Dirichlet Problem with L^1 Data for the Non-Homogeneous Minimal Surface Equation

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Introduction. We want to solve the equation

$$(0.1) \quad \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i}(x) (1 + |\text{grad } f(x)|^2)^{-1/2} \right] = A(x, f(x)), \quad x \in \Omega,$$

where Ω is a given open set of $\mathbf{R}^n$ and $A(x, t)$ is a given real function defined for $x \in \Omega$ and $t \in \mathbf{R}$. More precisely we want to find solutions of (0.1) satisfying a boundary condition

$$(0.2) \quad f(x) = g(x), \quad x \in \partial\Omega,$$

where $g(x)$ is a given real function defined for $x \in \partial\Omega$.

Throughout all this paper we will assume on the set Ω and on the functions $A(x, t)$, $g(x)$ the following

(H-1): Ω open and bounded, $\partial\Omega$ locally Lipschitz, i.e. $\forall x \in \partial\Omega$

$\exists$ an isometry $J : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $J(x) = 0$,

$$J(\Omega) \cap \{(y, z); y \in \mathbf{R}^{n-1}, |y| < a, z \in \mathbf{R}, |z| < b\} \\ = \{(y, z); |y| < a, \psi(y) < z < b\}$$

where $\psi : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ is a Lipschitz function, a and b are positive numbers with $b > \sup \{|\psi(y)| : |y| < a\}$.

(H-2): $A(x, t)$ is continuous and non-decreasing in t for each $x \in \Omega$, measurable in x for each $t \in \mathbf{R}$.

(H-3): $g(x)$ is Lebesgue integrable over $\partial\Omega$.

We will try to solve the problem (0.1), (0.2) by proving the existence of the minimum for the following functional

$$(0.3) \quad \mathfrak{F}(f) = \int_{\Omega} \left[(1 + |\text{grad } f(x)|^2)^{1/2} + \int_0^{f(x)} A(x, t) dt \right] dx \\ + \int_{\partial\Omega} |f(x) - g(x)| dH_{n-1}.$$

In section 1 we will prove the existence of the minimum for the functional $\mathfrak{F}$ in the class $BV(\Omega)$ of functions "with bounded variations on Ω ", under a suitable condition (H-4) for the function $A(x, t)$. In section 2 we will prove that the functions $f(x)$ minimizing the functional $\mathfrak{F}$ are continuous almost everywhere in Ω if the function $A(x, t)$ is locally bounded in $\Omega \times \mathbf{R}$. In the final section 3 we will prove that (0.2) is satisfied by the functions $f(x)$ minimizing $\mathfrak{F}$ if the set Ω and the functions $A(x, t)$, $g(x)$ satisfy a suitable hypothesis.

1. We denote by $BV(\Omega)$ the set of Lebesgue integrable functions $f(x)$ over Ω , whose gradient, in the sense of distributions, is a vector measure with finite total variation on Ω . In symbols we can write:

$$(1.1) \quad f(x) \in BV(\Omega) \Leftrightarrow f(x) \in L^1(\Omega) \quad \text{and} \\ \sup \left\{ \int_{\Omega} f(x) \operatorname{div} G(x) \, dx : G(x) \in [C_0^1(\Omega)]^n, |G(x)| \leq 1 \forall x \right\} < +\infty.$$

We will write the formulas containing the first derivatives of functions $f(x) \in BV(\Omega)$ as if these derivatives were Lebesgue integrable functions rather than measures. This abuse of language seems to be less misleading than the introduction of new notations. So by $\int_{\Omega} |\operatorname{grad} f(x)| \, dx$ we mean the total variation of the gradient of $f(x) \in BV(\Omega)$.

The following proposition (see e.g. [2]) about the "traces" of the functions $f(x) \in BV(\Omega)$ is known:

Proposition 1. *If $\Omega \subset \mathbf{R}^n$ satisfies (H-1), then for each $f(x) \in BV(\Omega)$ there exists a unique function in $L^1(\partial\Omega)$, which we call trace of $f(x)$ and denote with $Tf(x)$, such that*

$$(1.2) \quad \lim_{\rho \rightarrow 0} \rho^{-n} \int_{y \in \Omega, |y-x| < \rho} |f(y) - Tf(x)| \, dy = 0, \quad H_{n-1} - a.e. \quad \text{on } \partial\Omega,$$

and for each $\Phi(x) \in [C^1(\mathbf{R}^n)]^n$,

$$(1.3) \quad \int_{\Omega} f(x) \operatorname{div} \Phi(x) \, dx + \int_{\Omega} \Phi(x) \operatorname{grad} f(x) \, dx = \int_{\partial\Omega} Tf(x) \Phi(x) \nu(x) \, dH_{n-1},$$

where $\nu(x)$ is the outside normal to $\partial\Omega$ at the point x .

The functions $f(x) \in BV(\Omega)$ and their traces $Tf(x) \in L^1(\partial\Omega)$ are related to each other in a Sobolev type inequality which we state as

Proposition 2. *If $\Omega \subset \mathbf{R}^n$ satisfies (H-1), then for each $f(x) \in BV(\Omega)$*

$$(1.4) \quad \int_{\Omega} |f(x)| \, dx \leq n(\omega_n^{-1} \operatorname{meas} \Omega)^{1/n} \left\{ \int_{\Omega} |\operatorname{grad} f(x)| \, dx + \int_{\partial\Omega} |Tf(x)| \, dH_{n-1} \right\},$$

where ω_n is the measure of the unit ball of $\mathbf{R}^n$.

