Multi-Parameter Spectral Theory

PATRICK J. BROWNE

Communicated by RALPH PHILLIPS

1. Introduction. Let $H_1, H_2, \dots, H_k$ be separable Hilbert spaces and $H = \bigotimes_{1 \leq r \leq k} H_r$ their tensor product. In each space H_r we assume we have continuous operators $T_r, V_{rs} : H_r \rightarrow H_r, s = 1, 2, \dots, k$ such that

- (i) all the operators $T_r, V_{rs}, r, s = 1, 2, \dots, k$ are Hermitian,
- (ii) for any choice of $f_r \neq 0, f_r \in H_r, r = 1, 2, \dots, k$,

$$(1) \quad \det (V_{rs} f_r, f_r)_r > 0$$

where $(\cdot, \cdot)_r$ denotes the inner product in H_r .

Such a system has, in effect, been the basis for discussion of multi-parameter eigenvalue problems by Atkinson [2] and Browne [4]. The results obtained have been applied to multi-parameter systems of second order ordinary differential equations thus covering the work of Faierman [6]. In these investigations the operators V_{rs} have been assumed compact relative to the operators T_r or alternatively, the operators T_r have been self adjoint with compact resolvent. The differential equation application has been extended to cover so called "singular" cases—see Browne [5]. Here the operators T_r are of Sturm-Liouville type over a half line and thus while being self adjoint do not necessarily have compact resolvent. This suggests the possibility of a more general multi-parameter spectral theory in which the compactness assumptions are removed. It is the aim of this paper to develop, at least partially, such a theory.

2. Operators from determinantal tensor expansions. Following Atkinson [2] we define the operators $\Delta_s : H \rightarrow H, s = 0, 1, \dots, k$ as follows. Let $f = f_1 \otimes \dots \otimes f_k$ be a decomposable element of H and let $\alpha_0, \alpha_1, \dots, \alpha_k$ be arbitrary complex numbers. Then $\Delta_0 f, \dots, \Delta_k f$ are defined by the equation

$$\sum_{s=0}^k \alpha_s \Delta_s f = \bigotimes \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_k \\ T_1 f_1 & V_{11} f_1 & \cdots & V_{1k} f_1 \\ \cdots & \cdots & \cdots & \cdots \\ T_k f_k & V_{k1} f_k & \cdots & V_{kk} f_k \end{vmatrix}$$

where this determinant is to be expanded formally using the tensor product. Thus, for example,

$$\begin{aligned}\Delta_0 f &= \bigotimes \begin{vmatrix} V_{11}f_1 & \cdots & V_{1k}f_1 \\ \cdots & \cdots & \cdots \\ V_{k1}f_k & \cdots & V_{kk}f_k \end{vmatrix} \\ &= \sum_{\sigma} \epsilon_{\sigma} V_{1\sigma(1)}f_1 \otimes \cdots \otimes V_{k\sigma(k)}f_k\end{aligned}$$

where σ runs through all permutations of $\{1, 2, \dots, k\}$ and ϵ_{σ} is $+1$ or -1 according as σ is even or odd. This defines $\Delta_0 f$ for decomposable $f \in H$ and we extend the definition to arbitrary $f \in H$ by linearity and continuity. The operators $\Delta_1, \dots, \Delta_k$ are obtained in a similar fashion.

We note that the definiteness condition (1) implies that $(\Delta_0 f, f) > 0$ for all decomposable $f \in H$ and so also for finite linear combinations of decomposable elements (see [2], Chapt. 7 and §11.8, p. 194). Henceforth we shall assume

$$(2) \quad \inf_{\|f\|=1} (\Delta_0 f, f) = \mu > 0,$$

thus ensuring that $\Delta_0^{-1} : H \rightarrow H$ exists as a bounded linear operator.

