On Some Properties of Composition Operators

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1. Introduction. In his thesis [6] Schwartz showed that analytic functions mapping the unit disc into itself induce bounded linear operators (composition operators) on the Hardy space H^2 . He obtained several properties of these operators and in particular obtained a characterization of the compact composition operators, (see all J. V. Ryff [5]). This work has also been extended in several directions. E. Nordgren in [4] describes the spectrum of the composition operators whenever the analytic function inducing the operator is an automorphism of the disc.

Deddens in [2] has also described the spectrum of a composition operator induced by a linear polynomial mapping the disc into the disc. In a more recent work Shapiro and Taylor [7] have obtained conditions on the analytic mapping function in order that the operator be Hilbert-Schmidt or Trace Class. They also have several interesting results concerning when the analytic mapping induces or fails to induce a compact operator. Many of the Shapiro and Taylor results are obtained by placing geometric conditions on the range of the mapping function. In our work we characterize the composition operators which are Fredholm operators. We also obtain a sharp result on when a composition operator has closed range. Our results are stated in terms of the boundary values of the analytic mapping and not in terms of its range. We include a few pertinent examples.

2. Notations and definitions. We let Δ denote the open unit disc in $\mathbf{C}$ and $\Gamma = \{|\lambda| = 1\}$ its boundary. For $1 \leq p < \infty$ the space $H^p(\Delta) = H^p$ consists of the analytic functions f in Δ (with pointwise operations) which satisfy

$$(2.1) \quad \sup_{0 \leq r < 1} \left(\int |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

The linear space H^p equipped with the norm (2.1) is called the Hardy space for Δ . If ϕ is an analytic mapping of Δ into Δ then ϕ induces a linear operator C_ϕ on H^p as follows. For $f \in H^p$

$$(C_\phi f)(z) = f \circ \phi(z) = f(\phi(z)), \quad z \in \Delta.$$

The function $f \circ \phi$ is analytic in Δ and C_ϕ satisfies

$$\|C_\phi f\| \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{1/p} \|f\|.$$

Hence, the composition operators form a semigroup of the bounded operators on H^p . The algebras H^∞ and A are the algebra's of bounded analytic functions on Δ and the algebra of continuous functions on $\Delta \cup \Gamma$ (analytic in Δ), respectively. An operator A on a Hilbert space X is called a Fredholm operator if AX is closed and if the dimensions of the kernel and the cokernel are finite. Every function f in H^p has a factorization

$$f(\lambda) = B(\lambda)S(\lambda)F(\lambda)$$

where $B(\lambda)$ is a Blaschke product, $S(z)$ is a singular inner function and F is a non-zero function in H^p , $\|F\| = \|f\|$. Each function in H^p has well defined boundary values *a.e.*, in particular, if $\phi: \Delta \rightarrow \Delta$ we set

$$\hat{\phi}(e^{i\theta}) = \lim_{r \rightarrow 1} \phi(re^{i\theta}) \quad \text{a.e.} \quad \theta \in [0, 2\pi].$$

Then $\hat{\phi} \in L^\infty(\Gamma)$ and we will say that ϕ is an inner function if $|\hat{\phi}(e^{i\theta})| = 1$ *a.e.* Finally, the symbols $\mathcal{R}(C_\phi)$, $\mathcal{R}(C_\phi)^\perp$, $[x, y, \dots]$ shall mean the range of C_ϕ , the orthogonal complement of $\mathcal{R}(C_\phi)$ and the closed linear span of the vectors $x, y, \dots$, respectively.

3. The composition Fredholm operators. In [6] Schwartz has shown that C_ϕ is invertible if and only if ϕ is a conformal automorphism of the disc.

Theorem 1. *Let ϕ be an analytic mapping of Δ into Δ and C_ϕ the corresponding composition operator. C_ϕ is a Fredholm operator if and only if ϕ is a conformal automorphism of Δ .*

Proof. Clearly, if ϕ is a conformal automorphism then C_ϕ is a Fredholm operator by the above mentioned result of Schwartz. When ϕ is non-trivial, the kernel of C_ϕ is always the zero vector. We assume then that $\mathcal{R}(C_\phi)$ is closed and that $\mathcal{R}(C_\phi)^\perp$ is finite dimensional. The function ϕ has a canonical factorization as

$$\phi(\lambda) = B(\lambda)S(\lambda)F(\lambda).$$

We shall show first that $|S(\lambda)| \equiv 1$ and that $B(\lambda)$ has only a finite number of factors in its product.

