A Generalization of Weighted Shifts

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1. Introduction. A weighted shift in ℓ^2 is a bounded linear transformation T of the form

$$(1.1) \quad T(c_0, c_1, c_2, \dots) = (0, c_0 a_1, c_1 a_2, \dots)$$

for a fixed never-zero complex sequence $(a_1, a_2, \dots)$. One possible generalization is to let formula (1.1) define an operator T in a Frechet sequence space $\mathcal{S}$ (a Frechet space whose elements are sequences, which contains all finitely non-zero sequences and for which the mappings

$$(c_0, c_1, c_2, \dots) \rightarrow c_i$$

are continuous for all j). Call such an operator T a sequence shift.

The j^{th} element of the "usual basis" in $\mathcal{S}$ is the sequence e_i with 1 in the j^{th} position, 0's elsewhere. In [1, p. 539], using the assumption that $\{e_i\}_0^\infty$ is a Schauder basis for $\mathcal{S}$, it was shown that two operators that commute with T commute with each other. The same commutation property will be proven here for operators satisfying the weaker requirements.

$$(1.2) \quad \begin{array}{l} \text{i) } \mathcal{C}\ell(T^n \mathcal{S}) \text{ is of co-dimension } n \text{ in } \mathcal{S}, \\ \text{ii) } \bigcap_n \mathcal{C}\ell(T^n \mathcal{S}) = \{0\}, \end{array}$$

where $\mathcal{C}\ell$ denotes closure. Call an operator satisfying (i) and (ii) an M -shift operator. Note that for any sequence shift (ii) holds and also the co-dimension of $\mathcal{C}\ell(T^n \mathcal{S})$ in $\mathcal{S}$ is not less than n .

On the other hand it will be shown that if $\mathcal{S} = \ell^\infty$, the space of bounded sequences in the sup norm, and for a proper choice of $(a_1, a_2, \dots)$, two operators may commute with T but not with each other. This answers a question raised by Lambert and Kerlin in [2].

The present paper is mostly concerned with the properties of M -shift operators.

2. M-shifts. We will call $\mathcal{S}$ an M -shift space if it is a Frechet space whose elements are power series $f = b_0 + b_1z + b_2z^2 + \cdots$ including the polynomials such that

- (a) The mappings $(\cdot)_n : f \rightarrow b_n = (f)_n$ are continuous linear functionals.
- (b) $f \in \mathcal{S}$ implies $zf \in \mathcal{S}$.
- (c) $(f)_i = 0$, all $i < n$ implies $f \in \mathcal{C}\ell(z^n\mathcal{S})$.
- (d) $(f)_i = 0$, all i implies $f = 0$.

Proposition 2.1. Let $A = (a_{ij})$, $i, j \geq 0$ be an infinite complex-valued matrix. Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are M -shift spaces, and for each $f = b_0 + b_1z + b_2z^2 + \cdots \in \mathcal{S}_1$, the series $Af = \sum_{i=0}^{\infty} (\sum_{j=0}^{\infty} a_{ij}b_j)z^i$ is an element of $\mathcal{S}_2$ (the inner sum is required to converge in the usual sense). Then A defines a linear continuous map from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Proof. The linear functionals $f \rightarrow \sum_{i=0}^n a_{ij}b_j$ converge as $n \rightarrow \infty$ for each $f \in \mathcal{S}_1$ so by the uniform boundedness theorem $f \rightarrow \sum_{i=0}^{\infty} a_{ij}b_j$ defines a continuous linear functional F_i on $\mathcal{S}_1$. $Af = \sum_{i=0}^{\infty} F_i(f)z^i$. We show A is closed: If $f_n \rightarrow f$ and $Af_n \rightarrow g$ then $F_i(f_n) \rightarrow F_i(f)$ but also $F_i(f_n) = (Af_n)_i \rightarrow (g)_i$. Now by 2(d), $g = \sum_{i=0}^{\infty} F_i(f)z^i = Af$. A is closed and defined everywhere, hence is continuous.