Proof. Denote with $\tau_h(x)$, $h = 1, 2, 3, \dots$, a sequence of functions in $C^1_0(\mathbf{R}^n)$ satisfying

$$(1.5) \quad \tau_h(x) \geq 0, \quad \tau_h(x) = 0 \quad \text{for} \quad |x| > h^{-1}, \quad \int_{\mathbf{R}^n} \tau_h(x) dx = 1,$$

and put

$$(1.6) \quad f_h(x) = \int_{\Omega} f(y) T_h(x - y) dy.$$

we have

$f_h(x) \in C^1(\mathbf{R}^n)$ for each h (see *e.g.* [4]) and

$$(1.7) \quad \int_{\mathbf{R}^n} |f_h(x)| dx \rightarrow \int_{\Omega} |f(x)| dx,$$

$$(1.8) \quad \int_{\mathbf{R}^n} |\text{grad } f_h(x)| dx \rightarrow \int_{\Omega} |\text{grad } f(x)| dx + \int_{\partial\Omega} |Tf(x)| dH_{n-1}.$$

We also have for each h

$$(1.9) \quad \int_{\mathbf{R}^n} |f_h(x)| dx = \int_{s>0} \text{meas } \{x \in \mathbf{R}^n : |f_h(x)| > s\} ds.$$

The isoperimetric inequality (see [6] and [5]) applied to the set $\{x \in \mathbf{R}^n; |f_h(x)| > s\}$ gives

$$(1.10) \quad \text{meas } \{x \in \mathbf{R}^n; |f_h(x)| > s\} \leq n\omega_n^{-1/n} P(\{x \in \mathbf{R}^n; |f_h(x)| > s\}) (\text{meas } \Omega_h)^{1/n}$$

where $\Omega_h = \{x \in \mathbf{R}^n; \text{dist}(x, \Omega) < h^{-1}\}$ and $P(E)$ denotes the total variation over $\mathbf{R}^n$ of the gradient of the characteristic function φ_E of E , *i.e.*

$$(1.11) \quad \begin{aligned} P(E) &= \int_{\mathbf{R}^n} |\text{grad } \varphi_E| dx \\ &= \sup \left\{ \int_E \text{div } G(x) dx; G(x) \in [C^1_0(\mathbf{R}^n)]^n, |G(x)| \leq 1 \right\}. \end{aligned}$$

In [5] it was also proved that

$$(1.12) \quad \int_{0 < s} P(\{x; |f_h(x)| > s\}) ds \leq \int_{\mathbf{R}^n} |\text{grad } f_h(x)| dx.$$

Integrating (1.10) with respect to s and using (1.9), (1.12) we have for all h

$$(1.13) \quad \int_{\mathbf{R}^n} |f_h(x)| dx \leq n(\omega_n^{-1} \text{meas } \Omega_h)^{1/n} \int_{\mathbf{R}^n} |\text{grad } f_h(x)| dx.$$

Letting $h \rightarrow \infty$ and recalling (1.7), (1.8) we obtain (1.4).

Q.E.D.

In order to extend the definition (0.3) of the functional $\mathfrak{F}$ to the functions $f(x) \in BV(\Omega)$ we assume an extra hypothesis to be satisfied by the function $A(x, t)$. This hypothesis is of a type introduced by J. Serrin in [7]:

(H-4): $\exists$ two non negative measurable functions $\alpha(x)$, $\beta(x)$ defined for $x \in \Omega$ and a non negative constant Γ such that

$$\forall x \in \Omega : A(x, \alpha(x)) \geq -\Gamma, A(x, -\beta(x)) \leq \Gamma, \quad \text{and}$$

$$\int_{\Omega} [\alpha(x)A^-(x, 0) - \beta(x)A^+(x, 0)] dx > -\infty,$$

where $A^-(x, 0) = \min(A(x, 0), 0)$ and $A^+(x, 0) = \max(A(x, 0), 0)$. An easy consequence of (H-4) is

$$(1.14) \quad \int_0^{f(x)} A(x, t) dt \geq \alpha(x)A^-(x, 0) - \beta(x)A^+(x, 0) - \Gamma |f(x)|, \quad \forall x \in \Omega,$$

which implies that the definition of $\mathfrak{F}$ can be extended to the functions $f(x) \in BV(\Omega)$ if we accept the value $+\infty$ and put

$$(1.15) \quad \mathfrak{F}(f) = \int_{\Omega} \left[(1 + |\text{grad } f(x)|^2)^{1/2} + \int_0^{f(x)} A(x, t) dt \right] dx \\ + \int_{\partial\Omega} |Tf(x) - g(x)| dH_{n-1},$$

where

$$\int_{\Omega} (1 + |\text{grad } f(x)|^2)^{1/2} dx \\ = \sup \left\{ \int_{\Omega} [f(x) \text{div } G(x) + G_0(x)] dx; \right. \\ \left. G(x) \in [C_0^1(\Omega)]^n, G_0(x) \in C_0^1(\Omega), |G(x)|^2 + |G_0(x)|^2 \leq 1 \quad \forall x \right\}$$

Combining (1.14) with Proposition 2 we have

$$(1.16) \quad \mathfrak{F}(f) \geq [1 - \Gamma n(\omega_n^{-1} \text{ meas } \Omega)^{1/n}] \\ \cdot \left[\int_{\Omega} (1 + |\text{grad } f(x)|^2)^{1/2} dx + \int_{\partial\Omega} |Tf(x)| dH_{n-1} \right] \\ - \int_{\partial\Omega} |g(x)| dH_{n-1} + \int_{\Omega} [\alpha(x)A^-(x, 0) - \beta(x)A^+(x, 0)] dx,$$

which allows the proof of the following existence theorem:

Theorem 1. *If the hypothesis (H-1) – (H-4) are satisfied together with the inequality*

$$(1.17) \quad \Gamma n(\omega_n^{-1} \text{ meas } \Omega)^{1/n} < 1,$$

then the functional $\mathfrak{F}$ defined by (1.15) attains its minimum in $BV(\Omega)$.