Before continuing we give an example of a situation in which condition (2) is satisfied. Let $[a_r, b_r]$, $r = 1, 2, \dots, k$ be finite subintervals of the real line and put $H_r = L^2([a_r, b_r])$ — (Lebesgue measure). Suppose $a_{rs}(x_r)$, $x_r \in [a_r, b_r]$, $r, s = 1, 2, \dots, k$ are continuous real valued functions satisfying the condition $\det (a_{rs}(x_r)) > 0$. This determinant being a continuous function of the variable $x \in \times_{1 \leq r \leq k} [a_r, b_r]$, there must exist positive constants m, M such that

$$(3) \quad m < \det (a_{rs}(x_r)) < M.$$

We now define $V_{rs} : H_r \rightarrow H_r$ by $(V_{rs}f_r)(x_r) = a_{rs}(x_r)f_r(x_r)$ from which it readily follows that $\Delta_0 : H \rightarrow H$ is given by $(\Delta_0 f)(x) = (\det a_{rs}(x_r))f(x)$. Note that $H = L^2(\times_{1 \leq r \leq k} [a_r, b_r])$. That (2) is satisfied in this situation is an immediate consequence of (3).

In the main, we shall not use the inner product $(\cdot, \cdot)$ in H generated by the inner products $(\cdot, \cdot)_r$ in H_r , but rather the inner product given by $(\Delta_0 \cdot, \cdot)$ which we shall denote by $[\cdot, \cdot]$. The norms induced by these inner products are equivalent and so we may discuss topological concepts such as continuity of operators and convergence of sequences unambiguously without reference to a particular inner product. Algebraic concepts, however, may depend on the inner product. If $A : H \rightarrow H$ is a continuous linear operator, we shall denote by $A^{\#}$ the adjoint of A with respect to $[\cdot, \cdot]$; i.e. $A^{\#}$ is defined by the equation

$$[Af, g] = [f, A^{\#}g] \quad \text{for all } f, g \in H.$$

The customary notation A^* will denote the adjoint of A with respect to $(\cdot, \cdot)$. Further, for $r = 1, 2, \dots, k$, Γ_r will denote the operator $\Delta_0^{-1}\Delta_r : H \rightarrow H$.

Theorem 1. $\Gamma_r^{\#} = \Gamma_r$, $r = 1, 2, \dots, k$.

Proof. We first note that $\Delta_r^* = \Delta_r$, $r = 0, 1, \dots, k$, since these operators are formed from the Hermitian operators T_r , $V_{r,s}$. Then for $f, g \in H$ and $r = 1, 2, \dots, k$, we have

$$\begin{aligned} [\Gamma_r f, g] &= (\Delta_0 \Delta_0^{-1} \Delta_r f, g) = (f, \Delta_r g) \\ &= (f, \Delta_0 \Delta_0^{-1} \Delta_r g) = (\Delta_0 f, \Gamma_r g) \\ &= [f, \Gamma_r g], \end{aligned}$$

thus proving the result.

For a continuous operator $A : H_r \rightarrow H_r$, we denote by $A^\dagger$ the continuous operator $A^\dagger : H \rightarrow H$ defined on decomposable elements $f = f_1 \otimes \dots \otimes f_k$ by $A^\dagger f = f_1 \otimes \dots \otimes f_{r-1} \otimes A f_r \otimes f_{r+1} \otimes \dots \otimes f_k$ and extended to arbitrary $f \in H$ by continuity and linearity.

Lemma 1. Let $A_r : H_r \rightarrow H_r$, $r = 1, 2, \dots, k$ be continuous linear operators. Then

$$\bigcap_{r=1}^k \text{Ker}(A_r^\dagger) = \bigotimes_{r=1}^k \text{Ker}(A_r).$$

Proof. This result has been proved by Atkinson [2, Theorem 4.7.2, p. 72] in the case that the spaces H_r , $r = 1, 2, \dots, k$ are all finite dimensional. A similar argument using orthonormal bases in the spaces H_r shows it to be true in the present setting.

We now make a further assumption. We shall assume that for each $f \in H$ we can find $g^1, g^2, \dots, g^k \in H$ such that

$$T_r^\dagger f + \sum_{s=1}^k V_{r,s}^\dagger g^s = 0, \quad r = 1, 2, \dots, k.$$

It is easy to show that the example cited earlier in connection with assumption (2) also satisfies this condition.