The range of C_ϕ is equal to the closed linear span of the powers of ϕ . For $f \in H^\infty$ let

$$fH^2 \equiv \{g \mid g = fh, h \in H^2\} \subseteq H^2.$$

Then

$$(3.1) \quad [\phi, \phi^2, \phi^3, \dots] \subseteq BH^2 \cap SH^2.$$

Note that BH^2 and SH^2 are closed subspaces of H^2 . Also if B_n is a Blaschke product with n factors in its representation then $\text{codimension } B_n H^2 = n$ so that if B is an infinite Blaschke product then BH^2 has infinite codimension in H^2 . Further, given any singular inner function S and any integer n we may write S as a product $S = S_1 \cdot S_2 \cdot \dots \cdot S_n$ where each S_i is a singular inner function. Letting T_s be the Toeplitz operator on H^2 given by $T_s : f \mapsto Sf$, we see that each T_s is a pure (non-unitary) isometry. Hence, $T_s H^2 = SH^2$ has infinite codimension. We see from equation (3.1) that we may assume that $\phi = B_k F$, where B_k is a Blaschke product with k factors and F is an outer function in H^∞ . We show now that $k = 1$ and $|F(\lambda)| = 1$ all $\lambda \in \Delta$.

Let us assume $k > 1$ and that

$$B_k(\lambda) = \prod_{i=1}^k \left(\frac{\lambda - \alpha_i}{1 - \bar{\alpha}_i \lambda} \right), \quad \lambda \in \Delta,$$

$|\alpha_i| < 1$, $j = 1, 2, \dots, k$. For each $n = 1, 2, 3, \dots$,

$$[\phi^n, \phi^{n+1}, \phi^{n+2}, \dots] \subseteq B^n H^2$$

and $\text{codim } (B^n H^2) \geq kn \geq 2n$. Hence, $\text{codim } \mathcal{R}(C_\phi) = \text{codim } [1, \phi, \phi^2, \dots] \geq n - 1$. This contradicts $\text{codim } \mathcal{R}(C_\phi)$ being finite so we must have $k = 1$ and

$$\phi(\lambda) = \left(\frac{\lambda - \alpha}{1 - \bar{\alpha} \lambda} \right) F(\lambda); \quad |\alpha| < 1.$$

We need only consider $\|\phi\|_\infty = 1$ since in the case $\|\phi\|_\infty < 1$ we know C_ϕ is a compact operator. Suppose there is a compact set A contained in Γ with positive measure such that

$$(3.2) \quad |\hat{F}(e^{i\theta})| \leq \rho < 1, \quad e^{i\theta} \in A.$$

The closed unit ball of H^2 is a normal family (in the sense of Montel). Hence, if $f \in H^2$ and $\|f\|_2 \leq 1$, there is a constant $K = K(\rho)$ such that

$$(3.3) \quad |f(\lambda)| \leq K, \quad |\lambda| \leq \rho.$$

We claim that $C_\phi H^2$ contains only those H^2 functions that are bounded on A . For if $f \in H^2$ and $\Delta(\theta)$ is any angle in Δ at $e^{i\theta} \in A$, then we have

$$\lim f(\lambda) = \hat{f}(e^{i\theta})(\lambda \rightarrow e^{i\theta}), \quad \lambda \in \Delta(\theta).$$

If $g = C_\phi f$ we see that

$$\begin{aligned} \lim g(\lambda) &= f \left(\left(\frac{e^{i\theta} - \alpha}{1 - \bar{\alpha} e^{i\theta}} \right) \hat{F}(e^{i\theta}) \right) = f(\beta) \\ &(\lambda \rightarrow e^{i\theta}, \lambda \in \Delta(\theta)), \end{aligned}$$

where

$$|\beta| = \left| \left(\frac{e^{i\theta} - \alpha}{1 - \bar{\alpha} e^{i\theta}} \right) \hat{F}(e^{i\theta}) \right| \leq \rho.$$

