On The Iversen-Tsuji Theorem

JOHN B. GARNETT

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1. Let D be an open set in the complex plane, let $z_0 \in \partial D$, and let $f(\zeta)$ be a function defined on D . The cluster set of f at z_0 is

$$\text{Cl}(f, z_0) = \bigcap_{r>0} \overline{f(D \cap \{|\zeta - z_0| < r\})}.$$

We write $\sup |S|$ for $\sup \{w : w \in S\}$. The Iversen-Tsuji theorem [4], [6] states:

Theorem 1. *Let E be a compact subset of ∂D with capacity zero. Let $z_0 \in \overline{(\partial D) \setminus E}$, $z_0 \notin E$. Then whatever $f(z)$ is analytic on D*

$$(1.1) \quad \sup |\text{Cl}(f, z_0)| \leq \lim_{\substack{z \rightarrow z_0 \\ z \in (\partial D) \setminus E}} \sup |\text{Cl}(f, z)|.$$

Equivalent formulations of this theorem can be found in [4 p. 17].

Recently two new proofs of Theorem 1 have been given, by Gamelin [1] and by Martio and Rickman [3]. The proof in [3] uses path families and yields a more general theorem on quasiregular mappings in $\mathbf{R}^n$. The proof in [1] uses some special properties, from [2], of the Banach algebra $H^\infty(D)$. It yields a refinement of the theorem in which (1.1) holds when each $\text{Cl}(f, z)$, $z \in (\partial D) \setminus E$ is replaced by a smaller cluster set determined by restricting the approach to z from within D . It also yields a similar theorem for product domains in $\mathbf{C}^n$.

We give a direct proof of a theorem containing Theorem 1 and related results for [1]. This research arose from our trying to simplify the work in [1] and our debt to that paper is large.

Theorem 1 can be viewed as a local version (at a boundary point) of the maximum principle. Our proof uses Vitushkin's localization operator ([7], [2]) and the subharmonicity of $\log |f(\zeta)|$.

The notation $f \in H^\infty(D)$ means that f is a bounded analytic function on D . Clearly it suffices to prove Theorem 1 for $f \in H^\infty(D)$. It also suffices to assume $z_0 \in \partial D$ is an *essential boundary point*, which means by definition that not all $f \in H^\infty(D)$ extend analytically to a neighborhood of z_0 . We write $\|f\|$ for $\sup |f(\zeta)|$, and $\Delta(z, \delta)$ for the disc $\{|\zeta - z| < \delta\}$. We write $\sup_{z \in S} |\text{Cl}(f, z)|$ for $\sup_{z \in S} \sup \{|w| : w \in \text{Cl}(f, z)\}$, where $\sup \emptyset = -\infty$.

2. For each $z \in \partial D$ choose a subset $\Gamma(z)$ of D , possibly void, and define, for any function f on D ,

$$\text{Cl}_\Gamma(f, z) = \bigcap_{r>0} \overline{f(\Gamma(z) \cap \Delta(z, r))}.$$

We assume: for every $\zeta \in D$ and every relatively open subset V of ∂D there exists $\lambda = \lambda(\zeta, V)$, $0 \leq \lambda \leq 1$ such that

$$(2.1) \quad \log |f(\zeta)| \leq \lambda \sup_{z \in V} \text{Cl}_\Gamma(\log |f|, z) + (1 - \lambda) \sup_{z \in (\partial D) \setminus V} \text{Cl}_\Gamma(\log |f|, z)$$

for all $f \in H^\infty(D)$.

Theorem 2. Assume the sets $\Gamma(z)$ satisfy (2.1). Let $z_0 \in \partial D$ be an essential boundary point. Then for each $f \in H^\infty(D)$,

$$\sup |\text{Cl}(f, z_0)| \leq \overline{\lim_{\substack{z \rightarrow z_0 \\ z \in \partial D}}} \sup |\text{Cl}_\Gamma(f, z)|.$$

Theorem 2 easily implies a small extension of Theorem 1. Let E be a Borel subset of ∂D with zero harmonic measure. Set $\Gamma(z) = D$ if $z \notin E \cup \{z_0\}$; $\Gamma(z) = \emptyset$ if $z \in E \cup \{z_0\}$. Take $\lambda(\zeta, V)$ to be the harmonic measure $\omega(\zeta, V)$. Then (2.1) follows easily. Indeed for $f \in H^\infty(D)$, let $u(\zeta)$ be the solution of the Dirichlet problem with boundary function

$$u(z) = \begin{cases} 2 \log ||f|| & z \in E \\ \sup_{V \setminus E} \text{Cl}(\log |f|, w) & z \in V \setminus E \\ \sup_{(\partial D) \setminus (V \cup E)} \text{Cl}(\log |f|, w) & z \in (\partial D) \setminus (V \cup E). \end{cases}$$

Then $\overline{\lim}_{\zeta \rightarrow z} \log |f(\zeta)| \leq u(z)$ for all $z \in \partial D$. By the Perron solution of Dirichlet's problem, $\log |f(\zeta)| \leq u(\zeta)$; and this is (2.1) in this case.

