interesting to know if there exist cases for which $N < \infty$ and $F(x)$ is continuous. If X is discrete with infinitely many atoms, the ergodic theorem gives $N < \infty$, but of course, $E_\mu N = \infty$.

Now let α be counter-based, with base measure $\nu(dt)$, and define $M^* : \Omega \rightarrow \mathbf{N} \cup \{\infty\}$ by $M^* = \min \{n \in \mathbf{N} : \Delta_{R_n} = \Delta_{R_0}\}$. Notice that $M^*|_{\hat{\mathcal{A}}} = M \hat{P}_\alpha$ -a.s. with M as in §1.1. Assume $\mathfrak{D}$ is ergodic in (26)–(29). An immediate consequence of (22) is

(26) Theorem. Suppose ν is integrable. Then if $\hat{E}_\nu M < \infty$, the $\hat{P}_\alpha$ distribution of Δ_0 is purely atomic with only finitely many atoms, and conversely.

Proof. Apply (23) to the system $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}_\nu, \theta_{R_1})$ and the random variable Δ_0 .

(27) Corollary. Suppose ν is integrable. Then there exists a finite set G such that $P(\Delta_t \in G \text{ for all } t) = 1$ if and only if $EM^*(R_1 - R_0)^{-1} < \infty$.

Note. If α is itself a counter, then $G = \{0, 1\}$, $M^* \equiv 1$, and, as seen in §1.1, $E\nu(0, 1) = E(R_1 - R_0)^{-1}$.

Proof. Choosing $\xi = M^*/(R_1 - R_0)$ in (5):

$$\begin{aligned} E\left[\frac{M^*}{R_1 - R_0}\right] &= \hat{E}_\nu \int_0^{R_1} \left(\frac{M^*}{R_1 - R_0}\right) \circ \theta_t \, dt \\ &= \hat{E}_\nu M \end{aligned}$$

since $0 < t < R_1(\omega)$, $\omega \in \hat{\Omega} \Rightarrow M^*(\theta_t \omega) = M(\omega)$ and $(R_1 - R_0)(\theta_t \omega) = R_1(\omega)$.

By (26), when $\hat{E}_\nu(M) < \infty$, there is a finite set H with $\Delta_0 \in H \hat{P}_\alpha$ -a.s. Define a homogeneous measure $\beta(dt, \omega) = I_H(\Delta_t(\omega))\alpha(dt, \omega)$. Obviously, $d\hat{P}_\beta = I_H(\Delta_0)d\hat{P}_\alpha = d\hat{P}_\alpha$, so that [4] $\alpha(\cdot, \omega) = \beta(\cdot, \omega)$ P -a.s.. Now take $G = H \cup \{0\}$.

Turning to the general discrete case with $\tau < \infty \hat{P}_\alpha$ -a.s., the number of atoms until first mass repetition will generally be infinite. However, $K =$

$\#\{0 < t < \tau : (1/2)\Delta_0 \leq \Delta_t \leq (3/2)\Delta_0\}$ is finite on $\{\tau < \infty\}$ since $\alpha(dt, \omega)$ is finite on compacts.

(28) Theorem. *If α is locally integrable and $\hat{E}_\alpha K < \infty$, then the $\hat{P}_\alpha$ -distribution of Δ_0 is purely atomic with no finite, nonzero accumulation points.*

Proof. Let $B_n, n \in \mathbf{N}$, be an enumeration of the following partition of $(0, \infty)$: $(2/2^m, 3/2^m], (3/2^m, 4/2^m], m \in \mathbf{Z}$. Define $\alpha_n, \nu_n, R^n_i, \Delta^{(n)}_0$, etc, as in §1.2 relative to this partition. We apply (26) to each of the homogeneous measures ν_n , each of which is integrable since α is locally integrable. Thus, we need only check that

$$\hat{E}_{\nu_n}(M_n) < \infty \text{ for each } n \in \mathbf{N}$$

where M_n is defined in the proof of (15).

Equivalently, we need to show $\hat{E}_{\alpha_n} M_n < \infty$ for all $n \in \mathbf{N}$, since $d\hat{P}_{\alpha_n} = \Delta^{(n)}_0 dP_{\nu_n}$ and each $\Delta^{(n)}_0$ is bounded away from 0 and ∞ . Now

$$\hat{E}_{\alpha_n} M_n = \hat{E}_\alpha(M_n; \Delta_0 \in B_n) \leq \hat{E}_\alpha(K; \Delta_0 \in B_n) < \infty,$$

because $y \in B_n \Rightarrow [(1/2)y, (3/2)y] \supset B_n$, for all $n \in \mathbf{N}$, which in turn implies $M_n I_{\{\Delta_0 \in B_n\}} \leq K I_{\{\Delta_0 \in B_n\}}$ for all $\omega \in \hat{\Omega}, n \in \mathbf{N}$. Thus Δ_0 takes only finitely many values in B_n $\hat{P}_\alpha$ -a.s., hence $\hat{P}_\alpha$ -a.s., for each $n \in \mathbf{N}$.

(29) Corollary. *If α is locally integrable and $\hat{E}_\alpha K < \infty$, there is a countable set G_∞ with no finite nonzero accumulation points and $P(\Delta_t(\omega) \in G_\infty \text{ for all } t) = 1$.*

Proof. Add $\{0\}$ to the set of atoms of Δ_0 and proceed as in the proof of (27).

In the non-ergodic case the above results do not apply. Returning to Example 2, §1.2, let γ be a rational number. Taking, for instance, $\gamma = 1/4$, and $\Delta_0(x)$ any positive function symmetric around $x = 1/2$, we obtain a non-ergodic system in which Δ_0 need not be atomic, and with a repetition time ≤ 4 . In such a case we have $\overline{\sigma(\Delta_0)} \subset \bar{\mathcal{Q}}_\tau$ (strict inclusion).