Proof. Let $f_h(x) \in BV(\Omega)$ be a sequence of functions minimizing the functional $\mathfrak{F}$, i.e.

$$(1.18) \quad \lim_{h \rightarrow \infty} \mathfrak{F}(f_h) = \inf_{f \in BV(\Omega)} \mathfrak{F}(f).$$

We can assume for each h

$$(1.19) \quad \mathfrak{F}(f_h) \leq \mathfrak{F}(0) = \text{meas } \Omega + \int_{\partial\Omega} |g(x)| dH_{n-1};$$

hence from (1.16) we have

$$(1.20) \quad \int_{\Omega} |\text{grad } f_h(x)| dx + \int_{\partial\Omega} |Tf_h(x)| dH_{n-1} \leq [1 - n\Gamma(\omega_n^{-1} \text{ meas } \Omega)^{1/n}]^{-1} \cdot \left[2 \int_{\partial\Omega} |g(x)| dH_{n-1} + \text{meas } \Omega + \int_{\Omega} [-\alpha(x)A^-(x, 0) + \beta(x)A^+(x, 0)] dx \right].$$

Denote for each h by $\tilde{f}_h(x)$ the function defined as equal to $f_h(x)$ for $x \in \Omega$ and equal to 0 otherwise. The sequence $\tilde{f}_h(x)$ is a sequence of integrable functions over $\mathbf{R}^n$ with compact support and satisfying, as a consequence of Proposition 1 and (1.20),

$$(1.21) \quad \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_h(x)| dx = \int_{\Omega} |\text{grad } f_h(x)| dx + \int_{\partial\Omega} |Tf_h(x)| dH_{n-1} \leq \text{const} < +\infty;$$

thus, from a known compactness result (see e.g. [4]), we can assume that the sequence $\tilde{f}_h(x)$ itself (for simplicity of notations we do not pass to a subsequence) is convergent in $L^1(\mathbf{R}^n)$ to a function $\tilde{f}_0(x)$. By the definition (1.1) it is easy to verify that

$$(1.22) \quad \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_0(x)| dx \leq \liminf_{h \rightarrow \infty} \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_h(x)| dx < +\infty,$$

which implies that $f_0(x) \in BV(\Omega)$ and then we can calculate $\mathfrak{F}(f_0)$. We prove now that

$$(1.23) \quad \mathfrak{F}(f_0) \leq \liminf_{h \rightarrow \infty} \mathfrak{F}(f_h)$$

by proving the two inequalities

$$(1.24) \quad \int_{\Omega} |\text{grad } f_0(x)| dx + \int_{\partial\Omega} |Tf_0 - g| dH_{n-1} \leq \liminf_{h \rightarrow \infty} \left\{ \int_{\Omega} |\text{grad } f_h(x)| dx + \int_{\partial\Omega} |Tf_h - g| dH_{n-1} \right\},$$

$$(1.25) \quad \int_{\Omega} \left[\int_0^{f(x)} A(x, t) dt \right] dx \leq \liminf_{h \rightarrow \infty} \int_{\Omega} \left[\int_0^{f_h(x)} A(x, t) dt \right] dx.$$

To prove (1.24) we recall that $g(x)$ can be extended outside Ω as a function of $L^1(\mathbf{R}^n - \Omega)$, with first derivatives in $L^1(\mathbf{R}^n - \Omega)$. By extending $g(x)$ we intend that a function $G(x)$ can be defined in $\mathbf{R}^n - \Omega$ having $g(x)$ as trace on $\partial\Omega$ (for such an extension see *e.g.* [3]). If we now put $\tilde{f}_h(x) = f_h(x)$ for $x \in \Omega$ and $\tilde{f}_h(x) = G(x)$ for $x \in \mathbf{R}^n - \Omega$, we can write, using Proposition 1,

$$(1.26) \quad \int_{\Omega} |\text{grad } f_h(x)| dx + \int_{\partial\Omega} |Tf_h - g| dH_{n-1} \\ = \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_h(x)| dx - \int_{\mathbf{R}^n - \Omega} |\text{grad } G(x)| dx,$$

$$(1.27) \quad \int_{\Omega} |\text{grad } f_0(x)| dx + \int_{\partial\Omega} |Tf_0 - g| dH_{n-1} \\ = \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_0(x)| dx - \int_{\mathbf{R}^n - \Omega} |\text{grad } G(x)| dx.$$

By using definition (1.1) we have

$$(1.28) \quad \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_0(x)| dx \leq \liminf_{h \rightarrow \infty} \int_{\mathbf{R}^n} |\text{grad } \tilde{f}_h(x)| dx.$$

(1.26), (1.27) and (1.28) imply (1.24). The proof of (1.25) is a simple consequence of Fatou's Lemma and of the fact that we can assume (for simplicity we do not pass to a subsequence as we should)

$$(1.29) \quad |f_h(x)| \leq \Psi(x) \in L^1(\Omega), \quad \forall h; \quad \liminf_{h \rightarrow \infty} f_h(x) = f_0(x), \quad a.e. \quad x \in \Omega.$$

From (1.14) we obtain

$$(1.30) \quad \int_0^{f_h(x)} A(x, t) dt \geq \alpha(x)A^-(x, 0) - \beta(x)A^+(x, 0) - \Gamma \cdot \Psi(x), \quad \forall h,$$

which allows the application of Fatou's Lemma to the sequence $\int_0^{f_h(x)} A(x, t) dt$ to obtain (1.25). Q.E.D.

2. In this section we want to show how a regularity result for the boundaries of sets, written in [8], can be used to prove the continuity *a.e.* of the functions $f(x) \in BV(\Omega)$ minimizing $\mathcal{F}$. Let us first recall the regularity result for boundaries of sets: given an open and bounded set G in $\mathbf{R}^m$ ($m > 1$) and a measurable, bounded function $A(y)$ defined for $y \in G$, if for a measurable set $E \subset G$ the $\mathcal{L}(E) = \int_G |\text{grad } \varphi_E| dy + \int_E A(y) dy = \min < +\infty$ (*i.e.* $\mathcal{L}(E) \leq \mathcal{L}(E')$, $\forall E' \subset G$ measurable, whose symmetric difference with E is contained in a compact subset of G), then there exists an open subset G_0 of G such that $H_s(G - G_0) = 0$, $\forall$ real $s > m - 8$, and $\partial E \cap G_0$ is locally a graph of a $C^{1,\alpha}$ function defined in an open set of $\mathbf{R}^{m-1}$. The boundary ∂E is intended as the

essential boundary of E , i.e. $y \in \partial E \Leftrightarrow \forall \rho > 0, \text{meas } \{E \cap B_\rho(y)\} > 0$ and $\text{meas } \{(\mathbf{R}^n - E) \cap B_\rho(y)\} > 0$.