Theorem 2. The operators Γ_r , $r = 1, 2, \dots, k$ are pairwise commutative.

Proof. Let $f \in H$ and $g^1, g^2, \dots, g^k \in H$ be such that

$$T_r^\dagger f + \sum_{s=1}^k V_{r,s}^\dagger g^s = 0, \quad r = 1, 2, \dots, k.$$

Temporarily denoting f by g^0 , we see from [2, Theorem 6.4.2, p. 106] that

$$\Delta_p g^q = \Delta_q g^p, \quad p, q = 0, 1, \dots, k.$$

We should point out that the theorem referred to applies to the case of the spaces H_r being finite dimensional. Its extension to the case of countable dimension is a formality. In particular, we have $\Delta_0 g^q = \Delta_q f$; i.e. $g^q = \Gamma_q f$. Hence for $p, q = 1, 2, \dots, k$ we obtain

$$\Delta_p \Gamma_q f = \Delta_q \Gamma_p f, \quad \text{for all } f \in H.$$

The result follows by applying Δ_0^{-1} on the left throughout this equation.

As an immediate consequence we have

Corollary. $T_r^\dagger + \sum_{s=1}^k V_{rs}^\dagger \Gamma_s = 0, r = 1, 2, \dots, k.$

3. Multi-parameter spectral theory. Working with the inner product $[\cdot, \cdot]$ in H , the operators $\Gamma_r, r = 1, 2, \dots, k$ form a family of k commuting Hermitian operators. Let $\sigma(\Gamma_r)$ denote the spectrum of Γ_r (which necessarily will be a non-empty compact subset of the real line $\mathbb{R}$) and $\sigma_0 = \times_{1 \leq r \leq k} \sigma(\Gamma_r)$, the Cartesian product of the $\sigma(\Gamma_r), r = 1, 2, \dots, k$. Then σ is a non-empty compact subset of $\mathbb{R}^k$.

We now give a slight generalization of the usual concept of "numerical range". By the numerical range of the system $\{T_r, V_{rs}\}$ we shall mean the subset Ω of $\mathbb{R}^k$ given by

$$\begin{aligned} \Omega &= \{(\langle \Delta_1 f^1, f^1 \rangle, \dots, \langle \Delta_k f^k, f^k \rangle) \mid (\Delta_0 f^r, f^r) = 1, f^r \in H, r = 1, 2, \dots, k\} \\ &= \{([\Gamma_1 f^1, f^1], \dots, [\Gamma_k f^k, f^k]) \mid [f^r, f^r] = 1, f^r \in H, r = 1, 2, \dots, k\}. \end{aligned}$$

Theorem 3. (i) Ω is a convex set.
(ii) $\sigma_0 \subset \bar{\Omega}$.

Proof. (i) Ω is the Cartesian product of the sets $\{[\Gamma_r f, f] \mid [f, f] = 1\}, r = 1, 2, \dots, k$, each of which is convex [3, Corollary to Theorem 21.12, p. 388]. Consequently Ω is convex.

(ii) Suppose $\lambda = (\lambda_1, \dots, \lambda_k) \notin \bar{\Omega}$. Then there exists $r, 1 \leq r \leq k$ such that $\lambda_r \notin \{[\Gamma_r f, f] \mid [f, f] = 1\}$. Thus $\lambda_r \notin \sigma(\Gamma_r)$ and so $\lambda \notin \sigma_0$. We now have $\sigma_0 \subset \bar{\Omega}$.

We turn now to the theory of functions of several commuting Hermitian operators. This topic is discussed briefly by Akhiezer and Glazman [1, pp. 80–83] and more fully by Prugovečki [7, pp. 270–285]. It is the latter work which is our chief reference. We outline the main points of the construction.

