On Nonstationary Stokes Flow Past an Obstacle

JOHN G. HEYWOOD

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1. Introduction. Solutions of the exterior stationary problem for the Stokes equations do not generally tend to the prescribed limit at infinity sufficiently fast for the difference between a solution and its limit to be square-summable. The square-integral of this difference, which can be interpreted as the energy present in a flow, is known to be infinite in the case of Stokes flow past an obstacle to which fluid adheres. Therefore, however natural it may be in some respects to base the study of nonstationary solutions upon energy estimates, such studies are restricted to a class of nonstationary solutions (finite energy solutions) which tend more rapidly than do stationary solutions to their limits at infinity. In this paper we study a class of nonstationary solutions of the Stokes equations whose behavior at infinity is merely required to be similar to that of stationary solutions which possess a finite Dirichlet integral. Our purpose is to provide a natural solution class for the initial boundary value problem in the case of an exterior domain, to treat existence and uniqueness questions within this class, and finally to prove the convergence of nonstationary solutions to stationary ones as time $t \rightarrow \infty$. The last mentioned objective amounts to demonstrating the attainability of stationary solutions as limits of nonstationary solutions and it has been our primary goal. In earlier papers [3, 4] we studied similar problems for the nonlinear Navier–Stokes equations and in those papers our results were limited in scope by finite energy hypotheses which are dispensed with here for the Stokes equations.

Since energy estimates are impossible for the solutions of problems we will consider, we base our methods instead upon L^2 estimates for certain combinations of a solution's derivatives. The solutions we obtain belong to a Hilbert space H_0 whose norm involves these derivatives and which is defined by completion of a set of stationary solutions. A close connection thus results between the stationary and nonstationary theories. Section 2 below contains a preliminary investigation of the space H_0 .

In Section 3 the initial boundary value problem is posed with homogeneous boundary conditions, a class of generalized solutions belonging to the space H_0 is defined, and uniqueness is proved within this class of solutions. Existence of class H_0 solutions is proved in Section 4 under a variety of hypotheses, the important point being that neither the initial data nor the forces are required to be square-summable. In Section 5 it is shown that when the boundary conditions are homogeneous and the forces decay sufficiently rapidly, class H_0 solutions of the initial boundary value problem tend to zero as $t \rightarrow \infty$. Some nonhomogeneous boundary value problems are considered in Section 6. There it is shown that solutions of the exterior stationary problem may be obtained in a natural way as limits of nonstationary solutions.

Our theorems concern generalized solutions and have been made to read the same for any number of spacial dimensions $n \geq 2$. Their classical interpretation, however, is different if $n = 2$ than if $n \geq 3$ due to a differing behavior at infinity of the generalized solutions in the two cases. In contrast to the case $n \geq 3$ for which functions in H_0 tend to zero at infinity in a reasonable sense, if $n = 2$ there are functions in H_0 which equal nonzero constants or even grow infinite in a neighborhood of infinity. This gives rise to the Stokes paradox and to some related nonstationary paradoxes which are the subject of the paper [5] which will follow this one.

The methods used in this paper have other obvious applications. It may be seen immediately that every theorem given here for the Stokes equations has an analogue for the heat equation and in fact an analogue for any of a certain class of parabolic equations. Although we have not obtained similar results for the Navier-Stokes equations we have where practical used methods which can be applied to nonlinear problems.

2. Preliminaries. The region Ω occupied by fluid may be any open set in $\mathbf{R}^n$ with $n \geq 2$, except that if $n = 2$ we assume the complement of Ω contains at least an arc. We denote points in Ω by $x = (x_1, \dots, x_n)$ and the time variable by t . The space-time region $\Omega \times (\epsilon, T)$ with $0 \leq \epsilon < T \leq \infty$ is denoted by $Q_{\epsilon, T}$ or simply by Q_T if $\epsilon = 0$. The closure and boundary of a set G are denoted by $\bar{G}$ and ∂G respectively.

All functions in this paper are either $\mathbf{R}$ or $\mathbf{R}^n$ -valued; in the latter case they are denoted by bold faced letters. $L^2(\Omega)$ denotes the space of $\mathbf{R}$ -valued functions which are square-summable in Ω , and $\mathbf{L}^2(\Omega)$ denotes the space of $\mathbf{R}^n$ -valued functions square-summable in Ω . Similarly $L^2_{loc}(\Omega)$ denotes the space of $\mathbf{R}$ -valued functions which are square-summable in compact subsets of Ω , and $\mathbf{L}^2_{loc}(\Omega)$ denotes the corresponding space of $\mathbf{R}^n$ -valued functions. The spaces of smooth $\mathbf{R}$ and $\mathbf{R}^n$ -valued functions with compact support in Ω are denoted by $C_0^\infty(\Omega)$ and $\mathbf{C}_0^\infty(\Omega)$ respectively. Spaces $\mathbf{L}^2(Q_T)$, $L^2_{loc}(\bar{Q}_\infty)$, $\mathbf{C}_0^\infty(Q_{\epsilon, T})$, etc., over space-time regions are defined similarly.

We use the usual notation of vector analysis; thus the i^{th} component of $\Delta \mathbf{u}$ is $\sum_{j=1}^n \partial^2 u_i / \partial x_j^2$, $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i$, and $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i,j=1}^n \partial u_i / \partial x_j \cdot \partial v_i / \partial x_j$.

Further

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, & ||\mathbf{u}|| &= (\mathbf{u}, \mathbf{u})^{1/2}, \\(\nabla \mathbf{u}, \nabla \mathbf{v}) &= \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx, & ||\nabla \mathbf{u}|| &= (\nabla \mathbf{u}, \nabla \mathbf{u})^{1/2}, \\(\cdot, \cdot)_{Q_{\epsilon, T}} &= \int_{\epsilon}^T (\cdot, \cdot) \, dt, & ||\cdot||_{Q_{\epsilon, T}} &= (\cdot, \cdot)_{Q_{\epsilon, T}}^{1/2}.\end{aligned}$$

We need to introduce some additional function spaces which are defined in a similar way over either Ω or $Q_{\epsilon, T}$. For brevity we consider first and in more detail the spaces over Ω . To begin with, let

$$\begin{aligned}\mathbf{D}(\Omega) &= \{\phi: \phi \in \mathbf{C}_0^{\infty}(\Omega) \text{ and } \nabla \cdot \phi = 0\}, \\ \mathbf{J}(\Omega) &= \text{Completion of } \mathbf{D}(\Omega) \text{ in the norm } ||\cdot||, \\ \mathbf{G}(\Omega) &= \{\phi: \phi \in \mathbf{L}^2(\Omega) \text{ and } \phi = \nabla p \text{ for some } p \in L^2_{10c}(\Omega)\}, \\ \mathbf{J}_0(\Omega) &= \text{Completion of } \mathbf{D}(\Omega) \text{ in the norm } ||\nabla \cdot||.\end{aligned}$$

The spaces $\mathbf{J}(\Omega)$, $\mathbf{G}(\Omega)$, and $\mathbf{J}_0(\Omega)$ are Hilbert spaces. Lemmas 1 and 2 contain some well known facts about them.

Lemma 1. $\mathbf{G}(\Omega)$ is the orthogonal complement of $\mathbf{J}(\Omega)$ in $\mathbf{L}^2(\Omega)$. If $\mathbf{u} \in \mathbf{L}^2_{10c}(\Omega)$ then $(\mathbf{u}, \phi) = 0$ for all $\phi \in \mathbf{D}(\Omega)$ if and only if $\mathbf{u} = \nabla p$ for some $p \in L^2_{10c}(\Omega)$ with $\nabla p \in \mathbf{L}^2_{10c}(\Omega)$.

The first statement is proved in [6, pp. 23–30]. The second statement can be proved by considering a sequence of domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_k \subset \cdots$, each with compact closure in Ω , whose union equals Ω . We may assume that Ω and each Ω_k is connected. Certainly $\mathbf{u} \in \mathbf{G}(\Omega_k)$, so $\mathbf{u} = \nabla p_k$ in Ω_k for some $p_k \in L^2_{10c}(\Omega_k)$. Because p_i and p_k , $j < k$, can differ by at most a constant in Ω_i it is possible to redefine the sequence of functions $\{p_k\}$ so that they are all equal in common regions of definition, thereby determining a function $p \in L^2_{10c}(\Omega)$ with $\mathbf{u} = \nabla p$ in Ω .