Corollary 2.2. (a) The operator of multiplication by z , $M_z : f \rightarrow zf$ is a continuous operator on an M -shift space $\mathcal{S}$. (b) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two M -shift spaces, and as sets of power series $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then the mapping from $\mathcal{S}_1$ to $\mathcal{S}_2$ sending power series f to f is continuous. (c) The topology of an M -shift space is completely determined by the set of power series composing it. (d) Suppose g is a power series in $\mathcal{S}$ with the property that $g \cdot f \in \mathcal{S}$ for all $f \in \mathcal{S}$ (where $g \cdot f$ is the formal product of the series g and f). Then the mapping $M_g : f \rightarrow g \cdot f$ is a continuous operator commuting with M_z .

Proof. For (a), (b), and (d): The mappings are defined by matrices. The mapping in (d) clearly commutes with M_z . (c) follows from (b).

Theorem 2.3. An operator T on a Frechet space $\mathcal{S}$ can be represented as multiplication by z on an M -shift space iff T is an M -shift operator. The representation is determined completely by the choice of an arbitrary element $g \in \mathcal{S} - \mathcal{C}\ell(T\mathcal{S})$ to be represented by the series 1 in the M -shift space.

Proof. It is clear that the operator M_z on an M -shift space is an M -shift operator. Now suppose T is an M -shift operator on $\mathcal{S}$ and $g \in \mathcal{S} - \mathcal{C}\ell(T\mathcal{S})$. First it is necessary to check that $T^n g \in \mathcal{C}\ell(T^n\mathcal{S}) - \mathcal{C}\ell(T^{n+1}\mathcal{S})$. By 1.2i), $T^n\mathcal{S}$ is not contained in $\mathcal{C}\ell(T^{n+1}\mathcal{S})$ so pick $T^nh \in T^n\mathcal{S} - \mathcal{C}\ell(T^{n+1}\mathcal{S})$. Then $h \in \mathcal{S} - \mathcal{C}\ell(T\mathcal{S})$. Again by 1.2i), $g - ch \in \mathcal{C}\ell(T\mathcal{S})$ for some non-zero complex c . Then $T^ng - cT^nh \in \mathcal{C}\ell(T^{n+1}\mathcal{S})$ so by the choice of h , $T^ng \notin \mathcal{C}\ell(T^{n+1}\mathcal{S})$. It follows that no non-trivial linear combination of $g, Tg, \dots, T^ng$ lies in $\mathcal{C}\ell(T^{n+1}\mathcal{S})$ (or else by applying the proper power of T to the linear combination, $T^ng \in \mathcal{C}\ell(T^{n+1}\mathcal{S})$ is obtained). Thus, $g, Tg, \dots, T^{n-1}g$ spans an n -dimensional complement to $\mathcal{C}\ell(T^n\mathcal{S})$. For

each $f \in \mathcal{S}$ inductively define a sequence $b_0, b_1, \dots$ such that $b_0, b_1, \dots, b_{n-1}$ are the unique numbers such that $f - b_0g - b_1Tg - \dots - b_{n-1}T^{n-1}g \in \mathcal{C}\ell(T^n\mathcal{S})$. If g is represented by the series 1, f must be represented by the series $b_0 + b_1z + \dots$. It is easy to check that Tf is represented by $b_0z + b_1z^2 + \dots$. The continuity of the coefficient functionals (property 2(a)) is also easy to check. Property 2(d) follows from 1.2ii).

From now on let T denote the M -shift operator M_z in an M -shift space $\mathcal{S}$. Denote M_g by $g(T)$.

Theorem 2.4. *The operators $M_g = g(T)$ of 2.2(d) are the only continuous operators commuting with T . If S commutes with T and $S(1) = g$ then $S = g(T)$.*

Proof. $S(\mathcal{C}\ell(T^n\mathcal{S})) \subseteq \mathcal{C}\ell(S(T^n\mathcal{S})) = \mathcal{C}\ell(T^n(S\mathcal{S})) \subseteq \mathcal{C}\ell(T^n\mathcal{S})$. Suppose $S(1) = c_0 + c_1z + \dots$ and $f = b_0 + b_1z + \dots$. Note that $b_{i+1}z^{i+1} + b_{i+2}z^{i+2} + \dots \in \mathcal{C}\ell(T^{i+1}\mathcal{S})$. Therefore $(Sf)_i = (S(b_0 + b_1z + \dots + b_i z^i))_i = (S[b_0I + b_1T + \dots + b_i T^i](1))_i = ([b_0I + b_1T + \dots + b_i T^i]S(1))_i = b_0(S(1))_i + b_1(T(S(1)))_i + \dots + b_i(T^i(S(1)))_i = b_0c_i + b_1c_{i-1} + \dots + b_ic_0 = (S(1) \cdot f)_i$ for all i . So $Sf = S(1) \cdot f$.