Let $E_r(\cdot)$ denote the resolution of the identity for the operator Γ_r and let $M_r \subset \mathbb{R}$ be a Borel set, $r = 1, 2, \dots, k$. We define $E(M_1 \times \dots \times M_k) = E_1(M_1)E_2(M_2) \dots E_k(M_k)$. Note that the projections $E_r(\cdot)$ will commute since the operators Γ_r commute. In this way we obtain a spectral measure $E(\cdot)$ on the Borel subsets of $\mathbb{R}^k$ vanishing outside σ_0 . Thus for each $f, g \in H$, $[E(\cdot)f, g]$ is a complex valued Borel measure vanishing outside σ_0 . Measures of the type $[E(\cdot)f, f]$ will be non-negative totally finite Borel measures vanishing outside σ_0 .

We shall define the spectrum, σ , of the system $\{T_r, V_{rs}\}$ to be the support of the operator valued measure $E(\cdot)$; i.e. σ is the smallest closed set outside of which $E(\cdot)$ vanishes or alternatively, σ is the smallest closed set with the property $E(M) = E(M \cap \sigma)$ for all Borel sets $M \subset \mathbb{R}^k$. Then σ is a compact subset of $\mathbb{R}^k$ and, if $\lambda \in \sigma$, then for all non degenerate closed rectangles M with

$\lambda \in M$, $E(M) \neq 0$. Thus we see that the measures $[E(M)f, g]$, $f, g \in H$ actually vanish outside σ .

If $F(\lambda) = F(\lambda_1, \dots, \lambda_k)$ is a Borel function defined on σ we can define the operator $F(\Gamma) = F(\Gamma_1, \dots, \Gamma_k)$ by

(i) $\mathfrak{D}(F(\Gamma)) = \{f \in H \mid \int_{\sigma} |F(\lambda)|^2 [E(d\lambda)f, f] < \infty\}$,

(ii) for $f \in \mathfrak{D}(F(\Gamma))$ and arbitrary $g \in H$, $[F(\Gamma)f, g] = \int_{\sigma} F(\lambda)[E(d\lambda)f, g]$.

If $F(\lambda)$ is a bounded function, then $\mathfrak{D}(F(\Gamma)) = H$, whereas if $F(\lambda)$ is not bounded, $F(\Gamma)$ is densely defined. As mentioned earlier, full details of these results may be found in [7].

In previous discussions of multi-parameter eigenvalue problems (see e.g. [2], [4]) an eigenvalue has been defined as a k -tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ for which there exists a non zero decomposable element $f = f_1 \otimes \dots \otimes f_k \in H$ such that

$$T_r f_r + \sum_{s=1}^k \lambda_s V_{rs} f_r = 0, \quad r = 1, 2, \dots, k,$$

and we shall adopt this terminology. If $\lambda = (\lambda_1, \dots, \lambda_k)$ is an eigenvalue, then the self adjointness of the operators T_r, V_{rs} , $r, s = 1, 2, \dots, k$ forces each λ_r to be real [4, Theorem 1].

Theorem 4. *If $\lambda \in \sigma$ is such that $E(\{\lambda\}) \neq 0$, then λ is an eigenvalue. Conversely, if λ is an eigenvalue then $\lambda \in \sigma$ and $E(\{\lambda\}) \neq 0$.*

Proof. Suppose $\lambda = (\lambda_1, \dots, \lambda_k) \in \sigma$ is such that $E(\{\lambda\}) \neq 0$ and let $g \in E(\{\lambda\})H$, $g \neq 0$. Then since the operators $E_r(\{\lambda_r\})$, $r = 1, 2, \dots, k$, commute, $E_r(\{\lambda_r\})g = g$ for each r . From ordinary spectral theory we deduce that $\Gamma_r g = \lambda_r g$ and thus

$$T_r^\dagger g + \sum_{s=1}^k \lambda_s V_{rs}^\dagger g = \left(T_r^\dagger + \sum_{s=1}^k V_{rs}^\dagger \Gamma_s \right) g = 0,$$

$r = 1, 2, \dots, k$, by the Corollary to Theorem 2. Hence

$$0 \neq g \in \bigcap_{r=1}^k \text{Ker} \left(T_r + \sum_{s=1}^k \lambda_s V_{rs} \right)^\dagger = \bigotimes_{r=1}^k \text{Ker} \left(T_r + \sum_{s=1}^k \lambda_s V_{rs} \right),$$

and so there must exist a non zero decomposable element $f = f_1 \otimes \dots \otimes f_k \in H$ such that

$$T_r f_r + \sum_{s=1}^k \lambda_s V_{rs} f_r = 0, \quad r = 1, 2, \dots, k.$$

This shows that λ is an eigenvalue.