Lemma 2. $\mathbf{J}_0(\Omega) \subset \mathbf{L}^2_{10c}(\Omega)$. If Ω' is a bounded subset of Ω there exists a constant $C_{\Omega'}$ such that $\int_{\Omega'} \phi^2 \, dx \leq C_{\Omega'}^2 ||\nabla \phi||^2$ holds for all $\phi \in \mathbf{J}_0(\Omega)$.

If $n \geq 3$ the inequality

$$\int_{\mathbf{R}^n} \frac{\phi^2(x)}{|x-y|^2} \, dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbf{R}^n} (\nabla \phi)^2 \, dx$$

holds for all $\phi \in \mathbf{C}_0^{\infty}(\mathbf{R}^n)$ and every point $y \in \mathbf{R}^n$; see [6, p. 17]. Thus if Ω' is contained in a sphere of radius R the constant $C_{\Omega'}$ may be chosen equal to $2R/(n-2)$. If $n = 2$ smooth functions with compact support in $\{x: |x| > r\}$ satisfy the inequality

$$\int_{|x|>r} \frac{\phi^2(x) \, dx}{|x|^2 \log^2 |x/r|} \leq 4 \int_{|x|>r} (\nabla \phi)^2 \, dx;$$

see [6] again. Thus if the complement of Ω contains a disk of radius r and if the subset Ω' of Ω is contained in a concentric disk of radius R we may take $C_{\Omega'} = 2R \log R/r$. Even if the complement of $\Omega \subset \mathbf{R}^2$ contains only an arc a suitable constant $C_{\Omega'}$ can be found by methods often used to prove Poincaré's inequality; see for instance [1].

The inequalities just mentioned restrict the behavior at infinity of functions in $\mathbf{J}_0(\Omega)$. Because $|x - y|^{-2}$ is not integrable in a neighborhood of infinity the inequality for $n \geq 3$ implies that functions in $\mathbf{J}_0(\Omega)$ tend in some sense to zero at infinity. This is not implied if $n = 2$ because $|x|^{-2} \log^{-2} |x/r|$ is integrable in a neighborhood of infinity. Indeed, it will be shown in [5] that $\mathbf{J}_0(\Omega)$ contains elements which equal nonzero constant vectors in a neighborhood of infinity and even elements which grow infinite at infinity. However the inequality for $n = 2$ does imply that functions in $\mathbf{J}_0(\Omega)$ grow in some sense more slowly than $\log |x|$ at infinity. We shall say that functions in $\mathbf{J}_0(\Omega)$ *tend to zero at infinity in a generalized sense* if $n \geq 3$ and are *$o(\log |x|)$ at infinity in a generalized sense* if $n = 2$.

If $\mathbf{f} \in \mathbf{D}(\Omega)$, then $(\mathbf{f}, \phi)$ defines a bounded linear functional on $\phi \in \mathbf{J}_0(\Omega)$ because $|(\mathbf{f}, \phi)| \leq \|\mathbf{f}\| \cdot \|\phi\|_{\Omega'} \leq C_{\Omega'} \|\mathbf{f}\| \cdot \|\nabla \phi\|$ where Ω' is the support of $\mathbf{f}$. Clearly distinct functions $\mathbf{f} \in \mathbf{D}(\Omega)$ determine distinct functionals. These observations ensure that the map $\tilde{\Delta} : \mathbf{K}_0(\Omega) \rightarrow \mathbf{J}(\Omega)$ defined below is well defined, is one to one, and maps onto all of $\mathbf{D}(\Omega)$.

Definition. $\mathbf{K}_0(\Omega)$ is the set of all $\mathbf{u} \in \mathbf{J}_0(\Omega)$ such that $(\nabla \mathbf{u}, \nabla \phi) = (-\mathbf{f}, \phi)$, for some $\mathbf{f} \in \mathbf{D}(\Omega)$ and all $\phi \in \mathbf{J}_0(\Omega)$. The map $\tilde{\Delta} : \mathbf{K}_0(\Omega) \rightarrow \mathbf{J}(\Omega)$ is defined by setting $\tilde{\Delta} \mathbf{u} = \mathbf{f}$. $\mathbf{K}_0(\Omega)$ is given the inner product $(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\tilde{\Delta} \mathbf{u}, \tilde{\Delta} \mathbf{v})$ and associated norm $(\|\nabla \mathbf{u}\|^2 + \|\tilde{\Delta} \mathbf{u}\|^2)^{1/2}$.

The map $\tilde{\Delta}$ is closable; if $\{\mathbf{u}_n\}$ and $\{\tilde{\mathbf{u}}_n\}$ are two Cauchy sequences in $\mathbf{K}_0(\Omega)$, then $\lim_{n \rightarrow \infty} \|\nabla \mathbf{u}_n - \nabla \mathbf{u}_n\| = 0$ implies $\lim_{n \rightarrow \infty} \|\tilde{\Delta} \mathbf{u}_n - \tilde{\Delta} \tilde{\mathbf{u}}_n\| = 0$. It follows that the completion of $\mathbf{K}_0(\Omega)$ may be regarded as a subset of $\mathbf{J}_0(\Omega)$ and that the map $\tilde{\Delta}$ extends to the completion.

Definition. $\mathbf{H}_0(\Omega)$ is the completion of $\mathbf{K}_0(\Omega)$ in the norm $(\|\nabla \cdot\|^2 + \|\tilde{\Delta} \cdot\|^2)^{1/2}$, and $\tilde{\Delta} : \mathbf{H}_0(\Omega) \rightarrow \mathbf{J}(\Omega)$ is the closure of $\tilde{\Delta} : \mathbf{K}_0(\Omega) \rightarrow \mathbf{J}(\Omega)$.

Clearly $\mathbf{H}_0(\Omega)$ is a separable Hilbert space. The following lemma contains for future reference two conditions under which a certain "integration by parts" may be performed.

Lemma 3. *The equation $(\nabla \psi, \nabla \phi) = (\psi, -\tilde{\Delta} \phi)$ holds if either (i) $\phi \in \mathbf{K}_0(\Omega)$ and $\psi \in \mathbf{J}_0(\Omega)$, or if (ii) $\phi \in \mathbf{H}_0(\Omega)$ and $\psi \in \mathbf{J}_0(\Omega) \cap \mathbf{L}^2(\Omega)$.*

If $\phi \in \mathbf{K}_0(\Omega)$ the given equation defines the relation between ϕ and $\tilde{\Delta} \phi$. When $\psi \in \mathbf{J}_0(\Omega) \cap \mathbf{L}^2(\Omega)$ the equation can be extended to $\phi \in \mathbf{H}_0(\Omega)$ by passing to the limit from functions $\phi \in \mathbf{K}_0(\Omega)$.

Lemma 4. *If $\mathbf{u} \in \mathbf{H}_0(\Omega)$, then $\mathbf{u}$ has second derivatives $\mathbf{u}_{x_i x_j} \in \mathbf{L}^2_{loc}(\Omega)$ and $\Delta \mathbf{u} = \nabla p + \tilde{\Delta} \mathbf{u}$ for some $p \in \mathbf{L}^2_{loc}(\Omega)$ with $\nabla p \in \mathbf{L}^2_{loc}(\Omega)$.*

For $n = 3$, Ladyzhenskaya [6, pp. 38–9] has proved that a function $\mathbf{u} \in \mathbf{J}_0(\Omega)$ has locally square-summable second derivatives if $(\nabla \mathbf{u}, \nabla \phi) = (-\mathbf{f}, \phi)$ for some $\mathbf{f} \in \mathbf{L}^2_{100}(\Omega)$ and all $\phi \in \mathbf{D}(\Omega)$. We will give the proof here, generalizing it to any number of dimensions $n \geq 2$ and correcting an error of presentation occurring near the end of the proof given in [6].

