On the Cesaro Operator

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Introduction and summary of results. Let ν be a positive finite Borel measure on the complex plane with compact support, and let $H^2(\nu)$ be the closure of the complex polynomials in the metric of $L^2(\nu)$. The operator $M_z: f(z) \rightarrow zf(z)$ acting on $H^2(\nu)$ is subnormal, and it is well known that every subnormal operator with a cyclic vector is unitarily equivalent to such an operator M_z [18]. For example, if ν is normalized Lebesgue measure on the boundary of the open unit disk D , then $H^2(\nu)$ is the Hardy space H^2 and M_z is the simple unilateral shift.

The Cesaro operator C_0 is defined on the Hilbert space ℓ^2 of square summable complex sequences $\{a_n\}_{n=0}^\infty$ by $C_0\{a_n\} = \{b_n\}$, where $b_n = (n+1)^{-1} \sum_{i=0}^n a_i$. In [9], it was shown that C_0 is subnormal by exhibiting a measure μ such that $I - C_0$ is unitarily equivalent to M_z acting on $H^2(\mu)$. The measure μ is supported on the closure of D and the elements of $H^2(\mu)$ have representatives which are analytic in D . This paper is a study of M_z and $H^2(\mu)$, and continues the earlier investigations of C_0 by Brown, Halmos and Shields [2], Shields and Wallen [15], and ourselves [9].

We first show that M_z and thus $I - C_0$ is not similar to any weighted shift operator on ℓ^2 [7, p. 894]. The rest of the paper concerns invariant subspaces of M_z and related function theory. An intertwining operator between the simple unilateral shift U_+ and M_z associates inner functions in H^2 with certain invariant subspaces of M_z (Theorem 5). Other invariant subspaces are determined by non-Blaschke sequences of points in D which converge to 1 radially "faster than $\{1 - (2n)^{-1}\}$ " (Theorem 7 and following remark). A curious consequence is the existence of sequences $\{z_n\}$ and $\{w_n\}$ in D , each of which is the zero set for a function in $H^2(\mu)$, yet any f in $H^2(\mu)$ which vanishes at each z_n and each w_n must vanish identically.

We obtain a "Muntz-type" theorem concerning when the functions $\{(1-z)^{\lambda_n}\}_{n=0}^\infty$ span a dense subset of H^2 , where $\lambda_n > 0$ and $(\lambda_{n+1} - \lambda_n) > \delta > 0$ (Corollary 7). Finally, we investigate the spectra of the restrictions $M_z^*|_{\mathfrak{M}^\perp}$, where $\mathfrak{M}$ is an invariant subspace associated with an inner function. Results are analogous to those obtained for U_+^* by J. W. Moeller [12].

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1. Background and notation. It is convenient to also consider C_0 as acting on the Hardy space H^2 of analytic functions $\sum_{n=0}^{\infty} a_n z^n$ with $\sum |a_n|^2 < \infty$. (See [8] for information about H^p spaces.) Then $C_0(\sum a_n z^n) = \sum b_n z^n$, where $b_n = (n+1)^{-1} \sum_{i=0}^n a_i$. Let D denote the disk $|z| < 1$ and C the circle $|z| = 1$. By [2], $T = I - C_0$ has norm 1 and each w in D is an eigenvalue for T^* with corresponding eigenfunction $\varphi_w(z) = (1-z)^{w/(1-w)}$. The set of analytic functions of the form $F(z) = \langle f, \varphi_z \rangle$ (f in H^2) can be made into a Hilbert space $\mathcal{H}$ by defining $\|F\|_{\mathcal{H}} = \|f\|$. Then T is unitarily equivalent to the operator $S: F(z) \rightarrow zF(z)$ acting in $\mathcal{H}$ [9, 15]. The space $\mathcal{H}$ has the reproducing kernel function

$$K_w(z) = \Gamma\left(\frac{\bar{w}}{1-\bar{w}} + \frac{z}{1-z} + 1\right) / \Gamma\left(\frac{\bar{w}}{1-\bar{w}} + 1\right) \Gamma\left(\frac{z}{1-z} + 1\right);$$

i.e., $F(w) = \langle F, K_w \rangle$, for all w in D and F in $\mathcal{H}$.