Corollary 2.5. (a) *If S_1 and S_2 both commute with T , then S_1 commutes with S_2 .* (b) *If S commutes with T , then S has kernel $= \{0\}$ (or $S = 0$).* (c) *If P commutes with T and $P^2 = P$ then $P = 0$ or $P = I$.* (d) *Spectrum T is a connected set if $\mathcal{S}$ is a Banach space. $0 \in$ spectrum T .*

Proof. (a) Power series multiplication is commutative. (b) The product of two power series is 0 only if one of them is 0. (c) If $P \neq I$ then P has non-zero kernel. By (b), $P = 0$. (d) If spectrum T were not connected, the usual functional calculus [3, p. 421] would give a non-trivial projection commuting with T . $0 \in$ spectrum T because $TS \neq S$.

3. Analytic behavior. Let $\mathcal{S}$ be an M -shift space. Suppose U is a complex domain containing 0 such that every power series $f \in \mathcal{S}$ is the Taylor expansion around 0 of an analytic function defined in U . Then we call U a domain for $\mathcal{S}$. The elements of $\mathcal{S}$ can be considered to be functions defined on U .

Example 3.1. Let U be any complex domain containing 0. Let $\mathcal{S}_U$ be the space of all power series representing functions analytic in U , with the topology of uniform convergence on compact subsets. Then U is clearly a domain for the M -shift space $\mathcal{S}_U$.

Proposition 3.2. *Let U be a domain for $\mathcal{S}$. Then convergence in $\mathcal{S}$ implies uniform convergence on compact subsets for the associated functions on U .*

Proof. $\mathcal{S} \subseteq \mathcal{S}_U$ and by 2.2(b) the inclusion mapping is continuous.

Corollary 3.3. *Let U be a domain for $\mathcal{S}$. If $\lambda_i \in U$, $1 \leq i \leq n$, and m_i are positive integers then the set of functions in $\mathcal{S}$ vanishing to m_i or higher degrees at λ_i forms a T -invariant subspace.*

Proof. By 3.2 the set is closed. It is clearly invariant under multiplication by z .

For $f \in \mathcal{S}$: $\lambda \in U$, let $f(\lambda)$ denote the value of the "function" f at λ .

Corollary 3.4. For $\lambda \in U$, the mapping

$$\epsilon_\lambda : f \rightarrow f(\lambda) = \langle f, \epsilon_\lambda \rangle$$

is a non-zero continuous linear functional.

Proof. By 3.3, its kernel is closed (invariant subspace).

Note that if f, g and $f \cdot g$ are all in $\mathcal{S}$ then $\langle f \cdot g, \epsilon_\lambda \rangle = \langle f, \epsilon_\lambda \rangle \langle g, \epsilon_\lambda \rangle$.

Corollary 3.5. Let $f(T)$ be continuous. Then $f(U) = \{f(\lambda) : \lambda \in U\} \subseteq \text{spectrum } f(T)$.

Proof. $\langle (f(T) - f(\lambda)I)g, \epsilon_\lambda \rangle = \langle f \cdot g - f(\lambda)g, \epsilon_\lambda \rangle = f(\lambda)g(\lambda) - f(\lambda)g(\lambda) = 0$ so range $f(T) - f(\lambda)I$ isn't dense.

Corollary 3.6. If $\mathcal{S}$ has a domain U and S is a continuous operator on $\mathcal{S}$ then S commutes with T iff S leaves invariant the kernel of ϵ_λ for each $\lambda \in U$.

Proof. If S commutes with T then $\langle Sg, \epsilon_\lambda \rangle = \langle S(1) \cdot g, \epsilon_\lambda \rangle = \langle S(1)(\lambda)g(\lambda), \epsilon_\lambda \rangle = 0$ if $g(\lambda) = 0$. Conversely, for all $f \in \mathcal{S}$, $\lambda \in U$, $\langle f - f(\lambda)1, \epsilon_\lambda \rangle = 0$. Hence $0 = \langle S(f - f(\lambda)1), \epsilon_\lambda \rangle = (Sf)(\lambda) - (S(1)(\lambda))f(\lambda)$ for all $\lambda \in U$. $Sf = S(1) \cdot f$ as Taylor series, so S commutes with T .