Let Ω'' be any set with compact closure in Ω . Then there exists a larger open set Ω' also with compact closure in Ω such that $\bar{\Omega}'' \subset \Omega'$, and there exists a function $\zeta \in C_0^\infty(\Omega')$ with $\zeta = 1$ in Ω'' . We define Φ to be the function with i^{th} component

$$\Phi_i = \left\{ -\zeta^2 \Delta u_{\rho i} + \sum_{j, l, m=1}^n (\delta_i^l \delta_j^m - \delta_i^m \delta_j^l) \left(\frac{\partial}{\partial x_i} \zeta^2 \right) \left(\frac{\partial}{\partial x_l} u_{\rho m} \right) \right\}_\rho.$$

Here δ_i^j is one if $i = j$ and zero if $i \neq j$, and the subscript ρ denotes an averaging convolution $g_\rho(x) = \int g(x - \rho y) \omega(y) dy$ with kernel $\omega \in C_0^\infty(|x| < 1)$ satisfying $\int \omega(x) dx = 1$. For small enough ρ the support of Φ is in Ω' , and an easy calculation shows that $\Phi \in \mathbf{D}(\Omega)$. The identity $(\nabla \mathbf{u}, \nabla \Phi) = (-\mathbf{f}, \Phi)$ leads to the estimate $\|\zeta \Delta \mathbf{u}_\rho\|^2 \leq C_\zeta (\|\mathbf{f}\|_{\Omega'}^2 + \|\nabla \mathbf{u}\|^2)$ by using the well known identities $(\phi_\rho, \psi) = (\phi, \psi_\rho)$ and $(\phi_\rho)_{x_i} = (\phi_{x_i})_\rho$ for the averaging operation and the inequality $2ab \leq \epsilon a^2 + b^2/\epsilon$. One gets essentially the same estimate for $\|\Delta(\zeta \mathbf{u}_\rho)\|$ because $\Delta(\zeta \mathbf{u}_\rho) = \zeta \Delta \mathbf{u}_\rho + 2\nabla \zeta \cdot \nabla \mathbf{u}_\rho + (\Delta \zeta) \mathbf{u}_\rho$ and $\|\mathbf{u}\|_{\Omega'} \leq C_{\Omega'} \|\nabla \mathbf{u}\|$. Functions $\phi \in \mathbf{C}_0^\infty(\Omega')$ may, by a double integration by parts, be shown to satisfy $\|\Delta \phi\|_{\Omega'}^2 = \sum_{i,j=1}^n \|\phi_{x_i x_j}\|_{\Omega'}^2$. Thus $\|\mathbf{u}_{\rho x_i x_j}\|_{\Omega'} \leq \|(\zeta \mathbf{u}_\rho)_{x_i x_j}\|_{\Omega'} \leq \|\Delta(\zeta \mathbf{u}_\rho)\| \leq C_\zeta (\|\mathbf{f}\|_{\Omega'} + \|\nabla \mathbf{u}\|)$. Taking the limit as $\rho \rightarrow 0$ gives $\mathbf{u}_{x_i x_j} \in \mathbf{L}^2(\Omega'')$.

Finally, since $\mathbf{u}_{x_i x_j} \in \mathbf{L}^2_{100}(\Omega)$ we can integrate by parts and apply Lemma 3 to get $(\Delta \mathbf{u}, \phi) = -(\nabla \mathbf{u}, \nabla \phi) = (\tilde{\Delta} \mathbf{u}, \phi)$ for all $\phi \in \mathbf{D}(\Omega)$. It follows from Lemma 1 that $\Delta \mathbf{u} - \tilde{\Delta} \mathbf{u} = \nabla p$ for some $p \in L^2_{100}(\Omega)$.

Lemma 5. *The imbedding $\mathbf{J}_0(\Omega) \subset \mathbf{L}^2(\Omega')$ is compact if Ω' is a bounded subset of Ω .*

Proof. Let $\mathbf{W}_0(\Omega)$ be the completion of $\mathbf{C}_0^\infty(\Omega)$ in norm $\|\nabla \cdot\|$. Because $\mathbf{J}_0(\Omega)$ is a closed linear subspace of $\mathbf{W}_0(\Omega)$ any sequence which converges weakly in $\mathbf{J}_0(\Omega)$ also converges weakly in $\mathbf{W}_0(\Omega)$. Note too that the integral $(\mathbf{f}, \mathbf{u})$ defines a bounded linear functional on $\mathbf{u} \in \mathbf{W}_0(\Omega)$ if $\mathbf{f} \in \mathbf{C}_0^\infty(\Omega')$.

Now suppose $\{\mathbf{u}_k\}$ is a weakly convergent sequence in $\mathbf{J}_0(\Omega)$, say with limit zero. Then $\{\mathbf{u}_k\}$ is bounded in the Sobolev space $\mathbf{W}^{1,2}(\Omega')$ of $\mathbf{L}^2(\Omega')$ functions with first derivatives in $\mathbf{L}^2(\Omega')$ because it is bounded in the norm $\|\nabla \cdot\|$ of $\mathbf{J}_0(\Omega)$ and by Lemma 2 in the norm $\|\cdot\|_{\Omega'}$ of $\mathbf{L}^2(\Omega')$. Furthermore $(\mathbf{f}, \mathbf{u}_k) \rightarrow 0$ as $k \rightarrow \infty$ for every $\mathbf{f} \in \mathbf{C}_0^\infty(\Omega')$ because of the weak convergence in $\mathbf{W}_0(\Omega)$. Thus $\{\mathbf{u}_k\}$ converges weakly in $\mathbf{L}^2(\Omega')$, hence weakly in $\mathbf{W}^{1,2}(\Omega')$, and so by the Rellich theorem [2, p. 31] strongly in $\mathbf{L}^2(\Omega')$. The smoothness of $\partial\Omega'$ is obviously unimportant in this application of Rellich's theorem.

To each of the function spaces defined above over Ω there is a corresponding function space defined over $Q_{\epsilon,T}$. To be precise, we define $\mathbf{D}(Q_{\epsilon,T})$, $\mathbf{J}(Q_{\epsilon,T})$, $\mathbf{G}(Q_{\epsilon,T})$, $\mathbf{J}_0(Q_{\epsilon,T})$, $\mathbf{K}_0(Q_{\epsilon,T})$, and $\mathbf{H}_0(Q_{\epsilon,T})$ by substituting $Q_{\epsilon,T}$ for Ω , $(\cdot, \cdot)_{Q_{\epsilon,T}}$

for $(\cdot, \cdot)$, and $\|\cdot\|_{Q_{\epsilon, T}}$ for $\|\cdot\|$ in the corresponding definitions of these spaces over Ω . Clearly an element $\mathbf{u}(x, t)$ of any one of these spaces over $Q_{\epsilon, T}$ is an element of the corresponding space over Ω for almost every t . Since the operators ∇ and Δ denote differentiation with respect to the x variables only, elements of $\mathbf{J}(Q_{\epsilon, T})$, $\mathbf{G}(Q_{\epsilon, T})$, $\mathbf{J}_0(Q_{\epsilon, T})$, and $\mathbf{H}_0(Q_{\epsilon, T})$ need not be continuous with respect to t ; in particular functions in $\mathbf{H}_0(Q_T)$ need not vanish at $t = 0$. To ensure regularity with respect to t we will make appropriate assumptions about a function's weak time derivative. For instance, if $\mathbf{u}$ and $\mathbf{u}_t$ are both assumed to be in $\mathbf{J}_0(Q_T)$ one may easily see that $\mathbf{u}$ is continuous in $\mathbf{J}_0(\Omega)$ as a function of t , at least after redefinition on a set of t -measure zero. Finally, it is not hard to check that the proofs of lemmas 1, 2, 3, and 4 hold almost word for word if one substitutes $Q_{\epsilon, T}$ for Ω , etc.. Thus we have:

Lemma 6. *The analogues of lemmas 1, 2, 3, and 4 hold for the function spaces defined over $Q_{\epsilon, T}$.*