Gamelin's refinement of Theorem 1 [1, Theorem 3.1] also follows directly from Theorem 2. Let Δ be the unit disc and let $\pi: \Delta \rightarrow D$ be the uniformizer. Then π has a nontangential limit $\bar{\pi}(e^{i\theta})$ at almost every $\theta \in \partial \Delta$, and $\bar{\pi}$ is a Borel function from $\partial \Delta$ to ∂D . Fix an essential boundary point z_0 for D and fix $f \in H^\infty(D)$. Let S be a Borel subset of ∂D of full measure such that $\lim_{r \rightarrow 1} \pi(re^{i\theta})$ and $\lim_{r \rightarrow 1} f \circ \pi(re^{i\theta})$ exist. Set $G(z_0) = \emptyset$, and for $z \neq z_0$,

$$G(z) = \bigcup \{ \{ \pi(re^{i\theta}) : 0 < r < 1 \} : \theta \in S, \bar{\pi}(re^{i\theta}) = z \}.$$

Since $f \circ \pi$ is the Poisson integral of its boundary values, (2.1) holds for the sets $G(z)$ with $\lambda(\zeta, V) = \int_{\bar{\pi}^{-1}(V)} P_a(\theta) d\theta / 2\pi$, where $\pi(a) = \zeta$ and P_a is the Poisson kernel for $a \in \Delta$. Hence Theorem 2 implies the following.

Theorem 3. (Gamelin) Let z_0 be an essential boundary point of D and let $f \in H^\infty(D)$. Then

$$\sup |\text{Cl}(f, z_0)| \leq \overline{\lim_{z \rightarrow z_0}} \sup |\text{Cl}_G(f, z)|.$$

Theorem 3 implies Theorem 1 because if the Borel set $E \subset \partial D$ has harmonic measure zero, then $S = \pi^{-1}(E)$ has measure zero. Indeed Theorem 3 is stronger than Theorem 1 because in many cases $\text{Cl}_G(f, z)$ is either a singleton or void.

3. We prove Theorem 2. Let $f \in H^\infty(D)$, and assume $\|f\| \leq 1$. Let V be an open subset of ∂D , $z_0 \in V$. Write $m = \sup_{z \in V} |\text{Cl}_\Gamma(f, z)|$. Let $D \ni z_n \rightarrow z_0$ so that $f(z_n) \rightarrow p$. It suffices to show $|p| \leq m$. Write $\lambda_n = \lambda(z_n, V)$.

The case $\lambda_n \rightarrow 1$ is immediate. For by (2.1), $\log |f(z_n)| \leq \lambda_n \log m$, so that $\log |p| \leq \log m$.

For the case $\lambda_n \rightarrow 0$, we localize f about z_0 . Let $\delta_k \searrow 0$ and let $\varphi_k \in C^\infty$, $0 \leq \varphi_k \leq 1$; $\varphi_k \equiv 1$ on $\Delta(z_0, \delta_k/2)$, $\varphi_k = 0$ off $\Delta(z_0, \delta_k)$, $|\partial \varphi_k / \partial \bar{z}| \leq 4/\delta_k$. Define $f_k = 0$ on $\mathbb{C} \setminus D$ and write

$$f_k(w) = \varphi_k(w)f(w) + \frac{1}{\pi} \iint \frac{\partial \varphi_k}{\partial \bar{z}} \frac{f(z)}{z - w} dx dy.$$

Then the following are easily verified. See also [2, p. 459].

$$f_k \in H^\infty(D), \|f_k\| \leq 8.$$

$$f_k \text{ is analytic off } \Delta(z_0, \delta_k), f_k(\infty) = 0.$$

$$|f_k(w)| \leq \frac{8\delta_k}{\text{dist}(w, \Delta(z_0, \delta_k))}$$

$$f - f_k \text{ is analytic on } \Delta(z_0, \delta_{k/2}), \|f - f_k\| \leq 9.$$

$$A_k(f) = (f - f_k)(z_0) = -\frac{1}{\pi} \iint \frac{\partial \varphi_k}{\partial \bar{z}} \frac{f(z)}{z - z_0} dx dy$$

$$\text{satisfies } |A_k(f)| \leq 9.$$

Let $\epsilon_k = \sup \{|f_k(z)| : z \in (\partial D) \setminus V\}$; then $\epsilon_k \rightarrow 0$ by (iii). We have assumed $\lambda_n \leq \lambda < 1$, so that by (2.1)

$$(3.1) \quad \log |f_k(z_n)| \leq \lambda_n \log 8 + (1 - \lambda_n) \log \epsilon_k.$$