Conversely, if $\lambda = (\lambda_1, \dots, \lambda_k)$ is an eigenvalue with non zero decomposable eigenvector $f = f_1 \otimes \dots \otimes f_k$, we have

$$T_r^\dagger f + \sum_{s=1}^k \lambda_s V_{rs}^\dagger f = 0, \quad r = 1, 2, \dots, k.$$

Then from the proof of Theorem 2 we see that for $r = 1, 2, \dots, k$, $\Gamma_r f = \lambda_r f$. Hence $\lambda_r \in \sigma(\Gamma_r)$ and $E(\{\lambda_r\})f = f$. This shows that $E(\{\lambda\})f = f$ and the result is immediate.

The import of this theorem is that the set σ generalizes the concept of "eigenvalue" as used in [2] and [4].

We come now to the form of the Parseval equality and eigenvector expansion. The following result generalizes the Parseval equality and eigenvector expansion given by Atkinson [2, Theorems 7.9.2, p. 134 and 10.6.1, p. 180].

Theorem 5. *Let $f \in H$. Then*

$$(i) \quad (\Delta_0 f, f) = \int_{\sigma} [E(d\lambda)f, f] = \int_{\sigma} (E(d\lambda)f, \Delta_0 f),$$

$$(ii) \quad f = \int_{\sigma} E(d\lambda)f,$$

where this integral converges in the norm of H .

Proof. Let $F(\lambda_1, \dots, \lambda_k) = 1$. Then we have $F(\Gamma_1, \dots, \Gamma_k) = I$. Both results are now trivial consequences of the theory of functions of several commuting Hermitian operators referred to earlier.

4. The compact case. The usual (*i.e.* 1 parameter) spectral theory for compact Hermitian operators in Hilbert space is by now classical and well documented (see *e.g.* [3, §25.1, p. 445]). In this section we shall investigate the nature of the spectrum, σ , under the additional assumption that each of the operators $T_r : H_r \rightarrow H_r$, $r = 1, 2, \dots, k$ is compact. For $\lambda \in \mathbb{R}^k$ we define the operators $S_r(\lambda) : H_r \rightarrow H_r$ by

$$S_r(\lambda) = T_r + \sum_{s=1}^k \lambda_s V_{rs}, \quad r = 1, 2, \dots, k.$$

Theorem 6. (i) $\lambda \in \sigma$ if and only if 0 is in the spectrum of each $S_r(\lambda)$,
(ii) λ is an eigenvalue if and only if 0 is an eigenvalue of each $S_r(\lambda)$,
(iii) if each of the operators $T_r : H_r \rightarrow H_r$ is compact and $\lambda \neq 0$, then it is impossible for 0 to be in the continuous spectrum of each $S_r(\lambda)$.

Proof. (i) Suppose $\lambda \in \sigma$. Then $\lambda_r \in \sigma(\Gamma_r)$ and we may find a sequence $f^n \in H$, $\|f^n\| = 1$ such that $\|(\Gamma_r - \lambda_r)f^n\| \rightarrow 0$. (Such a sequence could be constructed by forming the intervals $I_r^n = [\lambda_r - 1/n, \lambda_r + 1/n]$ and selecting f^n from $E_1(I_1^n) \dots E_k(I_k^n)H$.) Then since $T_r^\dagger + \sum_{1 \leq s \leq k} V_{rs}^\dagger \Gamma_s = 0$, we see that

$$T_r^\dagger f^n + \sum_{s=1}^k \lambda_s V_{rs}^\dagger f^n \rightarrow 0, \quad r = 1, 2, \dots, k.$$

Hence 0 is in the spectrum of each $S_r^\dagger(\lambda)$ and since the spectra of $S_r(\lambda)$ and $S_r^\dagger(\lambda)$ coincide, we see that 0 is in the spectrum of each $S_r(\lambda)$.