3. The initial boundary value problem with homogeneous boundary conditions. Given $\mathbf{a} \in \mathbf{J}_0(\Omega)$ and $\mathbf{f} \in \mathbf{L}^2_{1oc}(\bar{Q}_\infty)$ we consider the problem of finding a solution pair $\mathbf{u}, p$ of

$$(1) \quad \begin{aligned} \mathbf{u}_t - \Delta \mathbf{u} &= -\nabla p + \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

in Q_∞ which satisfies the initial and boundary conditions

$$(2) \quad \begin{aligned} \mathbf{u}(x, 0) &= \mathbf{a}(x) \\ \mathbf{u}(x, t) &= 0 \quad \text{on} \quad \partial\Omega \times [0, \infty) \\ \mathbf{u}(x, t) &\rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty \end{aligned}$$

in a generalized sense. To be precise we call $\mathbf{u}$ a class $\mathbf{H}_0$ *generalized solution* of (1), (2) if and only if

$$(3) \quad \mathbf{u} \in \mathbf{H}_0(Q_T) \text{ and } \mathbf{u}_t \in \mathbf{J}_0(Q_{\epsilon, T}) \text{ for all } 0 < \epsilon < T < \infty,$$

$$(4) \quad \|\nabla \mathbf{u}(t) - \nabla \mathbf{a}\| \rightarrow 0 \text{ as } t \rightarrow 0, \text{ and}$$

$$(5) \quad \text{a scalar function } p \in L^2_{1oc}(Q_\infty) \text{ with } \nabla p \in \mathbf{L}^2_{1oc}(Q_\infty) \text{ exists such that (1) holds almost everywhere in } Q_\infty.$$

The generalized solutions just defined satisfy the condition at infinity, $\mathbf{u}(x, t) \rightarrow 0$ as $x \rightarrow \infty$, in the same generalized sense that elements of $\mathbf{H}_0(\Omega)$ do. As was shown in the discussion following Lemma 2 this is a reasonable enough generalized sense if $n \geq 3$. If $n = 2$ it was only shown that functions in $\mathbf{H}_0(\Omega)$ satisfy in a generalized sense the condition $\mathbf{u}(x) = o(\log |x|)$ as $x \rightarrow \infty$. Thus when $n = 2$ a limiting value for neither the pressure p nor velocity $\mathbf{u}$ is directly imposed on class $\mathbf{H}_0$ solutions at infinity. It is something of a paradox that uniqueness holds within this class of solutions when $n = 2$ just as it does in

higher dimensions. This matter, which is related to the Stokes paradox, is studied further in [5].

Lemma 7. *Suppose $\mathbf{u}$ satisfies (3). Then (5) holds if and only if*

$$(5') \quad \int_{Q_{\epsilon, T}} (\nabla \mathbf{u}_t : \nabla \phi + \tilde{\Delta} \mathbf{u} \cdot \tilde{\Delta} \phi + \mathbf{f} \cdot \tilde{\Delta} \phi) dx dt = 0$$

holds for all $\phi \in \mathbf{K}_0(Q_{\epsilon, T})$, and all $0 < \epsilon < T < \infty$.

Proof. Suppose (5) holds. In view of Lemmas 2 and 4 condition (3) implies $\mathbf{u}_t, \Delta \mathbf{u} \in \mathbf{L}^2_{100}(Q_\infty)$. Thus if $\phi \in \mathbf{K}_0(Q_{\epsilon, T})$, so that $\tilde{\Delta} \phi \in \mathbf{D}(Q_{\epsilon, T})$, we can multiply the first of equations (1) by $\Delta \phi$ and integrate over $Q_{\epsilon, T}$. Now $\int_{Q_{\epsilon, T}} \nabla p \cdot \tilde{\Delta} \phi dx dt = 0$ by Lemma 1, and $\int_{Q_{\epsilon, T}} \mathbf{u}_t \cdot \tilde{\Delta} \phi dx dt = - \int_{Q_{\epsilon, T}} \nabla \mathbf{u}_t : \nabla \phi dx dt$ by Lemma 3. Finally $\int_{Q_{\epsilon, T}} \Delta \mathbf{u} \cdot \tilde{\Delta} \phi dx dt = \int_{Q_{\epsilon, T}} \tilde{\Delta} \mathbf{u} \cdot \tilde{\Delta} \phi dx dt$ because Lemma 4 implies $\Delta \mathbf{u} = \tilde{\Delta} \mathbf{u} + \nabla q$ for some $q \in L^2_{100}(Q_\infty)$ with $\nabla q \in \mathbf{L}^2_{100}(Q_\infty)$, and Lemma 1 implies $\int_{Q_{\epsilon, T}} \nabla q \cdot \tilde{\Delta} \phi dx dt = 0$. Thus (5') is obtained.

Now suppose (5') holds for all $\phi \in \mathbf{K}_0(Q_{\epsilon, T})$. By Lemma 3 again, $\int_{Q_{\epsilon, T}} (\mathbf{u}_t - \tilde{\Delta} \mathbf{u} - \mathbf{f}) \cdot \tilde{\Delta} \phi dx dt = 0$. Because $\tilde{\Delta}$ maps $\mathbf{K}_0(Q_{\epsilon, T})$ onto $\mathbf{D}(Q_{\epsilon, T})$ it follows from Lemma 1 that $\mathbf{u}_t - \tilde{\Delta} \mathbf{u} - \mathbf{f} = -\nabla q$ for some $q \in L^2_{100}(Q_\infty)$. Finally Lemma 4 ensures that $\mathbf{u}_t - \Delta \mathbf{u} - \mathbf{f} = -\nabla p$ for some $p \in L^2_{100}(Q_\infty)$.

Theorem 1. *There is at most one class $\mathbf{H}_0$ solution of (1), (2). Solutions depend continuously in $\mathbf{H}_0(Q_T)$ upon the prescribed initial data in $\mathbf{J}_0(\Omega)$.*

Proof. Suppose $\mathbf{u}$ and $\bar{\mathbf{u}}$ are generalized solutions taking initial data $\mathbf{a}$ and $\bar{\mathbf{a}}$ respectively. Let $\mathbf{w} = \mathbf{u} - \bar{\mathbf{u}}$. Subtraction of (5') for $\bar{\mathbf{u}}$ from (5') for $\mathbf{u}$ yields $\int_{Q_{\epsilon, T}} (\nabla \mathbf{w}_t : \nabla \phi + \tilde{\Delta} \mathbf{w} \cdot \tilde{\Delta} \phi) dx dt = 0$ for all $\phi \in \mathbf{K}_0(Q_{\epsilon, T})$, and by taking a limit, even for all $\phi \in \mathbf{H}_0(Q_{\epsilon, T})$. Since $\mathbf{w} \in \mathbf{H}_0(Q_{\epsilon, T})$ we get $\int_{\epsilon}^T \int_{\Omega} (\nabla \mathbf{w}_t : \nabla \mathbf{w} + \tilde{\Delta} \mathbf{w} \cdot \tilde{\Delta} \mathbf{w}) dx dt = 0$ and consequently $(1/2) \|\nabla \mathbf{w}(T)\|^2 + \int_{\epsilon}^T \|\tilde{\Delta} \mathbf{w}\|^2 dt = (1/2) \|\nabla \mathbf{w}(\epsilon)\|^2$, at least after redefinition of $\mathbf{w}$ on a set of t -measure zero. From condition (4) it follows that $\|\nabla \mathbf{w}(\epsilon) - \nabla(\mathbf{a} - \bar{\mathbf{a}})\| \rightarrow 0$ and hence that $\|\nabla \mathbf{w}(\epsilon)\| \rightarrow \|\nabla(\mathbf{a} - \bar{\mathbf{a}})\|$ as $\epsilon \rightarrow 0$. Thus

$$(6) \quad \frac{1}{2} \|\nabla \mathbf{u}(T) - \nabla \bar{\mathbf{u}}(T)\|^2 + \int_0^T \|\tilde{\Delta}(\mathbf{u} - \bar{\mathbf{u}})\|^2 dt = \frac{1}{2} \|\nabla(\mathbf{a} - \bar{\mathbf{a}})\|^2.$$

This implies the continuous dependence in $\mathbf{H}_0(Q_T)$; we have $\|\nabla \mathbf{w}\|_{Q_T}^2 + \|\tilde{\Delta} \mathbf{w}\|_{Q_T}^2 \leq (1/2 + T) \|\nabla(\mathbf{a} - \bar{\mathbf{a}})\|^2$. Uniqueness follows by setting $\mathbf{a} = \bar{\mathbf{a}}$.