We can associate with a function $f(x) \in BV(\Omega)$ the set $E \subset \Omega \times \mathbf{R}$ of the points below the graph of $f(x)$, i.e. $E = \{(x, t); x \in \Omega, t \in \mathbf{R}, t < f(x)\}$. We will refer to such sets as "Cartesian sets"; in other words a Cartesian set is the union of parallel half lines. The characteristic function φ_E of the set E associated with the function $f(x)$ satisfies

$$(2.1) \quad \int_{\Omega \times \mathbf{R}} |\text{grad } \varphi_E| \, dx \, dt = \int_{\Omega} (1 + |\text{grad } f(x)|^2)^{1/2} \, dx.$$

If now $f(x)$ minimizes $\mathfrak{F}$, at a finite value, we have for each open bounded set $G' \subset G = \Omega \times R$ and for each Cartesian set E' satisfying $E \triangle E' \subset\subset G'^2$

$$(2.2) \quad \int_{G'} |\text{grad } \varphi_E| \, dx \, dt + \int_{G' \cap E} A(x, t) \, dx \, dt \\ \leq \int_{G'} |\text{grad } \varphi_{E'}| \, dx \, dt + \int_{G' \cap E'} A(x, t) \, dx \, dt.$$

The inequality (2.2) is satisfied also by non Cartesian sets E'' , namely if E'' is a measurable set such that $E \triangle E'' \subset\subset G'$, and if we denote by E' the Cartesian set defined by

$$(2.3) \quad E' = \{(x, t); x \in \Omega, t \in \mathbf{R}, t < \gamma(x)\},$$

where

$$(2.4) \quad \gamma(x) = f(x) + \int_{\mathbf{R}} (\varphi_{E''}(x, t) - \varphi_E(x, t)) \, dt,$$

we have (see [1])

$$(2.5) \quad \int_{\Omega} (1 + |\text{grad } \gamma(x)|^2)^{1/2} \, dx = \int_{\Omega \times \mathbf{R}} |\text{grad } \varphi_{E'}(x, t)| \, dx \, dt \\ \leq \int_{\Omega \times \mathbf{R}} |\text{grad } \varphi_{E''}(x, t)| \, dx \, dt,$$

and since $A(x, t)$ is non decreasing in t , we have

$$(2.6) \quad \int_{(\Omega \times \mathbf{R}) \cap E'} A(x, t) \, dx \, dt \leq \int_{(\Omega \times \mathbf{R}) \cap E''} A(x, t) \, dx \, dt.$$

The inequalities (2.5) and (2.6) imply (2.2) for E'' . We can then apply the regularity result for boundaries of sets to the set $E = \{(x, t); x \in \Omega, t \in \mathbf{R}, t < f(x)\}$ in the case $f(x)$ minimizes the functional $\mathfrak{F}$ at a finite value and under the assumption that the function $A(x, t)$ is locally bounded in $\Omega \times \mathbf{R}$. We can so claim that the singular set of $\partial E \cap (\Omega \times \mathbf{R})$ has Hausdorff dimension less or equal to $n - 7$ and the regular part is a $C^{1, \alpha}$ manifold of dimension n . These facts about the essential boundary of the set E imply the continuity a.e. of the function $f(x)$ as we want to show now.

For each $x \in \Omega$ put

$$(2.7) \quad \begin{aligned} \bar{f}(x) &= \operatorname{ess\,lim\,sup}_{y \rightarrow x} f(y) = \inf \{t \in \mathbf{R}: \\ &\exists \rho > 0, \operatorname{meas} \{y; |y - x| < \rho, f(y) > t\} = 0\}, \end{aligned}$$

$$(2.8) \quad \begin{aligned} \underline{f}(x) &= \operatorname{ess\,lim\,inf}_{y \rightarrow x} f(y) = \sup \{t \in \mathbf{R}: \\ &\exists \rho > 0, \operatorname{meas} \{y; |y - x| < \rho, f(y) < t\} = 0\}. \end{aligned}$$

We have for each $x \in \Omega$, recalling the definition of essential boundary of E ,

$$(2.9) \quad \{t \in \mathbf{R}: (x, t) \in \partial E\} = \{t \in \mathbf{R}: \underline{f}(x) \leq t \leq \bar{f}(x)\}.$$

On the other hand the relation (2.1) and the informations about the singularities of ∂E imply

$$(2.10) \quad H_n(\partial E \cap (\Omega \times \mathbf{R})) = \int_{\Omega} (1 + |\operatorname{grad} f(x)|^2)^{1/2} dx < +\infty.$$

Thus $\operatorname{meas} \{\partial E \cap (\Omega \times \mathbf{R})\} = 0$, which implies

$$(2.11) \quad \operatorname{meas} \{x \in \Omega: \underline{f}(x) < \bar{f}(x)\} = 0.$$

The relation (2.11) says that $f(x)$ is essentially continuous *a.e.* on Ω . Let us observe about the regular points of ∂E that the projection of Ω of a $C^{1,\alpha}$ regular point of ∂E is not necessarily a point where the function $f(x)$ is $C^{1,\alpha}$. What can actually occur is that at a regular point of ∂E the tangent hyperplane is vertical. We can summarize what we have proved in this section with the following statement.