Now suppose that 0 is in the spectrum of each $S_r(\lambda)$. Then we may find sequences $f_r^n \in H_r$, $\|f_r^n\| = 1$ such that $S_r(\lambda)f_r^n \rightarrow 0$. Putting $f^n = f_1^n \otimes \cdots \otimes f_k^n$ we have

$$\begin{aligned} \Delta_1 f^n &= \otimes \begin{vmatrix} 0 & 1 & \cdots & 0 \\ T_{11}f_1^n & V_{11}f_1^n & \cdots & V_{1k}f_1^n \\ \cdots & \cdots & \cdots & \cdots \\ T_{k1}f_k^n & V_{k1}f_k^n & \cdots & V_{kk}f_k^n \end{vmatrix} \\ &= \otimes \begin{vmatrix} \lambda_1 & 1 & \cdots & 0 \\ T_{11}f_1^n + \sum_{s=1}^k \lambda_s V_{1s}f_s^n & V_{11}f_1^n & \cdots & V_{1k}f_1^n \\ \cdots & \cdots & \cdots & \cdots \\ T_{k1}f_k^n + \sum_{s=1}^k \lambda_s V_{ks}f_s^n & V_{k1}f_k^n & \cdots & V_{kk}f_k^n \end{vmatrix} \\ &= \lambda_1 \Delta_0 f^n - \otimes \begin{vmatrix} T_{11}f_1^n + \sum_{s=1}^k \lambda_s V_{1s}f_s^n & V_{12}f_1^n & \cdots & V_{1k}f_1^n \\ \cdots & \cdots & \cdots & \cdots \\ T_{k1}f_k^n + \sum_{s=1}^k \lambda_s V_{ks}f_s^n & V_{k2}f_k^n & \cdots & V_{kk}f_k^n \end{vmatrix}. \end{aligned}$$

Thus we see that $\Delta_1 f^n - \lambda_1 \Delta_0 f^n \rightarrow 0$ and so $(\Gamma_1 - \lambda_1)f^n \rightarrow 0$. Similar arguments show that the sequence f^n has the property that $\|f^n\| = 1$ and $(\Gamma_r - \lambda_r)f^n \rightarrow 0$ for each r . Hence $\lambda \in \sigma$.

(ii) This statement is immediate from the definition of "eigenvalue".

(iii) Suppose that each of the operators $T_r : H_r \rightarrow H_r$, $r = 1, 2, \dots, k$ is compact and further that 0 is the continuous spectrum of each $S_r(\lambda)$ where $\lambda = (\lambda_1, \dots, \lambda_k) \neq 0$. For the sake of argument we assume $\lambda_1 \neq 0$. Then we can find sequences $f_r^n \in H_r$, $\|f_r^n\| = 1$ such that f_r^n tends weakly to zero and $S_r(\lambda)f_r^n$ tends strongly to zero. However since T_r is compact, $T_r f_r^n \rightarrow 0$ and so we see that $\sum_{1 \leq s \leq k} \lambda_s V_{rs} f_r^n \rightarrow 0$, $r = 1, 2, \dots, k$. Then putting $f^n = f_1^n \otimes \cdots \otimes f_k^n$,

$$\begin{aligned} (\Delta_0 f^n, f^n) &= \begin{vmatrix} (V_{11}f_1^n, f_1^n) & \cdots & (V_{1k}f_1^n, f_1^n) \\ \cdots & \cdots & \cdots \\ (V_{k1}f_k^n, f_k^n) & \cdots & (V_{kk}f_k^n, f_k^n) \end{vmatrix} \\ &= \frac{1}{\lambda_1} \begin{vmatrix} \left(\sum_{s=1}^k \lambda_s V_{1s}f_s^n, f_1^n \right) & (V_{12}f_1^n, f_1^n) & \cdots & (V_{1k}f_1^n, f_1^n) \\ \cdots & \cdots & \cdots & \cdots \\ \left(\sum_{s=1}^k \lambda_s V_{ks}f_s^n, f_k^n \right) & (V_{k2}f_k^n, f_k^n) & \cdots & (V_{kk}f_k^n, f_k^n) \end{vmatrix}. \end{aligned}$$