Inequality (3.3) above implies

$$|\hat{g}(e^{i\theta})| \leq K \|f\|_2, \quad e^{i\theta} \in A.$$

Let $\{A_i\}$ be a sequence of disjoint sets each of positive measure with $\bigcup_{i=1}^{\infty} A_i = A$. Further, for each $j = 1, 2, 3, \dots$, let g_j be a non-negative square summable function on Γ which is essentially unbounded on A_j . We may choose the g_j so that $\log g_j$ is summable. By Szegő's Theorem there exist functions f_j in H^2 which satisfy $|\hat{f}_j| = g_j$ a.e.. Let $H^2 = C_\phi H^2 \oplus N$ and let P be the orthogonal projection on N . Clearly, no linear combination of the f_j is in $C_\phi H^2$. Thus the $\{Pf_j\}$ are independent. (If $0 = \sum \alpha_i Pf_i = P \sum \alpha_i f_i$, then $\sum \alpha_i f_i \in C_\phi H^2$.) Hence $C_\phi H^2$ has infinite codimension.

This contradiction implies that ϕ is a Blaschke factor with a single factor.

4. Composition operators with closed range. It is known [4, Th. 1] that if θ is an inner function, then the composition operator C_ϕ is bounded below and hence has closed range. The converse of this statement *e.g.* if C_ϕ has closed range then ϕ is inner, is not true. We would like to give a few examples. We first note that if $\phi: \Delta \rightarrow \Delta$ is analytic and $f \in H^2$, then $\lim_{r \rightarrow 1} f(\phi(re^{i\theta}))$ exists a.e. and equals $\hat{f}(\hat{\phi}(e^{i\theta}))$ for $|\phi(e^{i\theta})| = 1$, and is $f(\hat{\phi}(e^{i\theta}))$ for $|\hat{\phi}(e^{i\theta})| < 1$ (see [5, Th. 2]). We will abuse notation and write $\hat{f}(\hat{\phi}(e^{i\theta}))$ for these boundary values.

Let Ω be the simply connected domain contained in Δ bounded by the curves

$$\Gamma_1 = \{|w| = 1, \quad -\pi/4 < \arg w < 5\pi/4\}$$

and

$$\Gamma_2 = \{\operatorname{Im} w = -1/2\}.$$

Let ψ be the Riemann mapping function of Δ into Ω with $\psi(0) = 0$ and $\psi'(0) > 0$, and suppose $|\alpha| = |\beta| = 1$ are such that $\psi(\alpha) = +1$, $\psi(\beta) = -1$. Define $\phi(\lambda) = [\psi(\lambda)]^2$ and observe that ϕ is not inner. The function ϕ is analytic and one to one on the arc of the unit circle determined by $[\alpha, \beta]$. It also has a non-vanishing derivative there. The computation below shows that C_ϕ has closed range.

$$\begin{aligned} \|C_\phi f\|_2^2 &= \int_{\Gamma} |\hat{f}(\hat{\phi}(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \geq \int_{(\alpha, \beta)} |\hat{f}(\hat{\phi}(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} |\hat{f}(e^{is})|^2 |\phi'|^{-1} ds \geq K(\|f\|_2^2) \end{aligned}$$

where $K = \min |\phi'|^{-1}$ on $[\alpha, \beta]$.

As a second example let us suppose $\phi: \Delta \rightarrow \Delta$ and that the range of $\hat{\phi}$ (on Γ) omits a set $K \subseteq \Gamma$ with $0 < mK < 2\pi$, where m denotes Lebesgue measure on Γ . We can obtain a function f in H^∞ with $|\hat{f}| = 1$ on K and $|\hat{f}| = 1/2$ of $\Gamma \setminus K$. It is clear by the Lebesgue dominated convergence theorem that $\|C_\phi f^n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. However $\|f^n\|_2 \geq mK$, so that C_ϕ is not bounded below. But C_ϕ is one to one, and thus C_ϕ does not have closed range. With these examples as motivation we proceed to Theorem 2. First, we need a result of Schwartz

[6, p. 51], (see also Nordgren [4]). If ϕ is analytic mapping Δ into Δ , then the function

$$\nu(A) \equiv m(\hat{\phi}^{-1}(A))$$

defined on the Borel subsets of Γ is a measure. Moreover, this measure is absolutely continuous with respect to Lebesgue measure and its Radon–Nikodym derivative is in L^∞ , in fact

$$\nu(A) \leq \int_A P(\phi(0), e^{it}) dt$$

where P is the Poisson kernel for Δ .