There is a positive Borel measure μ on $\bar{D} = D \cup C$ of total mass 1 such that T is unitarily equivalent to the operator $M_z: F(z) \rightarrow zF(z)$ acting in $H^2(\mu)$; $H^2(\mu)$ is identified with $\mathcal{H}$ by means of a unitary operator $U: H^2(\mu) \rightarrow \mathcal{H}$ which takes every polynomial onto itself. The measure μ has the following form: for any μ -integrable function f ,

$$\int_D f d\mu = \int_{\text{Re } z \geq 0} f\left(\frac{2z-1}{2z+1}\right) d\tau(z),$$

where $d\tau(z)$ is a measure on the half-plane $\text{Re } z \geq 0$ which is supported on the vertical lines $\text{Re } z = n/2$, $n = 0, 1, 2, 3, \dots$. The restriction of $d\tau$ to the line $\text{Re } z = x$ is

$$\frac{1}{2\pi} \frac{|\Gamma(x + (1/2) + iy)|^2}{\Gamma(2x+1)} dy, \quad x = 0, \frac{1}{2}, 1, \dots$$

The closed support of μ is therefore the union $\bigcup_{n=0}^{\infty} C_n$, where C_n is the circle of radius $1/(n+1)$ with center at $n/(n+1)$. The μ -measure of C_n is $2^{-(n+1)}$.

2. Non-similarity of $I - C_0$ with a weighted shift. As an operator on ℓ^2 , $T = I - C_0$ resembles the simple unilateral shift U_+ in several ways. Its spectrum is the closed unit disk and every point in the open disk is an eigenvector for T^* , $T - wI$ is bounded below for each $|w| < 1$ (Lemma 2 below), and T is unitarily equivalent to an operator M_z acting on a space $H^2(\mu)$. Thus it seems reasonable to conjecture that T is similar to U_+ , or at least similar to a weighted shift [7, p. 894]. We shall see below that this conjecture is false. Heuristically speaking, the difficulty is that the part of the measure μ on the unit circle approaches zero rapidly near the point 1.

Let $\{e_n\}_{n=0}^{\infty}$ be the usual orthonormal basis for ℓ^2 and let U_+ denote the simple shift operator on ℓ^2 , $U_+(\sum a_n e_n) = \sum a_n e_{n+1}$. A *weighted shift* is an operator W on ℓ^2 of the form $W(\sum a_n e_n) = \sum \lambda_n a_n e_{n+1}$, for some bounded sequence of complex numbers $\{\lambda_n\}$ called the *weights* of W . A necessary and sufficient condition that W and U_+ be similar is that the sequence of products $\{|\prod_{k=0}^n \lambda_k|\}_{n=0}^{\infty}$

be bounded away from both 0 and ∞ [6, p. 76]. The proof that $I - C_0$ is not similar to a weighted shift depends on the following result.

Lemma 1. *Let X be a contraction on a Hilbert space which is similar to a weighted shift with nonzero weights. If $\{X^n\}$ does not converge strongly to zero as $n \rightarrow \infty$, then X is similar to U_+ .*

Proof. We show that if a contraction X is similar to a weighted shift with nonzero weights but is not similar to U_+ , then $\{X^n\}$ converges strongly to zero. By hypothesis, there is an invertible operator A and a weighted shift W with nonzero weights $\{\lambda_i\}$ such that $AX = WA$. For each $k = 1, 2, \dots$, $W^k e_0 = (\prod_{i=0}^{k-1} \lambda_i) e_k$. Since $\|X\| \leq 1$, $\|W^k\| = \|AX^k A^{-1}\| \leq \|A\| \|A^{-1}\|$. Therefore $\{\prod_{i=0}^{n-1} \lambda_i\}$ is a bounded sequence. Since X is not similar to U_+ , it follows that this sequence is not bounded away from zero and hence a subsequence converges to zero, say $\prod_{i=1}^{n_k} \lambda_i \rightarrow 0$, as $k \rightarrow \infty$. We will show this implies that $\{W^{n_k+1}\}$ converges strongly to zero as $k \rightarrow \infty$, and therefore so does $X^{n_k+1} = A^{-1}W^{n_k+1}A$. Since $\|X\| \leq 1$, it will follow that $\{X^n\}$ converges strongly to zero.

Let $f = \sum a_n e_n$ belong to ℓ^2 . We want to show that $\|W^{n_k+1}f\| \rightarrow 0$ as $k \rightarrow \infty$. Let $\epsilon > 0$ and choose a positive integer N such that $\|\sum_{m=N+1}^{\infty} a_m e_m\| < \epsilon$. Then $\|W^{n_k+1}f\| < \|W^{n_k+1}(\sum_{m=0}^N a_m e_m)\| + \epsilon \|A\| \|A^{-1}\|$. Since ϵ is arbitrary, it is sufficient to show that $W^{n_k+1}(\sum_{m=0}^N a_m e_m) \rightarrow 0$, as $k \rightarrow \infty$. Now $W^{n_k+1}e_m = (\prod_{i=m}^{m+n_k} \lambda_i) e_{m+n_k+1}$, and so $\|W^{n_k+1}(\sum_{m=0}^N a_m e_m)\|^2 = \sum_{m=0}^N |a_m|^2 |\prod_{i=m}^{m+n_k} \lambda_i|^2$. Since all of the weights are nonzero and there are only finitely many terms in the last expression, it is sufficient to show that $\prod_{i=0}^{m+n_k} \lambda_i \rightarrow 0$ as $k \rightarrow \infty$, for each $m = 0, 1, 2, \dots$. This follows from the inequality $|\lambda_j| < \|W\|$ for all j , since