4. Existence. Our existence proofs are based on the method of Galerkin approximation. Since $\mathbf{H}_0(\Omega)$ is a separable Hilbert space with dense set $\mathbf{K}_0(\Omega)$ there exists a system of functions $\{\mathbf{a}'\}$ which is contained in $\mathbf{K}_0(\Omega)$ and complete in $\mathbf{H}_0(\Omega)$. It may be assumed that $\{\mathbf{a}'\}$ is orthonormal in $\mathbf{J}_0(\Omega)$. We let

$$\mathbf{u}^n(x, t) = \sum_{\ell=1}^n c_{\ell n}(t) \mathbf{a}'_{\ell}(x)$$

be the solution of the system ($\ell = 1, 2, \dots, n$) of ordinary differential equations

$$(7) \quad \int_{\Omega} (\nabla \mathbf{u}_t^n : \nabla \mathbf{a}' + \tilde{\Delta} \mathbf{u}^n \cdot \tilde{\Delta} \mathbf{a}' + \mathbf{f} \cdot \tilde{\Delta} \mathbf{a}') dx = 0$$

for which the coefficients $c_{\ell n}(t)$ take initial values $c_{\ell n}(0)$ prescribed below. Equations (7) can be written in the form

$$\frac{d}{dt} c_{\ell n}(t) + \sum_{k=1}^n c_{kn}(t) \int_{\Omega} \tilde{\Delta} \mathbf{a}^k \cdot \tilde{\Delta} \mathbf{a}' dx + \int_{\Omega} \mathbf{f} \cdot \tilde{\Delta} \mathbf{a}' dx = 0.$$

Because $\mathbf{f} \in \mathbf{L}^2_{100}(\bar{Q}_{\infty})$ this linear system is uniquely solvable on the whole time interval $[0, \infty)$. Obviously $\mathbf{u}^n(t) \in \mathbf{K}_0(\Omega)$ for all t , and $\mathbf{u}_t^n(t) \in \mathbf{K}_0(\Omega)$ for almost all t .

Theorem 2. *A class $\mathbf{H}_0$ solution of (1), (2) exists if $\mathbf{a} \in \mathbf{J}_0(\Omega)$, and if $\mathbf{f} \in \mathbf{J}_0(Q_T)$ for all $T > 0$.*

Proof. Assume first that $\mathbf{a} \in \mathbf{K}_0(\Omega)$. Then without further loss of generality we may assume $\mathbf{a}^1 = \|\nabla \mathbf{a}\|^{-1} \mathbf{a}$. Thus by prescribing initial values $c_{1n}(0) = \|\nabla \mathbf{a}\|$, and $c_{\ell n}(0) = 0$ for $\ell = 2, \dots, n$, we get $\mathbf{u}^n(x, 0) = \mathbf{a}(x)$ for all n . Multiplying equations (7) by $c_{\ell n}(t)$ and summing $\sum_{\ell=1}^n$ we obtain

$$(8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}^n\|^2 + \|\tilde{\Delta} \mathbf{u}^n\|^2 &= - \int_{\Omega} \mathbf{f} \cdot \tilde{\Delta} \mathbf{u}^n dx \\ &= \int_{\Omega} \nabla \mathbf{f} : \nabla \mathbf{u}^n dx \leq \|\nabla \mathbf{f}\| \cdot \|\nabla \mathbf{u}^n\|. \end{aligned}$$

It follows that $d/dt \|\nabla \mathbf{u}^n\| \leq \|\nabla \mathbf{f}\|$ and hence that

$$(9) \quad \|\nabla \mathbf{u}^n(t)\| \leq \|\nabla \mathbf{a}\| + \int_0^t \|\nabla \mathbf{f}\| d\tau.$$

Multiplying (7) by $dc_{\ell n}(t)/dt$ and summing $\sum_{\ell=1}^n$ we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\Delta} \mathbf{u}^n\|^2 + \|\nabla \mathbf{u}_t^n\|^2 &= - \int_{\Omega} \mathbf{f} \cdot \tilde{\Delta} \mathbf{u}_t^n dx \\ &= \int_{\Omega} \nabla \mathbf{f} : \nabla \mathbf{u}_t^n dx \leq \frac{1}{2} \|\nabla \mathbf{f}\|^2 + \frac{1}{2} \|\nabla \mathbf{u}_t^n\|^2. \end{aligned}$$

Therefore

$$(10) \quad \frac{d}{dt} \|\tilde{\Delta} \mathbf{u}^n\|^2 + \|\nabla \mathbf{u}_t^n\|^2 \leq \|\nabla \mathbf{f}\|^2,$$

which upon integration yields

$$(11) \quad \|\tilde{\Delta} \mathbf{u}^n(t)\|^2 + \int_0^t \|\nabla \mathbf{u}_\tau^n\|^2 d\tau \leq \|\tilde{\Delta} \mathbf{a}\|^2 + \int_0^t \|\nabla \mathbf{f}\|^2 d\tau.$$

Clearly $\{\mathbf{u}^n\}$ is bounded in $\mathbf{H}_0(Q_T)$ and $\{\mathbf{u}_t^n\}$ is bounded in $\mathbf{J}_0(Q_T)$ for all $T > 0$. Thus a subsequence can be selected such that $\{\mathbf{u}^{n_k}\}$ and $\{\mathbf{u}_t^{n_k}\}$ have

weak limits $\mathbf{u} \in \mathbf{H}_0(Q_T)$ and $\mathbf{u}_t \in \mathbf{J}_0(Q_T)$ respectively for all $T > 0$. Obviously $\mathbf{u}_t$ is the weak time derivative of $\mathbf{u}$. It can be assumed, after redefinition on a set of t -measure zero if necessary, that $\mathbf{u}$ and its derivatives satisfy for every t those estimates which hold for every $\mathbf{u}^n$.

Since $\mathbf{f} \in \mathbf{J}_0(Q_T)$, equation (5') with $\epsilon = 0$ is equivalent to

$$\int_{Q_T} (\nabla \mathbf{u}_t : \nabla \phi + \tilde{\Delta} \mathbf{u} \cdot \tilde{\Delta} \phi - \nabla \mathbf{f} : \nabla \phi) dx dt = 0.$$

In order to check this condition for all $\phi \in \mathbf{K}_0(Q_T)$ we need only consider ϕ of the form $\phi(x, t) = \sum_{i=1}^m d_i(t) \mathbf{a}'(x)$ with arbitrary coefficients $d_i(t)$. The validity of

$$\int_0^T \sum_{i=1}^m d_i(t) \int_{\Omega} (\nabla \mathbf{u}_t : \nabla \mathbf{a}' + \tilde{\Delta} \mathbf{u} \cdot \tilde{\Delta} \mathbf{a}' - \nabla \mathbf{f} : \nabla \mathbf{a}') dx dt = 0$$

follows from (7) and from the weak $L^2(Q_T)$ convergence of components of $\nabla \mathbf{u}_t^{n_k}$ and $\tilde{\Delta} \mathbf{u}^{n_k}$ to those of $\nabla \mathbf{u}_t$ and $\tilde{\Delta} \mathbf{u}$ respectively.

To verify that $\|\nabla \mathbf{u}(t) - \nabla \mathbf{a}\| \rightarrow 0$ as $t \rightarrow 0$, observe that estimates (9) and (11) imply

$$\begin{aligned} \|\nabla \mathbf{u}^n(t) - \nabla \mathbf{a}\|^2 &= \int_0^t \frac{d}{d\tau} \|\nabla \mathbf{u}^n - \nabla \mathbf{a}\|^2 d\tau \\ &= \int_0^t \int_{\Omega} 2 \nabla(\mathbf{u}^n - \mathbf{a}) : \nabla \mathbf{u}_\tau^n dx d\tau \\ &\leq 2 \|\nabla \mathbf{u}^n - \nabla \mathbf{a}\|_{Q_t} \cdot \|\nabla \mathbf{u}_t^n\|_{Q_t} \rightarrow 0 \quad \text{as } t \rightarrow 0 \end{aligned}$$

uniformly in n . The proof is complete for $\mathbf{a} \in \mathbf{K}_0(\Omega)$.

