On the Absolute Convergence of Lacunary Fourier Series of Lipschitz Functions

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1. Introduction. A classical theorem of S. N. Bernstein states that functions on the circle which satisfy a Lipschitz condition of order greater than $1/2$ have absolutely convergent Fourier series. On the other hand for functions with a sufficiently sparse spectrum (*i.e.*, a Sidon set) continuity suffices to insure the absolute convergence. In this paper we investigate spectral conditions which give absolute convergence of the Fourier series for functions satisfying Lipschitz conditions of various orders. Specifically, a set $E \subset \mathbb{Z}^s$ will be said to be a Lip α set if every function on T^s which satisfies a Lipschitz condition of order α and has its spectrum in E also has an absolutely convergent Fourier series. Since functions which satisfy Lipschitz conditions of order greater than 1 are constant, it follows trivially that every set E is a Lip α set when $\alpha > 1$. Therefore, we only consider the case $0 < \alpha \leq 1$. It turns out that there are density type conditions which are sufficient for E to be a Lip α set and there are similar conditions which are necessary. However, for $\alpha < 1$ and for sets with a sufficiently coherent arithmetical structure the sufficient condition is also necessary.

2. The sufficient condition. For a positive integer m and a set $E \subset \mathbb{Z}^s$ let

$$E_m = \{\mathbf{n} \in E \mid 2^m \leq \|\mathbf{n}\|_\infty < 2^{m+1}\}.$$

Using the standard proof of Bernstein's theorem [6, p. 240], one can obtain the following sufficient condition.

Theorem 1. *If $\sum_{m=0}^\infty (\text{card } E_m)^{1/2} 2^{-\alpha m} < \infty$, then E is a Lip α set.*

Remarks. (1) In [5] Rudin defined $\Lambda(p)$ sets. For $p > 2$, a $\Lambda(p)$ set is a Lip α set for all $\alpha > s/p$. This can be seen from theorem 2 of [1] which implies that $\text{card } E_m \leq K 2^{m 2s/p}$ where K depends only on the set E .

(2) It is well known that $\Lambda(p)$ sets cannot contain arbitrarily arithmetic progressions, but it is easy to construct sets which have arbitrarily long arithmetic progressions and still satisfy the hypothesis of theorem 1 for every $\alpha > 0$. Hence there are sets which are Lip α for every $\alpha > 0$ but not $\Lambda(p)$ for any p .

3. The necessary condition. Here we will need the s -dimensional generalization of Kahane's theorem 2 [2, p. 66]. The modifications in the proof are minor, and therefore the proof will be omitted.

Theorem 2. Consider the random series

$$(*) \quad f(\mathbf{x}) \sim \sum_{\mathbf{n} \in \mathbb{Z}^s} \epsilon_{\mathbf{n}} a_{\mathbf{n}} e^{i(\mathbf{n} \cdot \mathbf{x})}$$

where the $\epsilon_{\mathbf{n}}$ are the Rademacher random variables. If $\beta < 1$ and

$$\left(\sum_{2^m \leq \|\mathbf{n}\|_{\infty} < 2^{m+1}} |a_{\mathbf{n}}|^2 \right)^{1/2} = O(2^{-\beta m}),$$

then $(*)$ represents almost surely a function in $\text{Lip } \alpha$ for all $\alpha < \beta$.

Theorem 3. Suppose E is a $\text{Lip } \alpha$ set with $\alpha < 1$, then for all $\epsilon > 0$

$$\sum_{m=0}^{\infty} (\text{card } E_m)^{1/2} 2^{-(\alpha+\epsilon)m} < \infty.$$

Proof. Put $b_m = 2^{-(\alpha+\epsilon)m} (\text{card } E_m)^{-1/2}$ and let $a_{\mathbf{n}} = b_m$ when $\mathbf{n} \in E_m$. Then

$$\left(\sum_{E_m} |a_{\mathbf{n}}|^2 \right)^{1/2} = 2^{-(\alpha+\epsilon)m}.$$

By theorem 2 there exist $\pm$ signs so that

$$\sum_{\mathbf{n} \in \mathbb{Z}^s} \pm a_{\mathbf{n}} e^{i(\mathbf{n} \cdot \mathbf{x})}$$

represents a function in $\text{Lip } \alpha$. But $\sum |a_{\mathbf{n}}| = \sum_m (\text{card } E_m)^{1/2} 2^{-(\alpha+\epsilon)m}$ which is finite since E is a $\text{Lip } \alpha$ set.

Remarks. (1) Given α it is now easy to construct sets which are $\text{Lip } \beta$ for all $\beta \geq \alpha$ but not $\text{Lip } \beta$ for any $\beta < \alpha$. Take E to be any set so that $\text{card } E_m =$ greatest integer in $(2^{2\alpha m} m^{-4})$ then

$$\frac{1}{2} \sum_{m=0}^{\infty} 2^{(\alpha-\beta)m} m^{-2} \leq \sum_{m=0}^{\infty} (\text{card } E_m)^{1/2} 2^{-\beta m} \leq \sum_{m=0}^{\infty} 2^{(\alpha-\beta)m} m^{-2}.$$

The right inequality shows that E is $\text{Lip } \beta$ for $\beta \geq \alpha$ while the left shows that E is not $\text{Lip } \beta$ for $\beta < \alpha$.