As a further example, consider a real-valued stationary smooth ergodic Gaussian process $x(t)$ having a finite second spectral moment λ_2 . The derivative $\dot{x}(t)$ satisfies $P\{\dot{x}(t) = 0\} = 0$ thus the process has an occupation time density $\alpha^x(B)$:

$$\int_\Gamma \alpha^x(B) dx = \int_B I_\Gamma(x(s)) ds, \quad \Gamma \in \mathcal{B}, \quad B \in \mathcal{B}.$$

This is given explicitly by $\alpha^x(B) = \int_B |\dot{x}(s)|^{-1} \nu^x(ds)$ [3], where $\nu^x(ds) = \#\{t \in ds : x(t) = x\}$ is a counter. Thus, for a.e. x , we have a counter-based homogeneous measure of discrete type, with successive masses $|\dot{x}(R_n)|^{-1}$ where the R_n are the successive crossing times of the level x . The Palm measure $\hat{P}_{\alpha^x}$ is, for a.e. x , the ordinary conditional probability $P(\cdot | x(0) = x)$. From (29) we conclude that: at successive crossings of x , the size of the derivative $|\dot{x}(R_n)|$ does not repeat itself, or, if it does, the repetition index is not integrable. (The former is likely the case.)

Finally, suppose that $P(L_\alpha) = 1$, so that P and $\hat{P}_\alpha$ are equivalent on $\mathcal{A}$, and put $A_x = \{\omega : \Delta_t(\omega) = x \text{ for some } t \in \mathbf{R}\}$, $F(x) = \hat{P}_\alpha\{\Delta_0 \leq x\}$.

(30) Theorem. *For each $x > 0$, $A_x \in \mathcal{F}$ and $P(A_x) = 0$ if and only if $F(\{x\}) = 0$.*

Proof. Let $\{B_n\}$, $n \in \mathbf{N}$, partition $(0, \infty)$ as in §1.2. With $n_1 \in \mathbf{N}$ such that $x \in B_{n_1}$,

$$(31) \quad A_x = \bigcup_{j \in \mathbf{Z}} \hat{\theta}_{(n_1, j)}^{-1} \{\Delta_0 = x\} \in \mathcal{A}.$$

Suppose that $F(\{x\}) = \hat{P}_\alpha(\Delta_0 = x) = 0$. Then $\hat{P}_{\alpha_{n_1}}(\Delta_0 = x) = 0$ and hence $\hat{P}_{\nu_{n_1}}(\Delta_0 = x) = 0$, from which follows $\hat{P}_{\nu_{n_1}}(\hat{\theta}_{(n_1, j)}^{-1}(\Delta_0 = x)) = 0$ for all $j \in \mathbf{Z}$ since $R^{n_1}_j$ is a translation time for $\hat{P}_{\nu_{n_1}}$. By (31), then, $\hat{P}_{\nu_{n_1}}(A_x) = 0$, which in turn implies $P(A_x \cap L_{\nu_{n_1}}) = 0$. Finally, the observation that $A_x \subset L_{\nu_{n_1}}$, except for a null set, shows $P(A_x) = 0$. Conversely, since $(\Delta_0 = x) \subset A_x$, $P(A_x) = 0 \Rightarrow \hat{P}_\alpha(A_x) = 0 \Rightarrow F(\{x\}) = 0$, and we are finished.

REFERENCES

1. L. BREIMAN, *Probability*. Addison-Wesley, Reading, Massachusetts, 1968.
2. K. L. CHUNG, *A course in probability theory*. Harcourt, Brace and World, New York, 1968.
3. D. GEMAN & J. HOROWITZ, *Occupation times for smooth stationary processes*. Ann. of Probability **1**, (1973), 131–137.
4. D. GEMAN & J. HOROWITZ, *Remarks on Palm measures*. Ann. Inst. H. Poincaré **9** (1973), 215–232.
5. A. HANEN, *Processus ponctuels stationnaires et flots spéciaux*. Ann. Inst. H. Poincaré **7** (1971), 23–30.
6. C. HILICO, *Processus ponctuels marqués stationnaires. Application à l'interaction sélective de deux processus ponctuels stationnaires*, Ann. Inst. H. Poincaré **9** (1973), 177–192.
7. A. I. KHINCHIN, *Statistical mechanics*. Dover, New York, 1949.
8. B. A. LOGAN, *Stationary random measures*, Ph.D. Thesis, Univ. Southern Calif., 1970.
9. G. MARUYAMA, *Transformations of flows*. J. Math. Soc. Japan **18** (1966), 303–330.
10. K. MATTHES, *Stationäre zufällige Punktfolgen I*. Jahresbericht d. Deutsch. Math. Verein. **66** (1963–4), 66–79.
11. J. MECKE, *Invarianzeigenschaften allgemeiner Palmischer Masse*. (to appear in Teor. Veroj. i Prim.)
12. J. NEVEU, *Temps d'arrêt d'un système dynamique*. Z. Wahrscheinlichkeitstheorie verw. Geb. **13** (1969), 81–94.
13. J. NEVEU, *Sur la structure des processus ponctuels stationnaires*. C. R. Acad. Sci. Paris **267** (1968), 561–564.
14. C. RYLL-NARDZEWSKI, *Remarks on processes of calls*. Proc. Fourth Berkeley Symp. Math. Stat. and Prob. **2** (1961), 455–466.
15. I. M. SLIVNYAK, *Some properties of stationary flows of homogeneous random events*. Theor. Probability Appl. **7** (1962), 336–341.
16. H. TOTOKI, *Time changes of flows*. Mem. Fac. Sci. Kyūshū Univ. Ser. A. **20** (1966), 27–55.

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