If $\mathbf{a} \in \mathbf{J}_0(\Omega)$, a sequence $\{\alpha^k\} \subset \mathbf{K}_0(\Omega)$ exists such that $\|\nabla \alpha^k - \nabla \mathbf{a}\| \rightarrow 0$ as $k \rightarrow \infty$. Corresponding to each α^k there is a generalized solution $\mathbf{u}^k$ which takes α^k as initial data. By Theorem 1 the solutions $\mathbf{u}^k$ converge strongly in $\mathbf{H}_0(Q_T)$ to a limit $\mathbf{u}$ for all $T > 0$. In fact (6) yields

$$(12) \quad \frac{1}{2} \|\nabla \mathbf{u}^k(t) - \nabla \mathbf{u}(t)\|^2 + \int_0^t \|\tilde{\Delta}(\mathbf{u}^k - \mathbf{u})\|^2 d\tau = \frac{1}{2} \|\nabla \alpha^k - \nabla \mathbf{a}\|^2.$$

We now show that $\{\mathbf{u}^k\}$ is bounded in $\mathbf{J}_0(Q_{\epsilon, T})$ for all $0 < \epsilon < T < \infty$, thus ensuring that $\mathbf{u}$ has a time derivative $\mathbf{u}_t \in \mathbf{J}_0(Q_{\epsilon, T})$ and that a subsequence of $\{\mathbf{u}^k\}$ converges weakly to $\mathbf{u}_t$ in $\mathbf{J}_0(Q_{\epsilon, T})$. To obtain the bound we first combine (8) and (9) to get

$$\begin{aligned} \frac{1}{2} \|\nabla \mathbf{u}^k(t)\|^2 \\ + \int_0^t \|\tilde{\Delta} \mathbf{u}^k\|^2 d\tau \leq \frac{1}{2} \|\nabla \alpha^k\|^2 + \int_0^t \|\nabla \mathbf{f}\| \left(\|\nabla \alpha^k\| + \int_0^\tau \|\nabla \mathbf{f}\| d\sigma \right) d\tau. \end{aligned}$$

The right side is, for all k , less than some continuous function $F(t)$ defined on $[0, \infty)$. Thus $\int_0^\epsilon \|\tilde{\Delta} \mathbf{u}^k\|^2 d\tau \leq F(\epsilon)$, and consequently for some $\tau_k \in (0, \epsilon)$ there holds $\|\tilde{\Delta} \mathbf{u}^k(\tau_k)\|^2 \leq F(\epsilon)/\epsilon$. Finally, integration of (10) yields

$$||\tilde{\Delta}\mathbf{u}^k(T)||^2 + \int_{\tau_k}^T ||\nabla\mathbf{u}_t^k||^2 dt \leq ||\tilde{\Delta}\mathbf{u}^k(\tau_k)||^2 + \int_{\tau_k}^T ||\nabla\mathbf{f}||^2 dt,$$

and therefore

$$\int_0^T ||\nabla\mathbf{u}_t^k||^2 dt \leq \frac{1}{\epsilon} F(\epsilon) + \int_0^T ||\nabla\mathbf{f}||^2 dt.$$

By a weak convergence argument, the limit $\mathbf{u}$ can now be shown to satisfy (5') for all $\phi \in \mathbf{K}_0(Q_{\epsilon, T})$.

To complete the proof, we check that $\mathbf{u}$ assumes the initial data $\mathbf{a}$. Given any $\epsilon > 0$, choose k so that $||\nabla\alpha^k - \nabla\mathbf{a}|| \leq \epsilon/3$ and then choose $\tau > 0$ so that $||\nabla\mathbf{u}^k(t) - \nabla\alpha^k|| < \epsilon/3$ for $t \in (0, \tau)$. Taking note of (12) also, we have $||\nabla\mathbf{u}(t) - \nabla\mathbf{a}|| \leq ||\nabla\mathbf{u}(t) - \nabla\mathbf{u}^k(t)|| + ||\nabla\mathbf{u}^k(t) - \nabla\alpha^k|| + ||\nabla\alpha^k - \nabla\mathbf{a}|| < \epsilon$ for all $t \in (0, \tau)$.

The force considered in Theorem 2, although not necessarily square-summable, is assumed to be solenoidal and to vanish on $\partial\Omega$. These assumptions are unnecessary if alternative conditions are met, most importantly, if $\mathbf{f}$ and $\mathbf{f}_t$ are square-summable.

Theorem 3. *A class $\mathbf{H}_0$ solution of (1), (2) exists if $\mathbf{a} \in \mathbf{J}_0(\Omega)$, and if $\mathbf{f}, \mathbf{f}_t \in \mathbf{L}^2(Q_T)$ for all $T > 0$.*

Proof. We assume first that $\mathbf{a}(x) \equiv 0$ and $\mathbf{f}(x, 0) \equiv 0$. Since $\mathbf{a} = 0$, we set initial values $c_{\ell n}(0) = 0$ for $\ell = 1, \dots, n$. Multiplying (7) by $c_{\ell n}(t)$ and summing we obtain

$$\frac{1}{2} \frac{d}{dt} ||\nabla\mathbf{u}^n||^2 + ||\tilde{\Delta}\mathbf{u}^n||^2 \leq \frac{1}{2} ||\mathbf{f}||^2 + \frac{1}{2} ||\tilde{\Delta}\mathbf{u}^n||^2.$$

Since $\mathbf{u}^n(0) = 0$, integration yields

$$(13) \quad ||\nabla\mathbf{u}^n(t)||^2 + \int_0^t ||\tilde{\Delta}\mathbf{u}^n||^2 d\tau \leq \int_0^t ||\mathbf{f}||^2 d\tau.$$

By differentiating (7) with respect to t , then multiplying by $dc_{\ell n}(t)/dt$, and summing, we get

$$\frac{1}{2} \frac{d}{dt} ||\nabla\mathbf{u}_t^n||^2 + ||\tilde{\Delta}\mathbf{u}_t^n||^2 \leq \frac{1}{2} ||\mathbf{f}_t||^2 + \frac{1}{2} ||\tilde{\Delta}\mathbf{u}_t^n||^2.$$

Because both $\mathbf{u}^n$ and $\mathbf{f}$ vanish initially, it follows from (7) that $\nabla\mathbf{u}_t^n(0)$ vanishes also. So integration yields

$$(14) \quad ||\nabla\mathbf{u}_t^n(t)||^2 + \int_0^t ||\tilde{\Delta}\mathbf{u}_t^n||^2 d\tau \leq \int_0^t ||\mathbf{f}_t||^2 d\tau.$$

Using (13) and (14) one can readily show, as in the proof of Theorem 2, that a subsequence of $\{\mathbf{u}^n\}$ converges to a solution $\mathbf{u}$.

If $\mathbf{a}(x)$ and $\mathbf{f}(x, 0)$ don't vanish, we can extend $\mathbf{f}$'s domain of definition to $[-1, \infty)$ in such a way that $\mathbf{f}(x, -1) \equiv 0$ and $\mathbf{f}, \mathbf{f}_t \in \mathbf{L}^2(\Omega \times [-1, T])$ for all

$T > -1$. Then a generalized solution $\mathbf{u}_1$ exists on $[-1, \infty)$ which is subject to the force $\mathbf{f}$ and which vanishes at $t = -1$. By Theorem 2, a generalized solution $\mathbf{u}_2$ exists on $[0, \infty)$ which is subject to zero force, and which equals $\mathbf{a}(x) - \mathbf{u}_1(x, 0)$ at $t = 0$. The sum $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ is defined on $[0, \infty)$ and solves the original problem.