Since $f_k(z_n) \rightarrow p - A_k(f)$ ($n \rightarrow \infty$), we also have

$$(3.2) \quad \log |p - A_k(f)| \leq \lambda \log 8 + (1 - \lambda) \log \epsilon_k,$$

and $A_k(f) \rightarrow p$ ($k \rightarrow \infty$). Since $A_k(f)$ does not depend on the sequence $\{z_n\}$ we have proved

$$\text{Cl}(f, z_0) \subset \{|w| \leq m\} \cup \{p\}.$$

We assume $|p| > m$ and obtain a contradiction. Since $\text{Cl}(f, z_0)$ is not connected there is $\delta_0 > 0$ such that for $\delta < \delta_0$

$$U_\delta = D \cap \Delta(z_0, \delta) \cap f^{-1}(|w| > (|p| + m)/2)$$

is a proper nonvoid open-closed subset of $D \cap \Delta(z_0, \delta)$. We have $\lambda(\zeta, V) \leq a < 1$ for all $\zeta \in U_\delta$, because otherwise $\inf_{U_\delta} |f(\zeta)| = m$ by (2.1). Since $D \cap \Delta(z_0, \delta)$ is not connected there is a continuum $J_\delta \subset \Delta(z_0, \delta) \cap \partial U_\delta$; let $h: S^1 \setminus J_\delta$

$\rightarrow \Delta(0, 1)$ be a conformal mapping with $h(\infty) = 0$. Then $h \in H^\infty(D)$, $|h(\zeta_n)| \rightarrow 1$ if $\zeta_n \rightarrow z \in J_\delta$, and by Schwarz's lemma,

$$|h(\zeta)| \leq \frac{\delta}{\text{dist}(\zeta, \Delta(z_0, \delta))}.$$

Let $U_\delta \ni \zeta_n \rightarrow z \in J_\delta$. Then by (2.1)

$$0 = \lim \log |h(\zeta_n)| \leq (1 - a) \log \frac{\delta}{\text{dist}((\partial\Omega \setminus V), \Delta(z_0, \delta))} < 0,$$

if δ is sufficiently small. This contradiction proves $|p| \leq m$.

4. The key inequalities (3.1) and (3.2) hold uniformly in bounded subsets of H^∞ , and they therefore yield an extension of Theorem 3 to product domains. For simplicity, we work in $\mathbf{C}^2$, where we prove a special case of Theorem 9.2 of [1].

Lemma 1. *Let z_0 be an essential boundary point of D , let V open in ∂D , $z_0 \in V$. Let $D \ni z_n \rightarrow z_0$, and assume $\lambda_n = \lambda(z_n, V) \leq \lambda < 1$. For $f \in H^\infty(D)$, let $A_k(f)$ be as above. Then*

$$A(f) = \lim_{n \rightarrow \infty} f(z_n)$$

exists for all $f \in H^\infty(D)$, and

$$A(f) = \lim_{k \rightarrow \infty} A_k(f).$$

Moreover, both sequences converge to $A(f)$ uniformly on bounded subsets of $H^\infty(D)$ and the functional $A(f)$ is weak-star continuous:

$$A(f_i) \rightarrow A(f)$$

if f_i converges to f boundedly pointwise on D .

Proof. When both $\{f(z_n)\}$ and $\{A_k(f)\}$ converge, they have the same limit by (3.2). Being bounded, both sequences therefore have a unique limit point, and they converge to a common limit. By (3.2) $A_k(f) \rightarrow A(f)$ uniformly on bounded sets in $H^\infty(D)$, and by (v), $A_k(f)$ is weak-star continuous, so that $A(f)$ is also. Finally, if $\|f\| \leq 1$ and if $|z_n - z_0| < \delta_k$, then by (iv)

$$|(f - f_k)(z_n) - A_k(f)| \leq 18 |z_n - z_0|/\delta_k.$$

Then by (3.1)

$$|f(z_n) - A(f)| \leq 8^\lambda \epsilon_k^{1-\lambda} + 18 |z_n - z_0|/\delta_k + |A_k(f) - A(f)|.$$

Taking k large and $n \geq n(k)$, we see $|f(z_n) - A(f)|$ is uniformly small.