(2) If A and B are both $\text{Lip } \alpha$ sets, then $A \cup B$ is $\text{Lip } (\alpha + \epsilon)$ for any $\epsilon > 0$, since

$$\begin{aligned} \sum_{m=0}^{\infty} (\text{card } (A \cup B)_m)^{1/2} 2^{-(\alpha+\epsilon)m} \\ \leq \sum_{m=0}^{\infty} (\text{card } A_m)^{1/2} 2^{-(\alpha+\epsilon)m} + \sum_{m=0}^{\infty} (\text{card } B_m)^{1/2} 2^{-(\alpha+\epsilon)m}. \end{aligned}$$

4. Arithmetical structure.

Definition. Let $\mathbf{n}_1, \dots, \mathbf{n}_k$ be a set distinct elements from Z^s and consider the set

$$G(\mathbf{n}_1, \dots, \mathbf{n}_k) = \left\{ \sum_{i=1}^k \epsilon_i \mathbf{n}_i \mid \epsilon_i = 0 \text{ or } 1 \right\}.$$

If the cardinality of this set is 2^k , it is said to be independently generated. Let E be a finite subset of Z^s and consider a partition of $E = \bigcup_{i=1}^i (G_i + \mathbf{1}_i)$ by translates of independently generated sets G_i . Assume

$$(*) \quad \text{card } G_1 \geq \dots \geq \text{card } G_i.$$

Let $N(E, \mathcal{O})$ be the length of the longest string of equalities in $(*)$, where $\mathcal{O}$ stands for the partition. Define the arithmetic dispersion $\delta(E)$ of E to be the smallest $N(E, \mathcal{O})$ as $\mathcal{O}$ ranges over all partitions of E by translates of independently generated sets.

Remarks. (1) For any $\mathbf{n} \in Z^s$, $\delta(E) = \delta(\mathbf{n}E)$ where the product of elements in Z^s is defined by coordinatewise multiplication.

$$(2) \quad \delta(E + \mathbf{1}) = \delta(E).$$

$$(3) \quad \delta\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \delta(E_i) + n - 1.$$

$$(4) \quad 0 \leq \delta(E) \leq (\text{card } E) - 1.$$

(5) If E is independent in the sense that $\text{card } G(E) = 2^{(\text{card } E)}$ then $\delta(E) = (\text{card } E) - 1$.

(6) Suppose $E = \{1, 2, \dots, N\} \subset Z$, then $\delta(E) = 0$. For suppose $N = 2^{m_1} + \dots + 2^{m_i}$ where $m_1 > m_2 > \dots > m_i$. Then $E = \bigcup_{i=1}^i (\{1, 2, 3, \dots, 2^{m_i}\} + 2^{m_1} + \dots + 2^{m_{i-1}})$ and $\{1, 2, 3, \dots, 2^{m_i}\} = G(1, 2, 4, 8, \dots, 2^{m_{i-1}}) + 1$.

(7) In general consider a block $E = \{(\mathbf{n}_1, \dots, \mathbf{n}_s) \in Z^s \mid 1 \leq \mathbf{n}_i \leq N_i\}$. Suppose B_i is the set of exponents in the unique binary decomposition of N_i , and $r(B_1, \dots, B_s; k)$ is the number of ways one can write $k = m_1 + m_2 + \dots + m_s$ where $m_i \in B_i$. It is not hard to show that $\delta(E) \leq \sup_{k \in Z} r(B_1, \dots, B_s; k) - 1$.

(8) Let $E = Z^s$, then one can show that $\delta(E_m) \leq C$ where C depends only on s . This follows from writing E_m as a certain union of translates of blocks whose sides have lengths which are powers of 2 and then applying (2) (3) and (7).

Lemma. Consider a finite set $E \subset Z^s$, then there exists a trigonometric polynomial f satisfying the following:

$$(a) \quad \hat{f}(\mathbf{n}) = 0 \text{ for all } \mathbf{n} \notin E$$

$$(b) \quad \hat{f}(\mathbf{n}) = \pm 1 \text{ for all } \mathbf{n} \in E$$

$$(c) \quad \|f\|_\infty \leq 2 (\text{card } G_i)^{1/2} (\delta(E) + 1) / (1 - 2^{-1/2}) \text{ where } E = \bigcup_{i=1}^i (G_i + \mathbf{1}_i) \text{ is a partition of } E \text{ which gives } \delta(E).$$

Proof. Given $E = \bigcup_{i=1}^i (G_i + \mathbf{1}_i)$, fix i and consider $G_i + \mathbf{1}_i = G(\mathbf{n}_{i1}, \dots, \mathbf{n}_{ik_i}) + \mathbf{1}_i$. Construct the Rudin-Shapiro polynomials [4] in the following way.