Another condition on the force which ensures existence is that $\mathbf{f} = \nabla q$ for some $q \in L^2_{1,0c}(Q_\infty)$ with $\nabla q \in L^2_{1,0c}(\bar{Q}_\infty)$. In this case the force term in (1) can be absorbed into the pressure term. If, with such a force, the initial data $\mathbf{a}$ vanishes, then $\mathbf{u} \equiv 0$ is the unique generalized solution of (1), (2). The corresponding pressure p may be taken equal to q .

Corollary. A class $\mathbf{H}_0$ solution of (1), (2) exists if $\mathbf{a} \in \mathbf{J}_0(\Omega)$, and if $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3$ where (i) $\mathbf{f}_1 \in \mathbf{J}_0(Q_T)$ for all $T > 0$, (ii) $\mathbf{f}_2, \mathbf{f}_3 \in \mathbf{L}^2(Q_T)$ for all $T > 0$, and (iii) $\mathbf{f}_3 = \nabla q$ for some $q \in L^2_{1,0c}(Q_\infty)$ with $\nabla q \in L^2_{1,0c}(\bar{Q}_\infty)$.

5. Behavior as $t \rightarrow \infty$.

Theorem 4. Suppose $\mathbf{u}$ is a class $\mathbf{H}_0$ solution of (1), (2) with prescribed force $\mathbf{f} \equiv 0$. Then $\|\mathbf{u}(t)\|_{\Omega'} \rightarrow 0$ as $t \rightarrow \infty$ for every bounded subset Ω' of Ω . The same conclusion holds if $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3$ as in the corollary to Theorems 2 and 3 provided $\int_0^\infty \|\nabla \mathbf{f}_1\| dt$ and $\int_0^\infty \|\mathbf{f}_2\|^2 dt$ are finite.

Proof. Assume first that $\mathbf{f} \equiv 0$. Because solutions are unique, it suffices to consider the solution constructed by Galerkin's method in the proof of Theorem 2. We begin by showing that $\mathbf{u}$ satisfies

$$(15) \quad \|\tilde{\Delta} \mathbf{u}(t)\|^2 \leq \frac{1}{2} t^{-1} \|\nabla \mathbf{a}\|^2, \quad \text{and} \quad \|\nabla \mathbf{u}(t)\| \leq \|\nabla \mathbf{a}\|.$$

It follows from (10) that $\|\tilde{\Delta} \mathbf{u}^n(t)\|$ is a non-increasing function of t for every Galerkin approximation. Thus, from the integrated form

$$\frac{1}{2} \|\nabla \mathbf{u}^n(t)\|^2 + \int_0^t \|\tilde{\Delta} \mathbf{u}^n\|^2 d\tau = \frac{1}{2} \|\nabla \mathbf{a}\|^2$$

of (8), we see that $\|\tilde{\Delta} \mathbf{u}^n(t)\|^2 \leq (2t)^{-1} \|\nabla \mathbf{a}\|^2$ and also $\|\nabla \mathbf{u}^n(t)\| \leq \|\nabla \mathbf{a}\|$ hold for all $t \geq 0$ and for every approximation. The solution $\mathbf{u}$ satisfies these estimates because it is obtained as a limit of the Galerkin approximations if $\mathbf{a} \in \mathbf{K}_0(\Omega)$, or as a limit of solutions obtained as limits of the approximations if $\mathbf{a} \in \mathbf{J}_0(\Omega)$.

Since $\mathbf{u}(t)$ is bounded in $\mathbf{J}_0(\Omega)$, the weak convergence of $\mathbf{u}(t)$ to zero in $\mathbf{J}_0(\Omega)$ need only be tested by functions $\phi \in \mathbf{D}(\Omega)$. By (15),

$$\left| \int_\Omega \nabla \mathbf{u}(t) : \nabla \phi dx \right| = \left| \int_\Omega \tilde{\Delta} \mathbf{u}(t) \cdot \phi dx \right| \leq \|\tilde{\Delta} \mathbf{u}(t)\| \cdot \|\phi\| \rightarrow 0$$

as $t \rightarrow \infty$. Since the imbedding of $\mathbf{J}_0(\Omega)$ in $\mathbf{L}^2(\Omega')$ is compact, Lemma 5, we have $\|\mathbf{u}(t)\|_{\Omega'} \rightarrow 0$ as $t \rightarrow \infty$.

Now consider the general case with $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3$, and let $\epsilon > 0$ be given. Choose τ sufficiently large that $\int_{\tau}^{\infty} \|\nabla \mathbf{f}_1\| dt < \epsilon(4C_{\Omega'})^{-1}$ and $\int_{\tau}^{\infty} \|\mathbf{f}_2\|^2 dt < \epsilon^2(4C_{\Omega'})^{-2}$, where $C_{\Omega'}$ is the constant in Lemma 2. For $t \geq \tau$ we express $\mathbf{u}$ as the sum of two solutions $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. Here $\mathbf{u}_1$ is the solution on $[\tau, \infty)$ which vanishes at $t = \tau$ and is subject to the force $\mathbf{f}$. $\mathbf{u}_2$ is the solution on $[\tau, \infty)$ which equals $\mathbf{u}(\tau)$ at $t = \tau$ and is subject to zero force. It follows from estimates (9) and (13) that $\|\nabla \mathbf{u}_1(t)\| \leq \epsilon(2C_{\Omega'})^{-1}$, and hence that $\|\mathbf{u}_1(t)\|_{\Omega'} \leq \epsilon/2$ for all $t > \tau$. We have shown above that $\|\mathbf{u}_2(t)\|_{\Omega'} \rightarrow 0$ as $t \rightarrow \infty$. Therefore there exists a number $\tau^* > \tau$ such that $\|\mathbf{u}(t)\|_{\Omega'} \leq \|\mathbf{u}_1(t)\|_{\Omega'} + \|\mathbf{u}_2(t)\|_{\Omega'} < \epsilon$ for all $t > \tau^*$.

6. On the convergence to a steady limit of nonstationary stokes flow past an obstacle. In this section we assume Ω contains a neighborhood of infinity. We think of Ω as the exterior of an object and imagine it is occupied by a Stokesian fluid which adheres to the object's boundary. No assumptions about the regularity of $\partial\Omega$ are necessary because the boundary condition on $\partial\Omega$ will be considered in only a generalized sense.

The problem of steady Stokes flow past an object is that of finding a solution pair $\mathbf{w}(x)$, $p(x)$ of

$$(16) \quad \Delta \mathbf{w} = \nabla p \quad \text{in } \Omega$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega$$

$$(17) \quad \mathbf{w} = 0 \quad \text{on } \partial\Omega$$

$$\mathbf{w}(x) \rightarrow \mathbf{w}_{\infty} \quad \text{as } |x| \rightarrow \infty$$

where $\mathbf{w}_{\infty}$ is a prescribed constant vector, the velocity at infinity. In order to reduce this problem to one with a homogeneous condition at infinity a function $\mathbf{b}(x)$ is introduced which is smooth, solenoidal, equal to zero in a neighborhood of $\partial\Omega$, and equal to $\mathbf{w}_{\infty}$ in a neighborhood of infinity. Such functions are easy to construct. For example, if $\mathbf{w}_{\infty} = (1, 0, \dots, 0)$, one can take $\mathbf{b}(x) = \{\partial(x_2 \zeta(x))/\partial x_2, -\partial(x_2 \zeta(x))/\partial x_1, 0, \dots, 0\}$ where $\zeta \in C^{\infty}(\Omega)$ satisfies $\zeta \equiv 0$ in a neighborhood of $\partial\Omega$ and $\zeta \equiv 1$ in a neighborhood of infinity. In this section $\mathbf{b}$ will always denote a function with the properties listed above. We call $\mathbf{w}(x)$ a *class $\mathbf{H}_0$ generalized solution* of (16), (17) if and only if (i) $\mathbf{w} - \mathbf{b} \in \mathbf{H}_0(\Omega)$ for some $\mathbf{b}$ with the properties just mentioned, and (ii) a scalar $p(x)$ with $p \in L^2_{1,00}(\Omega)$ exists such that $\Delta \mathbf{w} = \nabla p$ in Ω . Evidently, if $\mathbf{w} - \mathbf{b} \in \mathbf{H}_0(\Omega)$ for some function $\mathbf{b}$ with the specified properties, then $\mathbf{w} - \mathbf{b} \in \mathbf{H}_0(\Omega)$ for every such function $\mathbf{b}$. Generalized solutions are unique because if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two solutions then $\mathbf{w}_1 - \mathbf{w}_2 \in \mathbf{H}_0(\Omega)$ and $\Delta(\mathbf{w}_1 - \mathbf{w}_2) = \nabla(p_1 - p_2)$. Hence $(\nabla(\mathbf{w}_1 - \mathbf{w}_2), \nabla \phi) = 0$ for all $\phi \in \mathbf{D}(\Omega)$, and by taking a limit for $\phi = \mathbf{w}_1 - \mathbf{w}_2$. Existence of a solution is easily demonstrated; simply take $\mathbf{w} = \bar{\mathbf{w}} + \mathbf{b}$ where $\bar{\mathbf{w}}$ is in $\mathbf{H}_0(\Omega)$ and satisfies $\Delta \bar{\mathbf{w}} = -\Delta \mathbf{b}$.