Let D_1, D_2 be plane domains, let $(z_0, w_0) \in \partial D_1 \times \partial D_2$, where z_0 and w_0 are essential boundary points, and let $\Gamma_1(z)$, $z \in \partial D_1$, $\Gamma_2(w)$, $w \in \partial D_2$ be subsets satisfying (2.1). Let V_i be an open subset of ∂D_i , $(z_0, w_0) \in V_1 \times V_2$. Then we have

Lemma 2. For every $f \in H^\infty(D_1 \times D_2)$

$$(4.1) \quad \sup |\text{Cl}(f, (z_0, w_0))| \leq \sup_{z \in V_1} \overline{\lim}_{\substack{\zeta \rightarrow z \\ \zeta \in \Gamma_1(z)}} \sup_{w \in V_2} |\text{Cl}_\Gamma(f(\zeta, \cdot), w)|.$$

Proof. Let $D_1 \ni z_n \rightarrow z_0$; $D_2 \ni w_n \rightarrow w_0$ so that $p = \lim f(z_n, w_n)$, exists, and $|p| = \sup |\text{Cl}(f, (z_0, w_0))|$. Let $\lambda_n = \lambda(z_n, V_1)$, $\mu_n = \lambda(w_n, V_2)$. If $\lambda_n \rightarrow 1$, $\mu_n \rightarrow 1$, (4.1) is easily verified, and so we assume $\lambda_n \leq \lambda < 1$.

Write $f_n(z) = f(z, w_n)$. We can assume $f_n(z) \rightarrow \tilde{f}(z)$ boundedly pointwise on D_1 . Then by Lemma 1, $f(z_n, w_n) = f_n(z_n) \rightarrow A(\tilde{f})$. Hence by Theorem 2,

$$\begin{aligned} \sup |\text{Cl}(f, (z_0, w_0))| &= |A(\tilde{f})| \leq \sup_{z \in V_1} \overline{\lim}_{\substack{\zeta \rightarrow z \\ \zeta \in \Gamma_1(z)}} |\tilde{f}(\zeta)| \\ &= \sup_{z \in V_1} \overline{\lim}_{\substack{\zeta \rightarrow z \\ \zeta \in \Gamma_1(z)}} \lim_{n \rightarrow \infty} |f(\zeta, w_n)| \leq \sup_{z \in V_1} \overline{\lim}_{\substack{\zeta \rightarrow z \\ \zeta \in \Gamma_1}} \sup_{w \in V_2} |\text{Cl}(f(\zeta, \cdot), w)|. \end{aligned}$$

Note that in using (4.1) we can take $\Gamma_2(w)$ to depend on $z \in \partial D_1$.

To state the next theorem, we let $\pi_i : \Delta \rightarrow D_i$ be the uniformizer and write $\pi = \pi_1 \times \pi_2 : \Delta_1 \times \Delta_2 \rightarrow D_1 \times D_2$. For almost every $(\theta, \varphi) \in \partial \Delta \times \partial \Delta$, $\lim_{s \rightarrow 1} \lim_{r \rightarrow 1} \pi(re^{i\theta}, se^{i\varphi}) = \tilde{\pi}(e^{i\theta}, e^{i\varphi})$ exist; and $\tilde{\pi}$ is a Borel mapping from $\partial \Delta \times \partial \Delta$ to $\partial D_1 \times \partial D_2$. Fix $f \in H^\infty(D_1 \times D_2)$ and let $g(re^{i\theta}, se^{i\varphi}) = f \circ \pi(re^{i\theta}, se^{i\varphi})$. Then $\lim_{s \rightarrow 1} \lim_{r \rightarrow 1} g(re^{i\theta}, se^{i\varphi}) = g(e^{i\theta}, e^{i\varphi})$ exists for almost every $(\theta, \varphi) \in \partial \Delta \times \partial \Delta$ [5, p. 69]. Write $\text{ess sup}_S |g(e^{i\theta}, e^{i\varphi})|$ for the L^∞ norm of the restriction of g to $S \subset \partial \Delta \times \partial \Delta$. The special case of 9.2 of [1] is

Theorem 4. (Gamelin) Let V be an open subset of $\partial D_1 \times \partial D_2$. Let $(z_0, w_0) \in V$ with z_0, w_0 essential boundary points. Let $f \in H^\infty(D_1 \times D_2)$ and let $g = f \circ \pi$. Then

$$(4.2) \quad \sup |\text{Cl}(f, (z_0, w_0))| \leq \text{ess sup}_{\tilde{\pi}^{-1}(V)} |g(e^{i\theta}, e^{i\varphi})|.$$