Define T^{-1} to be the mapping sending the power series f to $(f - (f)_0 1)/z$. If $(f - (f)_0 1)/z \in \mathcal{S}$ for all $f \in \mathcal{S}$ then T^{-1} is continuous. $T^{-1}Tf = f$. $TT^{-1}f = f - (f)_0 1$. T^{-1} is continuous iff $\mathcal{C}\ell(T\mathcal{S}) = T\mathcal{S}$.

Proposition 3.7. Let $\mathcal{S}$ be an M -shift Banach space. If T^{-1} is bounded then the set $U = \{\lambda : 1/\lambda \in \text{unbounded connected component of resolvent } T^{-1}\} \cup \{0\}$ is a domain for $\mathcal{S}$.

Proof. For each $f = b_0 + b_1 z + \dots \in \mathcal{S}$, $f(\lambda) = (1/\lambda)((1/\lambda)I - T^{-1})^{-1}f)_0$ is an analytic function on U . For $\lambda < \|T^{-1}\|^{-1}$, $f(\lambda) = ((I - \lambda T^{-1})^{-1}f)_0 = ((I + \lambda T^{-1} + \lambda^2 (T^{-1})^2 + \dots)f)_0 = b_0 + b_1 \lambda + b_2 \lambda^2 + \dots$.

4. Invariant subspaces.

Proposition 4.1. $\mathcal{C}\ell(T^n \mathcal{S})$ is an invariant subspace. T restricted to $\mathcal{C}\ell(T^n \mathcal{S})$ is an M -shift operator.

Proof. Obvious.

For $f \in \mathcal{S}$ define $\mathcal{C}_f =$ the smallest T -invariant subspace of $\mathcal{S}$ containing f . It will be shown that T restricted to $\mathcal{C}_f$ is an M -shift operator. On the other hand it is possible for T restricted to an invariant subspace to not be an M -shift.

Proposition 4.2. (a) $\mathcal{C}\ell(T\mathcal{C}_f) = \mathcal{C}_{Tf}$. (b) If $(f)_0 \neq 0$ then $\mathcal{C}\ell(T\mathcal{C}_f) = \mathcal{C}_{Tf} = \mathcal{C}_f \cap \mathcal{C}\ell(T\mathcal{S})$.

Proof. By 4.1, without loss of generality we may assume $(f)_0 \neq 0$ in (a). $Tf \in T\mathcal{C}_f$, hence $\mathcal{C}_{Tf} \subseteq \mathcal{C}\ell(T\mathcal{C}_f)$. $T\mathcal{C}_f \subseteq \mathcal{C}_f$ and $T\mathcal{C}_f \subseteq T\mathcal{S}$, hence $\mathcal{C}\ell(T\mathcal{C}_f) \subseteq \mathcal{C}_f \cap \mathcal{C}\ell(T\mathcal{S})$. Now assume $h \in \mathcal{C}_f \cap \mathcal{C}\ell(T\mathcal{S})$. Then $(h)_0 = 0$ and $h \in \mathcal{C}_f$. Hence $p_nf \rightarrow h$ in $\mathcal{S}$ for some polynomials p_n . Then $(p_n)_0(f)_0 = (p_nf)_0 \rightarrow (h)_0 = 0$. Because $(f)_0 \neq 0$, $(p_n)_0 \rightarrow 0$. $\lim_{n \rightarrow \infty} (p_n - (p_n)_0 1)f = \lim_{n \rightarrow \infty} p_nf - \lim_{n \rightarrow \infty} (p_n)_0 f = h - 0 = h$. But $p_n - (p_n)_0 1 = q_n z$ for some polynomials q_n . $h = \lim_{n \rightarrow \infty} q_n z f = \lim_{n \rightarrow \infty} q_n Tf$, so $h \in \mathcal{C}_{Tf}$.