Thus $(\Delta_0 f^n, f^n) \rightarrow 0$ which contradicts the fact that $(\Delta_0 f^n, f^n) > \mu > 0$. This completes the proof of the theorem. As immediate consequences we have

Corollary 1. *Suppose each of the operators $T_r : H_r \rightarrow H_r$, $r = 1, 2, \dots, k$ is compact. If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a non zero point in σ then at least one of the equations*

$$T_r f_r + \sum_{s=1}^k \lambda_s V_{rs} f_r = 0, \quad r = 1, 2, \dots, k,$$

has a non trivial solution.

Corollary 2. *Suppose each of the operators $T_r : H_r \rightarrow H_r$, $r = 1, 2, \dots, k$ is compact. Then $0 \notin \sigma$.*

We note that when specialized to the case $k = 1$, Corollary 1 yields the familiar result that the non zero part of the spectrum of a compact Hermitian operator in Hilbert space consists entirely of eigenvalues. One might be tempted to conjecture that in our present setting the non zero part of the spectrum consists only of eigenvalues. The following example shows this need not be the case.

Let $H_1 = L^2(0, 1)$ -Lebesgue measure and $T_1 : H_1 \rightarrow H_1$ a compact operator whose spectrum consists of two points, say 0 and 1 both of which will be eigenvalues. Let $H_2 = L^2(0, 1)$ -Lebesgue measure and define the operators V_{21} , $V_{22} : H_2 \rightarrow H_2$ by $(V_{21}g)(y) = yg(y)$, $(V_{22}g)(y) = (y + 1)g(y)$. Consider the multi-parameter array

$$\begin{bmatrix} T_1 & I & 0 \\ 0 & V_{21} & V_{22} \end{bmatrix}.$$

Now $H = H_1 \otimes H_2 = L^2((0, 1) \times (0, 1))$ and the operators Γ_1, Γ_2 are given by

$$(\Gamma_1 h)(x, y) = (T_1^\dagger h)(x, y), \quad (\Gamma_2 h)(x, y) = \left(\frac{y}{y+1} \right) (T_1^\dagger h)(x, y).$$

Then the spectrum of Γ_1 consists of 2 points, viz. 0 and 1 both of which are eigenvalues of Γ_1 with infinite multiplicity, whereas the spectrum of Γ_2 is the interval $[0, 1/2]$, 0 being the only eigenvalue. The spectrum σ of this array consists of the union of two disjoint degenerate intervals,

$$\sigma = \{(0, \lambda_2) \mid 0 \leq \lambda_2 \leq 1/2\} \cup \{(1, \lambda_2) \mid 0 \leq \lambda_2 \leq 1/2\}$$

from which $(0, 0)$ is the only eigenvalue.

REFERENCES

1. N. I. AKHIEZER & I. M. GLAZMAN, *Theory of linear operators in Hilbert space; Vol. II*, translated from the Russian by M. Nestell, Frederick Ungar Publishing Co., New York, 1963.
2. F. V. ATKINSON, *Multi-parameter eigenvalue problems, Vol. I: Matrices and compact operators*, Academic Press, New York, 1972.

3. G. BACHMAN and L. NARICI, *Functional analysis*, Academic Press, New York, 1966.
4. P. J. BROWNE, *A Multi-parameter eigenvalue problem*, J. Math. Anal. Appl. **38** (1972), 553–568.
5. P. J. BROWNE, *A Singular multi-parameter eigenvalue problem in second order ordinary differential equations*, J. Differential Equations **12** (1972), 81–94.
6. M. FAIERMAN, *The completeness and expansion theorems associated with the multi-parameter eigenvalue problem in ordinary differential equations*, J. Differential Equations **5** (1969), 197–213.
7. E. PRUGOVEČKI, *Quantum mechanics in Hilbert space*, Academic Press, New York, 1971.

University of Calgary

Date communicated: NOVEMBER 5, 1973