The boundary conditions (17) are satisfied by these generalized solutions in the same senses that elements of $\mathbf{H}_0(\Omega)$ tend to zero on $\partial\Omega$ and at ∞ . It follows from our earlier remarks that the condition at infinity may not be satisfied in a reasonable sense if $n = 2$. In fact it will be shown in [5] that if $n = 2$ the unique class $\mathbf{H}_0$ solution of (16), (17) is $\mathbf{w} \equiv 0$. This is the Stokes paradox, at least with respect to class $\mathbf{H}_0$ solutions.

It is generally presumed that solutions of steady flow problems occur as limits of solutions of nonstationary flow problems. We will consider two problems for the nonstationary Stokes equations which are related to the stationary problem (16), (17) in such a way that their solutions may be expected to converge to those of (16), (17) as $t \rightarrow \infty$. The first and simpler of these problems is the initial boundary value problem

$$\begin{aligned}
 (18) \quad & \mathbf{v}_t - \Delta \mathbf{v} = -\nabla p \quad \text{in } Q_\infty \\
 & \nabla \cdot \mathbf{v} = 0 \quad \text{in } Q_\infty, \\
 & \mathbf{v}(x, 0) = \mathbf{b}(x) \quad \text{in } \Omega \\
 (19) \quad & \mathbf{v}(x, t) = 0 \quad \text{on } \partial\Omega \times [0, \infty) \\
 & \mathbf{v}(x, t) \rightarrow \mathbf{w}_\infty \quad \text{as } |x| \rightarrow \infty,
 \end{aligned}$$

with the prescribed initial velocity $\mathbf{b}$ equal to $\mathbf{w}_\infty$ in a neighborhood of infinity and the prescribed limit at infinity held equal to $\mathbf{w}_\infty$ for all time.

Problem (18), (19) can be reduced to the homogeneous problem (1), (2) as follows. We say $\mathbf{v}$ is a *class $\mathbf{H}_0$ solution* of (18), (19) if and only if $\mathbf{v}(x, t) = \bar{\mathbf{v}}(x, t) + \mathbf{b}(x)$ where $\bar{\mathbf{v}}$ is a class $\mathbf{H}_0$ solution of (1), (2) with prescribed initial velocity $\mathbf{a} \equiv 0$ and force $\mathbf{f} \equiv \Delta \mathbf{b}$. Since $\Delta \mathbf{b} \in \mathbf{D}(\Omega)$, Theorems 1 and 2 assure the unique solvability of (18), (19). That $\mathbf{v}(x, t)$ converges to the solution $\mathbf{w}(x) = \bar{\mathbf{w}}(x) + \mathbf{b}(x)$ of (16), (17) as $t \rightarrow \infty$ is shown as follows. The difference $\mathbf{v}(x, t) - \mathbf{w}(x) = \bar{\mathbf{v}}(x, t) - \bar{\mathbf{w}}(x)$ is a generalized solution of (1), (2) with prescribed initial velocity $\mathbf{a}(x) = -\bar{\mathbf{w}}(x)$ and force $\mathbf{f}(x) \equiv 0$. Thus Theorem 4 applies and gives $\|\mathbf{v}(t) - \mathbf{w}\|_{\Omega'} \rightarrow 0$ as $t \rightarrow \infty$ for every bounded subset Ω' of Ω . We have proved:

Theorem 5. *The nonstationary problem (18), (19) has a unique class $\mathbf{H}_0$ solution and it converges in $\mathbf{L}^2_{loc}(\Omega)$ as $t \rightarrow \infty$ to the solution of the stationary problem (16), (17).*

The second nonstationary problem we consider models the following idealized experiment. Initially an object and a space filling fluid which surrounds it are both at rest. Subsequently the object is smoothly accelerated to a state of uniform translation which is then maintained indefinitely. We study the fluid motion in a reference frame attached to the object. Let the object's velocity relative to the inertial frame of initial rest be $-\alpha(t)\mathbf{w}_\infty$, where α is a smooth function such that $\alpha(0) = 0$ and $\alpha(t) = 1$ for all sufficiently large t . Then the governing equations written in the reference frame attached to the object

contain a fictitious force density $\alpha'(t)\mathbf{w}_\infty$, and the fluid velocity at infinity relative to this frame is $\alpha(t)\mathbf{w}_\infty$. Thus we consider the problem

$$\begin{aligned}
 (20) \quad & \mathbf{v}_t - \Delta \mathbf{v} = -\nabla p + \alpha'(t)\mathbf{w}_\infty \quad \text{in } Q_\infty \\
 & \nabla \cdot \mathbf{v} = 0 \quad \text{in } Q_\infty, \\
 & \mathbf{v}(x, 0) = 0 \quad \text{in } \Omega \\
 (21) \quad & \mathbf{v}(x, t) = 0 \quad \text{on } \partial\Omega \times [0, \infty) \\
 & \mathbf{v}(x, t) \rightarrow \alpha(t)\mathbf{w}_\infty \quad \text{as } |x| \rightarrow \infty.
 \end{aligned}$$

We call $\mathbf{v}$ a *class $\mathbf{H}_0$ solution* of (20), (21) if and only if $\mathbf{v}(x, t) = \bar{\mathbf{v}}(x, t) + \alpha(t)\mathbf{b}(x)$ where $\mathbf{b}$ has the properties listed earlier and $\bar{\mathbf{v}}$ is a class $\mathbf{H}_0$ solution of (1), (2) with prescribed initial velocity $\mathbf{a} \equiv 0$ and force $\mathbf{f}(x, t) = \alpha(t)\Delta\mathbf{b}(x) + \alpha'(t)(\mathbf{w}_\infty - \mathbf{b}(x))$. Existence of a solution follows immediately from Theorem 3. Uniqueness is verified by showing that the difference of two solutions is a solution of (1), (2) with $\mathbf{a} \equiv 0$ and $\mathbf{f} \equiv 0$. If $\mathbf{w}(x) = \bar{\mathbf{w}}(x) + \mathbf{b}$ is the solution of (16), (17), then $\mathbf{v}(x, t) - \alpha(t)\mathbf{w}(x) = \bar{\mathbf{v}}(x, t) - \alpha(t)\bar{\mathbf{w}}(x)$ is a solution of (1), (2) with prescribed force $\mathbf{f}(x, t) = \alpha'(t)(\mathbf{w}_\infty - \mathbf{b}(x) - \bar{\mathbf{w}}(x))$. This force vanishes for large t and therefore, by Theorem 4, $\|\mathbf{v}(t) - \mathbf{w}\|_{\Omega'} \rightarrow 0$ as $t \rightarrow \infty$ for bounded subsets Ω' of Ω . We have:

Theorem 6. *The nonstationary problem (20), (21) has a unique class $\mathbf{H}_0$ solution and it converges in $\mathbf{L}^2_{1,00}(\Omega)$ as $t \rightarrow \infty$ to the solution of the stationary problem (16), (17).*

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University of British Columbia

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