Theorem 2. *If $f(x) \in BV(\Omega)$ minimizes the functional $\mathfrak{F}$ at a finite value, and if the function $A(x, t)$ satisfies (H-2) and is locally bounded in $\Omega \times \mathbf{R}$, then $f(x)$ is continuous *a.e.* in Ω .*

3. We shall now prove that if $A(x, t)$, Ω and $g(x)$ satisfy a suitable condition at a boundary point $x_0 \in \partial\Omega$, then any function $f(x) \in BV(\Omega)$ minimizing the functional $\mathfrak{F}$ takes the value $g(x_0)$ at the point $x_0 \in \partial\Omega$. The argument of the proof is essentially based on an extension of the following maximum principle for the minimal surface operator.

Theorem 3. *If $\psi(y)$ and $\gamma(y)$ are two Lipschitz functions defined for $y \in \mathbf{R}^m$, and if for some $a > 0$ we have*

$$(3.1) \quad \psi(y) \leq \gamma(y), \quad \forall |y| < a.$$

$$(3.2) \quad \begin{aligned} &\sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi(y)}{\partial y_i} (1 + |\operatorname{grad} \psi(y)|^2)^{-1/2} \right] \\ &\geq \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \gamma(y)}{\partial y_i} (1 + |\operatorname{grad} \gamma(y)|^2)^{-1/2} \right], \quad \forall |y| < a, \end{aligned}$$

then either $\psi(y) = \gamma(y) \forall |y| < a$ or $\psi(y) < \gamma(y) \forall |y| < a$.

Proof. Denoting, for $p \in \mathbf{R}^m$ and $i = 1, 2, \dots, m$,

$$(3.3) \quad F_i(p) = p_i(1 + p^2)^{-1/2},$$

the inequality (3.2) can also be written as

$$(3.4) \quad \sum_{1 \leq i \leq m} \int_{|y| < a} [F_i(\text{grad } \gamma(y)) - F_i(\text{grad } \psi(y))] \frac{\partial \varphi(y)}{\partial y_i} dy \geq 0,$$

$$\forall \varphi \in C_0^1(\{|y| < a\}), \quad \varphi(y) \geq 0.$$

If we denote by $z(y)$ the difference $\gamma(y) - \psi(y)$, we have

$$(3.5) \quad F_i(\text{grad } \gamma(y)) - F_i(\text{grad } \psi(y))$$

$$= \sum_{1 \leq i \leq m} \frac{\partial z(y)}{\partial y_i} \int_{0 < t < 1} [F_{i,i}(\text{grad } \psi(y) + t \text{ grad } z(y))] dt,$$

where we have indicated by $F_{i,i}$ the derivative of the function F_i with respect to its j^{th} argument. Putting

$$(3.6) \quad a_{i,i}(y) = \int_{0 < t < 1} [F_{i,i}(\text{grad } \psi(y) + t \text{ grad } z(y))] dt$$

and combining (3.4) and (3.5) we obtain

$$(3.7) \quad \int_{|y| < a} \sum_{1 \leq i, j \leq m} a_{i,i}(y) \frac{\partial z(y)}{\partial y_i} \frac{\partial \varphi(y)}{\partial y_j} dy \geq 0, \quad \forall \varphi \in C_0^1(\{|y| < a\}), \quad \varphi(y) \geq 0.$$

It is now an easy consequence of Harnack's Theorem proved in [9] that the non-negative function $z(y)$ is either identically zero or strictly positive for $|y| < a$. Q.E.D.

The theorem we have just proved can be given the following geometric formulation: given two non intersecting graphs of Lipschitz functions, if the mean curvature of the upper graph calculated with respect to the upper normal is smaller than the curvature of the lower graph, then either they coincide or they do not touch at any interior point. This statement can be generalized to the case where one of the two graphs is substituted by a more general hypersurface such as the "boundary of a set".

Theorem 4. Assume that $\psi(y)$ is a Lipschitz function defined for $y \in \mathbf{R}^m$ and assume that for some $\lambda \in \mathbf{R}$ and $a > 0$

$$(3.8) \quad \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi(y)}{\partial y_i} (1 + |\text{grad } \psi(y)|^2)^{-1/2} \right] \geq \lambda, \quad \forall |y| < a.$$

For a given fixed $\epsilon > 0$ denote by Γ and Γ_+ the following subsets of $\mathbf{R}^{m+1}$:

$$\Gamma = \{(y, z): y \in \mathbf{R}^m, |y| < a, z \in \mathbf{R}, |z - \psi(y)| < \epsilon\},$$

$$\Gamma_+ = \{(y, z): |y| < a; 0 < z - \psi(y) < \epsilon\}.$$

Assume that $A(y, z)$ is a given measurable and bounded function defined for $(y, z) \in \Gamma$ and satisfying

$$(3.9) \quad A(y, z) \geq -\lambda, \quad \forall (y, z) \in \Gamma.$$

Assume that E is a measurable subset of Γ_+ satisfying

$$(3.10) \quad \mathcal{G}(E) = \int_{\Gamma} |\text{grad } \varphi_E(y, z)| \, dy \, dz + \int_E A(y, z) \, dy \, dz \leq \mathcal{G}(M),$$

$$\forall M \text{ s.t. } M \Delta E \subset \subset \Gamma.$$

Then $\forall \rho$ in $0 < \rho < a \exists \delta > 0$ s.t. either $E \cap \{(y, z); |y| < \rho, 0 < z - \psi(y) < \delta\} = \emptyset$ or $E \supset \{(y, z); |y| < \rho, 0 < z - \psi(y) < \delta\}$.