Theorem 2. *Suppose $\phi: \Delta \rightarrow \Delta$ is analytic and nonconstant. Then C_ϕ has closed range if and only if $d(m\phi^{-1})/dm$ is essentially bounded away from 0.*

Proof. If $d(m\phi^{-1})/dm$ is bounded away from 0 then a straightforward calculation shows that C_ϕ is bounded below.

Suppose for the converse that $h = dm\phi^{-1}/dm$ is not bounded away from 0. Then for each positive integer j the set B_j ,

$$B_j \equiv \{e^{i\theta} \mid \hat{h}(e^{i\theta}) < 1/j\}$$

has positive measure. Let the preimage of B_j under ϕ be A_j and observe that

$$m(A_j)/m(B_j) \rightarrow 0, \quad j \rightarrow \infty.$$

Choose now $f_j \in H^\infty$ with

$$|\hat{f}_j| = \begin{cases} 1 & \text{on } B_j \\ 1/2 & \text{on } \Gamma - B_j \end{cases}$$

Then $\|f_j^n\|_2 \rightarrow mB_j$ whereas

$$\|C_\phi \hat{f}_j^n\|_2^2 = \int_{A_j} |\hat{f}_j^n(\phi)|^2 dm + \int_{\Gamma - A_j} |\hat{f}_j^n(\phi)|^2 dm \leq 2mA_j$$

if n is suitably large, (say $n = n_j$). This implies that

$$\|C_\phi f_j^{n_j}\|_2 / \|f_j^{n_j}\|_2 \rightarrow 0,$$

and that C_ϕ is not bounded below.

To illustrate the sharpness of Theorem 2 we consider the following example. Define a mapping $\phi: \Delta \rightarrow \Delta$ by

$$\begin{aligned} \phi(\lambda) &= \frac{-i \log \left(\frac{1-\lambda}{1+\lambda} \right) + \left(\frac{\pi}{2} - 1 \right)}{-i \log \left(\frac{1-\lambda}{1+\lambda} \right) + \left(\frac{\pi}{2} + 1 \right)} \\ &\equiv \frac{-i \log \left(\frac{1-\lambda}{1+\lambda} \right) + \alpha}{-i \log \left(\frac{1-\lambda}{1+\lambda} \right) + \beta}. \end{aligned}$$

This is a conformal mapping of Δ into a domain $\Omega \subseteq \Delta$, where Ω is obtained from Δ by removing a disk internally tangent to Δ at $\lambda = 1$.

Let η be the inverse for ϕ on Ω and let ν be the measure $\nu = m\hat{\phi}^{-1}$ on the Borel sets of Γ . Since

$$\nu(A) = \int_A |\eta'(s)| \, ds.$$

we see that the Radon–Nikodym derivative of ν with respect to m is

$$\frac{d\nu}{dm} = |\eta'| = 1/|\phi'(\eta(e^{i\theta}))|.$$

The derivative of ϕ is

$$\phi'(\lambda) = \frac{2i(\beta - \alpha)}{(1 + \lambda)(1 - \lambda) \left\{ -i \log \left(\frac{1 - \lambda}{1 + \lambda} \right) + \beta \right\}^2}$$

and this tends to zero as $\lambda \rightarrow 1$. Thus $\hat{\phi}(\Gamma) \supset \Gamma$, but $\mathcal{R}(C_\phi)$ is not closed.

We will close with the following query. Our treatment of the theorems and examples in this paper is concentrated on studying the boundary values of the functions in question. It would be interesting to characterize the composition operators with closed range by considering the range of the mapping on Δ rather than on Γ .

There are some related results on composition operators in [1]. Namely, we consider the question of when the span of a fixed vector $f \in H^2$ has the property that $[C_\phi f; \phi \text{ an automorphism of } \Delta]$ is dense in H^2 and it is shown that the weakly closed algebra A of $B(H^2)$ generated by these $\{C_\phi\}$ is reflexive.

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