Corollary 4.3. $\mathcal{C}\ell(T^n \mathcal{C}_f) = \mathcal{C}_{T^n f}$. If $(f)_0 \neq 0$ then $\mathcal{C}\ell(T^n \mathcal{C}_f) = \mathcal{C}_{T^n f} = \mathcal{C}_f \cap \mathcal{C}\ell(T^n \mathcal{S})$.

Proof. By induction using 4.1 and 4.2.

Corollary 4.4. T restricted to $\mathcal{C}_f$ is an M -shift operator.

Proof. By 4.1, without loss of generality assume $(f)_0 \neq 0$. Then by 4.3 no non-trivial linear combination of $f, Tf, \dots, T^{m-1}f$ lies in $\mathcal{C}\ell(T^m \mathcal{C}_f) = \mathcal{C}_f \cap \mathcal{C}\ell(T^m \mathcal{S})$. On the other hand, $\text{span}\{f, Tf, \dots, T^{m-1}f\} + \mathcal{C}_{T^m f}$ is $\mathcal{C}_f$. Thus $\{f, Tf, \dots, T^{m-1}f\}$ spans an n -dimensional complement of $\mathcal{C}\ell(T^m \mathcal{C}_f) = \mathcal{C}_{T^m f}$ in $\mathcal{C}_f$. Also $\mathcal{C}_f \cap \mathcal{C}\ell(T^m \mathcal{C}_f) \subseteq \mathcal{C}\ell(T^m \mathcal{S}) = \{0\}$.

Example 4.5. Let $\mathcal{S}$ be the space of series representing functions analytic over the Riemann surface in Figure A with the topology of uniform convergence on compact subsets and let $\mathcal{C} \subseteq \mathcal{S}$ be the subspace of series in $\mathcal{S}$ which represent functions with analytic continuations to the Riemann surface in Figure B. Then $\mathcal{S}$ is an M -shift space, $\mathcal{C}$ is a T -invariant subspace but $\mathcal{C}\ell(T\mathcal{C})$ is of co-dimension 2 in $\mathcal{C}$.

5. Shifts in ℓ^∞ . The following is an example where T is a sequence shift on ℓ^∞ , and of two operators commuting with T but not with each other.

FIGURE A

FIGURE B

Let $T(x_1, x_2, x_3, \dots) = (0, a_1x_1, a_2x_2, a_3x_3, \dots)$ for $a_n = 2^{-i}$ where 2^i is the largest power of 2 dividing n . The a 's have been chosen so that $\lim_n a_{2^n} = 0$ and so that if m is fixed, $a_{2^n+m} = a_m$ for all large n . Choose a generalized limit Lim . Define $S(x_1, x_2, x_3, \dots) = (\text{Lim } x_{2^n+1}, \text{Lim } x_{2^n+2}, \text{Lim } x_{2^n+3}, \dots)$. Then $TS(x_1, x_2, x_3, \dots) = (0, a_1 \text{Lim } x_{2^n+1}, a_2 \text{Lim } x_{2^n+2}, a_3 \text{Lim } x_{2^n+3}, \dots)$ while $ST(x_1, x_2, x_3, \dots) = (\text{Lim } a_{2^n}x_{2^n}, \text{Lim } a_{2^n+1}x_{2^n+1}, \text{Lim } a_{2^n+2}x_{2^n+2}, \dots)$ so $TS = ST$. Now let $\text{Lim}_1 x_n = \text{Lim } x_{2^n}$ and $\text{Lim}_2 x_n = \text{Lim } x_{2^n+1}$. Define S_1 and S_2 in the same manner as S , but using Lim_1 and Lim_2 respectively in place of Lim . Then S_1 and S_2 commute with T . Next define $(x_1, x_2, \dots)$ by

$$\begin{aligned} x_m &= 1 & 2^{2^n} < m < 2^{2^n} + n, \\ x_m &= 0 & \text{otherwise.} \end{aligned}$$

Then $S_1(x_1, x_2, \dots) = (1, 1, 1, \dots)$, $S_2S_1(x_1, x_2, \dots) = (1, 1, 1, \dots)$, $S_2(x_1, x_2, \dots) = (0, 0, 0, \dots)$, $S_1S_2(x_1, x_2, \dots) = (0, 0, 0, \dots)$, and hence $S_1S_2 \neq S_2S_1$.

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