Proof. It is sufficient to prove that if $(0, \psi(0)) \in \partial E^*$, then

$$\exists \sigma_1 > 0 \text{ and } \sigma_2 > 0 \text{ s.t. } \{(y, z); |y| < \sigma_1, 0 < z - \psi(y) < \sigma_2\} \subset E.$$

For simplicity assume that $\psi(0) = 0$ and consider a sequence ρ_h ($h = 1, 2, \dots$) of positive numbers diverging to $+\infty$ s.t.

$$(3.11) \quad \rho_h^{-1} \psi(\rho_h, y) \rightarrow \psi_{\infty}(y), \text{ loc. unif. in } \mathbf{R}^m,$$

$$(3.12) \quad \varphi_E(\rho_h y, \rho_h z) \rightarrow \varphi_{E_{\infty}}(y, z), \text{ in } L_{\text{loc}}^1(\mathbf{R}^{m+1}).$$

(3.11) is possible because of the Ascoli-Arzelà Theorem, (3.12) is possible because of a compactness result proved in [4]. For the limit functions $\psi_{\infty}(y)$ and $\varphi_{E_{\infty}}(y, z)$ we have, as consequence of (3.8) and (3.10),

$$(3.13) \quad \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi_{\infty}(y)}{\partial y_i} (1 + |\text{grad } \psi_{\infty}(y)|^2)^{-1/2} \right] \geq 0, \quad \forall y \in \mathbf{R}^m,$$

$$(3.14) \quad \int_{|y|^2 + z^2 < \alpha^2} |\text{grad } \varphi_{E_{\infty}}(y, z)| \, dy \, dz \leq \int_{|y|^2 + z^2 < \alpha^2} |\text{grad } \varphi_M(y, z)| \, dy \, dz,$$

$$\forall \alpha > 0, \quad \forall M \text{ s.t. } M \Delta E \subset \subset \{|y|^2 + z^2 < \alpha^2\}.$$

We also have

$$(3.15) \quad E_{\infty} \subset \{(y, z); y \in \mathbf{R}^m, z > \psi_{\infty}(y)\}$$

and

$$(3.16) \quad 0 \in \partial E_{\infty}.$$

(3.13) (3.16) together with the fact that E_{∞} is a cone, i.e. the function $\varphi_{E_{\infty}}(y, z)$ is homogeneous of degree zero, imply (cf. [10] and [11]) that $\psi_{\infty}(y)$ is a linear function and ∂E_{∞} coincides with graph ψ_{∞} . Recalling the result proved in [8] we have that 0 must be a regular point for ∂E , i.e.

$$\exists \alpha > 0, \beta_1 > 0, \beta_2 > 0, \gamma(y) \in C^{1,\alpha}(\{|y| < \beta_1\}) \text{ s.t.}$$

$$(3.17) \quad \sup \{|\gamma(y)| : |y| < \beta_1\} < \beta_2 \text{ and}$$

$$E \cap \{(y, z); |y| < \beta_1, |z| < \beta_2\} = \{(y, z); |y| < \beta_1, \gamma(y) < z < \beta_2\}.$$

The inequalities (3.8), (3.9) and (3.10) imply

$$(3.18) \quad \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi(y)}{\partial y_i} (1 + |\text{grad } \psi(y)|^2)^{-1/2} \right] \\ \geq \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \gamma(y)}{\partial y_i} (1 + |\text{grad } \gamma(y)|^2)^{-1/2} \right], \quad \forall |y| < \beta_1.$$

The fact that $E \subset \Gamma_+$ implies

$$(3.19) \quad \psi(y) \leq \gamma(y), \quad \forall |y| < \beta_1,$$

and the fact that $(0, \psi(0)) \in \partial E$ implies

$$(3.20) \quad \psi(0) = \gamma(0).$$

Theorem 4 can now be applied to complete the proof.

Q.E.D.

Remark 1. It is not necessary to satisfy the inequality (3.10) for all subsets M of Γ ; it is sufficient to consider only the subsets of Γ_+ . Indeed if M is a general measurable set satisfying $M \Delta E \subset \subset \Gamma$ and if we denote by K the difference $M - M \cap \Gamma_+$, we have the following isoperimetric inequality (see its proof in the Appendix).

$$(3.21) \quad \lambda \text{meas}_{m+1} K + \int_{|y| < a} T_{\varphi_K}(y, \psi(y)) (1 + |\text{grad } \psi(y)|^2)^{1/2} dy \\ \leq \int_{\{|y| < a, z < \psi(y)\}} |\text{grad } \varphi_K(y, z)| dy dz.$$

(3.21) together with (3.9) and a result proved in [2] about the decomposition of the total variation of a gradient imply

$$(3.22) \quad \mathfrak{G}(M) \geq \mathfrak{G}(M \cap \Gamma_+),$$

which proves that it is sufficient to satisfy (3.10) for the set $M \subset \Gamma_+$.

Theorem 4 together with Remark 1 imply the following result about the boundary behavior of a function $f(x) \in BV(\Omega)$ minimizing the functional $\mathfrak{F}$.

Theorem 5. With $x = (y, z)$, $y \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$, $t \in \mathbf{R}$, let Ω , $A(x, t)$, $g(x)$ satisfy (H-1), (H-2), (H-3) and let $f(x) \in BV(\Omega)$ minimize the functional at a finite value. Let $0 \in \partial\Omega$ and assume that $\exists \lambda \in \mathbf{R}$, $a > 0$, $b > 0$, $\psi : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ Lipschitz s.t.

$$(3.23) \quad b > \sup_{|y| < a} |\psi(y)| \quad \Omega \cap \{(y, z) : |y| < a, |z| < b\} \\ = \{(y, z) : |y| < a, \psi(y) < z < b\}$$

$$(3.24) \quad \sum_{1 \leq i \leq n-1} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi(y)}{\partial y_i} (1 + |\text{grad } \psi(y)|^2)^{-1/2} \right] \geq \lambda, \quad \forall |y| < a,$$

$$(3.25) \quad A(y, z, \bar{g}(0)) \geq -\lambda, \quad \forall |y| < a, \psi(y) < z < b,$$

and assume that $A(y, z, t)$ is locally bounded in $\{(y, z, t) : |y| < a, \psi(y) < z < b\}$.