Proof. Let V_i be open in ∂D_i so that $(z_0, w_0) \in V_1 \times V_2 \subset V$. Let E be a Borel subset of $\partial \Delta$ of full measure. For $z \in \partial D_1$, define

$$\Gamma_1(z) = \bigcup \{ \{ \pi_1(re^{i\theta}) : 0 < r < 1 \} : \lim_{r \rightarrow 1} \pi_1(re^{i\theta}) = z, \theta \in E \}.$$

When $\Gamma_1(z) \neq \emptyset$, z fixed, let F_θ be a Borel subset of $\partial \Delta$ of full measure and define

$$\Gamma_2(w) = \bigcup \{ \{ \pi_2(se^{i\varphi}) : 0 < \varphi < 1 \} : \lim_{s \rightarrow 1} \pi_2(se^{i\varphi}) = w, \varphi \in F_\theta \}.$$

Otherwise define $\Gamma_2(w) = \emptyset$. Then (2.1) holds for these sets. Let $Q_i = \tilde{\pi}^{-1}(V_i)$. Then by (4.1) we have

$$\sup |\text{Cl}(f, (z_0, w_0))| \leq \sup_{\theta \in E \cap Q_1} \overline{\lim}_{r \rightarrow 1} \sup_{\varphi \in F_\theta \cap Q_2} \overline{\lim}_{s \rightarrow 1} |g(re^{i\theta}, se^{i\varphi})|.$$

To finish the proof we choose the sets E and F_θ so that the last expression is dominated by $\text{ess sup}_{Q_1 \times Q_2} |g(e^{i\theta}, e^{i\varphi})|$.

Change $g(e^{i\theta}, e^{i\varphi})$ on a set of measure zero so that

$$\sup_{Q_1 \times Q_2} |g(e^{i\theta}, e^{i\varphi})| = \text{ess sup}_{Q_1 \times Q_2} |g(e^{i\theta}, e^{i\varphi})|.$$

Choose E and F_θ so that $E \cap Q_1$ is contained in the points of density of Q_1 , and so that for $\theta \in E$, $\varphi \in F_\theta$,

$$\lim_{s \rightarrow 1} g(re^{i\theta}, se^{i\varphi}) = g(re^{i\theta}, e^{i\varphi})$$

exists, and

$$g(re^{i\theta}, e^{i\varphi}) = \frac{1}{2\pi} \int P_r(\theta - t) g(e^{it}, e^{i\varphi}) dt$$

where P_r is the Poisson kernel. This can be done since

$$g(re^{i\theta}, se^{i\varphi}) = \frac{1}{4\pi^2} \iint P_r(\theta - t_1) P_s(\varphi - t_2) g(e^{it_1}, e^{it_2}) dt_1 dt_2.$$

Then we have

$$\begin{aligned} \sup_{\theta \in E \cap Q_1} \overline{\lim}_{r \rightarrow 1} \sup_{\varphi \in F_\theta \cap Q_2} \lim_{s \rightarrow 1} |g(re^{i\theta}, se^{i\varphi})| \\ \leq \sup_{\theta \in E \cap Q_1} \overline{\lim}_{r \rightarrow 1} \sup_{\varphi \in F_\theta \cap Q_2} \left| \frac{1}{2\pi} \int P_r(\theta - t) g(e^{it}, e^{i\varphi}) dt \right| \\ \leq \sup_{\theta \in E \cap Q_1} \overline{\lim}_{r \rightarrow 1} \frac{1}{2\pi} \int_{Q_1} P_r(\theta - t) \sup_{\varphi \in F_\theta \cap Q_2} |g(e^{it}, e^{i\varphi})| dt \\ + \sup_{\theta \in E \cap Q_1} \overline{\lim}_{r \rightarrow 1} \frac{1}{2\pi} \int_{(\partial \Delta) \setminus Q_1} P_r(\theta - t) ||g|| dt. \end{aligned}$$

When θ is a point of density of Q_1 , the limit in the second term is zero, and (4.2) is proved.

Finally we should remark that the functional $A(f)$ in Lemma 1 is closely related to the distinguished homomorphism studied in [2]. Indeed whenever $z_n \rightarrow z_0$ and $\lambda_n \leq \lambda < 1$, $A(f)$ coincides with the distinguished homomorphism at z_0 . However, there are cases in which $\lambda_n \rightarrow 1$, for all V , whenever $z_n \rightarrow z_0$, but the distinguished homomorphism still exists.

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University of California, Los Angeles

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