Set $Q_0(\mathbf{x}) = P_0(\mathbf{x}) = e^{i(\mathbf{1}_i \cdot \mathbf{x})}$, $P_{s+1}(\mathbf{x}) = P_s(\mathbf{x}) + e^{i(\mathbf{n}_{i,s} \cdot \mathbf{x})} Q_s(\mathbf{x})$ and $Q_{s+1}(\mathbf{x}) = Q_s(\mathbf{x}) - e^{i(\mathbf{n}_{i,s} \cdot \mathbf{x})} P_s(\mathbf{x})$. Then $\hat{P}_s$ and $\hat{Q}_s$ are non-zero precisely on $\mathbf{1}_i + G(\mathbf{n}_{i,1}, \dots, \mathbf{n}_{i,s})$ where they take on the values ± 1 . Moreover the parallelogram law gives $|P_{s+1}(\mathbf{x})|^2 + |Q_{s+1}(\mathbf{x})|^2 = 2(|P_s(\mathbf{x})|^2 + |Q_s(\mathbf{x})|^2) = 2^{s+1}$ and hence $\|P_{s+1}\|_\infty \leq 2^{(s+1)/2}$. Put $f = \sum_{i=1}^j P_{k_i}$. Clearly f satisfies (a) and (b). As for (c) one has

$$\|f\|_\infty \leq \sum_{i=1}^j \|P_{k_i}\|_\infty \leq \sum_{i=1}^j 2^{k_i/2} \leq 2^{k_1/2} \sum_{i=1}^j 2^{(k_i - k_1)/2}.$$

Collecting terms for which $k_i = k_{i+1}$ we have

$$\begin{aligned} \|f\|_\infty &\leq 2^{k_1/2} \left(\sum m_i 2^{(k_i - k_1)/2} \right) \\ &\leq 2^{k_1/2} (\delta(E) + 1) / (1 - 2^{-1/2}). \end{aligned}$$

Theorem 4. Let $E \subset \mathbb{Z}^s$ and suppose there exists a constant B depending only on E , so that $\delta(E_m) \leq B$ for $m = 0, 1, 2, \dots$. Then for $\alpha < 1$, E is Lip α if and only if

$$\sum_{m=0}^{\infty} (\text{card } E_m)^{1/2} 2^{-\alpha m} < \infty.$$

Proof. This proof follows the lines of [3, p. 34]. The “if” part follows from theorem 1, so suppose that $\sum_{m=0}^{\infty} (\text{card } E_m)^{1/2} 2^{-\alpha m} = \infty$. By the lemma there exist polynomials f_m so that

$$(a) \quad \hat{f}_m(\mathbf{n}) = \begin{cases} \pm 1, & \mathbf{n} \in E_m \\ 0, & \mathbf{n} \notin E_m \end{cases}$$

(b) $\|f_m\|_\infty \leq C (\text{card } E_m)^{1/2}$ where C depends only on E .
Put $\phi_m = f_m (\text{card } E_m)^{-1/2} 2^{-\alpha m}$, then $\|\phi_m\|_\infty \leq C 2^{-\alpha m}$ and

$$\sum_{\mathbf{n} \in \mathbb{Z}^s} |\hat{\phi}_m(\mathbf{n})| = (\text{card } E_m)^{1/2} 2^{-\alpha m}.$$

Letting $\phi = \sum_{m=0}^{\infty} \phi_m$, it follows that

$$\sum_{\mathbf{n} \in \mathbb{Z}^s} \hat{\phi}(\mathbf{n}) = \sum_{m=0}^{\infty} (\text{card } E_m)^{1/2} 2^{-\alpha m} = \infty.$$

Therefore the proof is complete once it is established that $\phi \notin \text{Lip } \alpha$.

Given $\mathbf{h} \in T^s$, take N so that $2^{-N-1} \leq \|\mathbf{h}\|_\infty < 2^{-N}$. Then

$$\|\phi_{\mathbf{h}} - \phi\|_\infty \leq \left(\sum_{m=0}^N + \sum_{m=N+1}^{\infty} \right) \|\phi_{m\mathbf{h}} - \phi_m\|_\infty.$$

But for $f(\mathbf{x}) = \sum_{\|\mathbf{n}\|_1 \leq k} a_{\mathbf{n}} e^{i(\mathbf{n} \cdot \mathbf{x})}$ one has

$$\|f_{\mathbf{h}} - f\|_\infty \leq 2 \|\mathbf{h}\|_\infty k \|f\|_\infty \quad [2, \text{p. 57}].$$

So since $\alpha < 1$ the first sum above is

$$\leq 2^{-N+1} \sum_{m=0}^N s 2^{m+1} C 2^{-\alpha m} = C' 2^{-(N+1)\alpha}.$$

The second sum is

$$\leq 2 \sum_{m=N+1}^{\infty} C 2^{-\alpha m} \leq C'' 2^{-(N+1)\alpha}.$$

Hence $\|\phi_h - \phi\|_{\infty} \leq C''' \|h\|_2^{\alpha}$.

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