Then

$$(3.26) \quad \limsup_{x \rightarrow 0} \bar{f}(x) \leq \bar{g}(0).$$

Proof. For any given $\mu > \bar{g}(0)$ we can find $a_\mu \in (0, a)$ s.t. $g(x) < \mu \forall x \in \partial\Omega$, $|x| < a_\mu$. According to Theorem 4 and Remark 1 we can take as E the set $\{(x, t): x \in \Omega, t < f(x)\}$ and we have that either $\partial E \supset \{(x, t): x \in \partial\Omega, |x| < a_\mu, t > \mu\}$ or $\partial E \cap \{(x, t): x \in \partial\Omega, |x| < a_\mu, t > \mu\} = \emptyset$. In the first case we would have (cf. Theorem 4) $Tf(x) = +\infty \forall x \in \partial\Omega, |x| < a_\mu$, which is impossible since $\int_{\partial\Omega} |Tf - g| dH_{n-1} < +\infty$ and $g(x) \in L^1(\partial\Omega)$. So we must have $\partial E \cap \{(x, t): x \in \partial\Omega, |x| < a_\mu, t > \mu\} = \emptyset$, which implies

$$(3.27) \quad \limsup_{x \rightarrow 0} \bar{f}(x) \leq \mu.$$

Since (3.27) holds for all $\mu > \bar{g}(0)$ we have (3.26).

Q.E.D.

Remark 2. If we substitute the hypothesis (3.25) with

$$(3.28) \quad A(y, z, g(0)) \leq \lambda, \forall |y| < a, \psi(y) < z < b,$$

we have the following thesis

$$(3.29) \quad \liminf_{x \rightarrow 0} f(x) \geq g(0).$$

The proof of this is reduced to the previous one by changing the signs of the functions $A(x, t)$, $g(x)$ and $f(x)$.

Combining together Theorem 5 with the Remark 2 we obtain the following continuity result:

Theorem 6. Let Ω , $A(x, t)$, $g(x)$ satisfy (H-1), (H-2), (H-3) and $f(x) \in BV(\Omega)$ minimize the functional $\mathfrak{F}$ at a finite value. Let $0 \in \partial\Omega$ and assume that $\exists \lambda \geq 0$, $a > 0$, $b > 0$, $\psi: \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ s.t. (3.23) and (3.24) hold. Assume that $g(x)$ is continuous at 0 and

$$(3.30) \quad |A(y, z, g(0))| \leq \lambda, \forall |y| < a, \psi(y) < z < b.$$

Then

$$(3.31) \quad \lim_{x \rightarrow 0} f(x) = g(0).$$

A consequence of Theorems 1, 2 and 6 is the following proposition:

If Ω is an open bounded set of $\mathbf{R}^n$ with a C^2 boundary, and if $A(x, t)$ is a continuous function for $x \in \bar{\Omega}$ and $t \in \mathbf{R}$ and if $g(x)$ is a continuous function defined for $x \in \partial\Omega$ and if for each point $x \in \partial\Omega$ the sum of the principal curvatures of $\partial\Omega$ at x is strictly greater than $|A(x, g(x))|$, then there exists a weak solution $f(x)$ of the equation

$$\sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} \left[\frac{\partial f(x)}{\partial x_i} (1 + |\text{grad } f(x)|^2)^{-1/2} \right] = A(x)f(x), \quad x \in \Omega,$$

and this solution $f(x)$ is continuous *a.e.* in Ω and is continuous at each point of $\partial\Omega$ and satisfies

$$f(x) = g(x), \quad x \in \partial\Omega.$$

Appendix.

Theorem A.1. *If $\psi : \mathbf{R}^m \rightarrow \mathbf{R}$ is a Lipschitz function and $\lambda \in \mathbf{R}, a > 0$ are s.r.*

$$(A.1) \quad \sum_{1 \leq i \leq m} \frac{\partial}{\partial y_i} \left[\frac{\partial \psi(y)}{\partial y_i} (1 + |\text{grad } \psi(y)|^2)^{-1/2} \right] \geq \lambda, \quad \forall |y| < a,$$

and if K is a measurable bounded set of $\mathbf{R}^{m+1}$ s.t.

$$(A.2) \quad \tilde{K} \subset \{(y, z) : |y| < a, z \leq \psi(y)\},$$

then

$$(A.3) \quad \lambda \text{ meas}_{m+1} K + \int_{|y| < a} T\varphi_K(y, \psi(y))(1 + |\text{grad } \psi(y)|^2)^{1/2} dy \\ \leq \int_{\{|y| < a, z < \psi(y)\}} |\text{grad } \varphi_K(y, z)| dy dz.$$

Proof. If we denote by $\tau_j (j = 1, 2, \dots)$ a sequence of $C_0^1(\mathbf{R}^{m+1})$ functions s.t.

$$(A.4) \quad \tau_j(y, z) \geq 0, \quad \tau_j(y, z) = 0 \quad \forall |y|^2 + z^2 > j^{-2}, \quad \int \tau_j(y, z) dy dz = 1,$$

and denote by φ_j the sequence of functions defined by

$$(A.5) \quad \varphi_j(y, z) = \int \varphi_K(y, z) \tau_j(y - \tilde{y}, z - \tilde{z}) d\tilde{y} d\tilde{z},$$

where φ_K is the characteristic function of the set K , we have for almost all $\epsilon > 0$ (see *e.g.* [1])

$$(A.6) \quad \int_{\{|y| < a, z < \psi(y) - \epsilon\}} |\text{grad } \varphi_j(y, z)| dy dz \\ \rightarrow \int_{\{|y| < a, z < \psi(y) - \epsilon\}} |\text{grad } \varphi_K(y, z)| dy dz \quad \text{as } j \rightarrow \infty,$$

$$(A.7) \quad \int_{|y| < a} \varphi_j(y, \psi(y) - \epsilon) (1 + |\text{grad } \psi(y)|^2)^{1/2} dy \\ \rightarrow \int_{|y| < a} \varphi_K(y, \psi(y) - \epsilon) (1 + |\text{grad } \psi(y)|^2)^{1/2} dy \quad \text{as } j \rightarrow \infty,$$

$$(A.8) \quad \int_{\{|y| < a, z < \psi(y) - \epsilon\}} \varphi_j(y, z) dy dz \\ \rightarrow \text{meas}_{m+1} [K \cap \{|y| < a, z < \psi(y) - \epsilon\}] \quad \text{as } j \rightarrow \infty.$$

Moreover as $\epsilon \rightarrow 0$ we have

$$(A.9) \quad \int_{\{|y| < a, z < \psi(y) - \epsilon\}} |\text{grad } \varphi_K(y, z)| \, dy \, dz \\ \rightarrow \int_{\{|y| < a, z < \psi(y)\}} |\text{grad } \varphi_K(y, z)| \, dy \, dz \quad \text{as } \epsilon \rightarrow 0.$$

$$(A.10) \quad \int_{|y| < a} \varphi_K(y, \psi(y) - \epsilon) \cdot (1 + |\text{grad } \psi(y)|^2)^{1/2} \, dy \\ \rightarrow \int_{|y| < a} T\varphi_K(y, \psi(y)) (1 + |\text{grad } \psi(y)|^2)^{1/2} \, dy \quad \text{as } \epsilon \rightarrow 0.$$

$$(A.11) \quad \text{meas}_{m+1} [K \cap \{(y, z) : |y| < a, z < \psi(y) - \epsilon\}] \rightarrow \text{meas}_{m+1} K \text{ as } \epsilon \rightarrow 0.$$

To prove our Theorem we only need to verify that for any function $\varphi(y, z) \geq 0$, $\varphi(y, z) \in C_0^1(\{|y| < a, z \in \mathbf{R}\})$, the following inequality holds:

$$(A.12) \quad \lambda \int_{\{|y| < a, z < \psi(y)\}} \varphi(y, z) \, dy \, dz + \int_{|y| < a} \varphi(y, \psi(y)) (1 + |\text{grad } \psi(y)|^2)^{1/2} \, dy \\ \leq \int_{\{|y| < a, z < \psi(y)\}} |\text{grad } \varphi(y, z)| \, dy \, dz.$$

To prove (A.12), for a given $\epsilon > 0$ denote by $\eta_\epsilon(y, z)$ the Lipschitz function

$$(A.13) \quad \eta_\epsilon(y, z) = \begin{cases} 0, & z > \psi(y) + \epsilon \\ 1, & z < \psi(y) \\ \epsilon^{-1}(\psi(y) + \epsilon - z), & \text{otherwise,} \end{cases}$$

and multiply the inequality (A.1) by the function $\int \eta_\epsilon(y, z) \varphi(y, z) \, dz$ and integrate by parts. We obtain

$$(A.14) \quad \int \eta_\epsilon(y, z) \sum_{1 \leq i \leq m} \frac{\partial \psi(y)}{\partial y_i} \frac{\partial \varphi(y, z)}{\partial y_i} (1 + |\text{grad } \psi(y)|^2)^{-1/2} \, dx \, dz \\ + \int_{\mathbf{R}^m} \epsilon^{-1} \left[\int_{\psi(y) < z < \psi(y) + \epsilon} \varphi(y, z) \sum_{1 \leq i \leq m} \left(\frac{\partial \psi(y)}{\partial y_i} \right)^2 (1 + |\text{grad } \psi(y)|^2)^{-1/2} \, dz \right] \, dy \\ + \lambda \int \eta_\epsilon(y, z) \varphi(y, z) \, dy \, dz \leq 0.$$

We can add $-\int (1 + |\text{grad } \psi|^2)^{-1/2} \partial/\partial z (\eta_\epsilon \varphi) \, dy \, dz$, which is zero, and obtain

$$(A.15) \quad \int \eta_\epsilon(y, z) (1 + |\text{grad } \psi|^2)^{-1/2} \left(\sum_{1 \leq i \leq m} \frac{\partial \psi(y)}{\partial y_i} \frac{\partial \varphi}{\partial y_i} - \frac{\partial \varphi}{\partial z} \right) \, dy \, dz \\ + \int \epsilon^{-1} \left[\int_{\psi(y) < z < \psi(y) + \epsilon} \varphi(y, z) (1 + |\text{grad } \psi|^2)^{-1/2} \, dz \right] \, dy \\ + \lambda \int \eta_\epsilon(y, z) \varphi(y, z) \, dy \, dz \leq 0.$$

By letting $\epsilon \rightarrow 0$ and denoting by $\nu(y)$ the $(m+1)$ -vector whose components are given by

$$\nu_{m+1}(y) = (1 + |\text{grad } \psi(y)|^2)^{-1/2}, \quad \nu_i(y) = -\frac{\partial \psi(y)}{\partial y_i} \nu_{m+1}(y) \quad \text{for } i \leq m,$$

we have

$$\begin{aligned} (A.16) \quad & - \int_{\mathbf{R}^m} \left[\int_{z < \psi(y)} \nu(y) \text{grad } \varphi(y, z) dz \right] dy \\ & + \int_{\mathbf{R}^m} \varphi(y, \psi(y)) (1 + |\text{grad } \psi(y)|^2)^{-1/2} dy \\ & + \lambda \int_{\mathbf{R}^m} \left[\int_{z < \psi(y)} \varphi(y, z) dz \right] dy \leq 0, \end{aligned}$$

which implies (A.12), since $|\nu(y)| = 1$.

Q.E.D.

FOOTNOTES

1. H_{n-1} denotes the $(n-1)$ dimensional Hausdorff measure.
2. $E \Delta E' \subset \subset G' \Leftrightarrow \text{clos}([E - E'] \cup [E' - E]) \subset G'$.
3. The inequality is to be understood in the sense of distributions.
4. Recall that by $(0, \psi(0)) \epsilon \partial E$ we intend that $\forall \delta > 0, 0 < \text{meas}_{m+1} \{(y, z) \in E: |y|^2 + |z - \psi(0)|^2 < \delta^2\} < \omega_{m+1} \delta^{m+1} = \text{meas}_{m+1} \{(y, z): |y|^2 + z^2 < \delta^2\}$.

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