$$\left| \prod_{i=0}^{m+n_k} \lambda_i \right| = \left| \prod_{i=0}^{n_k} \lambda_i \right| \left| \prod_{i=n_k+1}^{m+n_k} \lambda_i \right| \leq \|W\|^m \left| \prod_{i=0}^{n_k} \lambda_i \right| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This completes the verification that $W^{n_k+1}f \rightarrow 0$ as $k \rightarrow \infty$. The lemma follows.

The next theorem was also found by Stuart Clary, who has made a thorough study of operators M_s which are similar to U_+ [3].

Let σ denote Lebesgue measure on the unit circle C normalized so that $\sigma(C) = 1$.

Theorem 1. *Suppose that ν is a positive finite Borel measure on the closed unit disk $\bar{D}$. Let ρ be the restriction of ν to C and suppose that α is the part of ρ which is absolutely continuous with respect to σ . If*

$$\int \log \left(\frac{d\alpha}{d\sigma} \right) d\sigma = -\infty,$$

then the only bounded operator $X: H^2(\nu) \rightarrow H^2$ with $XM_s = U_+X$ is $X = 0$.

Proof. It follows from the hypothesis that $Xp = pq$ for every polynomial p , where $q = X(1)$. Then $\int |pq|^2 d\sigma \leq M \int_D |p|^2 d\nu$, where $M = \|X\|^2$. Since $z^n pq$ and pq have the same H^2 -norm for every $n = 1, 2, \dots$, $\int |pq|^2 d\sigma \leq M \int_D |z^n p|^2 d\nu$. Upon letting $n \rightarrow \infty$, we obtain by the dominated convergence theorem that $\int |pq|^2 d\sigma \leq M \int_C |p|^2 d\rho$ for every polynomial p . For each $|w| < 1$, $(1 - \bar{w}z)^{-1}$ is the uniform limit of polynomials, hence

$$\int \frac{1 - |w|^2}{|1 - \bar{w}z|^2} |q|^2 d\sigma \leq M \int_C \frac{1 - |w|^2}{|1 - \bar{w}z|^2} d\rho.$$

It follows from Fatou's Theorem, upon letting $w \rightarrow e^{i\theta}$ on C , that $|q(e^{i\theta})|^2 \leq M(d\alpha/d\sigma)(e^{i\theta})$ a.e. By the hypothesis on $d\alpha/d\sigma$,

$$\int \log |q| d\sigma = -\infty$$

so that $q = 0$ a.e. Thus $X = 0$ and the proof is complete.

Theorem 2. *The operator M_z on $H^2(\mu)$ (and thus $I - C_0$) is not similar to a weighted shift.*

Proof. Suppose that M_z is similar to a weighted shift W . Since M_z has zero kernel, all of the weights of W are non-zero. Moreover M_z is a contraction and $\|M_z^n 1\|^2 = \int_D |z^n|^2 d\mu(z) \geq \int_C |z^n|^2 d\mu(z) = \mu(C) = 1/2$ so that M_z^n does not tend strongly to zero as $n \rightarrow \infty$. Hence, by Lemma 1, M_z is similar to U_+ . If we put $\nu = \mu$ in Theorem 1 and refer to the formula for τ in section 1 we have

$$\frac{d\alpha}{d\sigma}(e^{i\theta}) = |1 - e^{i\theta}|^{-2} \left| \Gamma\left(\frac{1}{1 - e^{i\theta}}\right) \right|^2 = \frac{\pi}{2(1 - \cos \theta) \cosh \left[\frac{\pi \sin \theta}{2(1 - \cos \theta)} \right]},$$

so that $\int \log (d\alpha/d\sigma) d\sigma = -\infty$. By Theorem 1, M_z cannot be similar to U_+ , a contradiction. This completes the proof.

3. Some invariant subspaces of C_0 . The operator $I - C_0$ in ℓ^2 has certain obvious invariant subspaces, namely for each $k = 1, 2, \dots$, the subspace $\mathfrak{M}_k$ consisting of those sequences $\{a_n\}$ with $a_n = 0$ for $n < k$. Upon mapping into $H^2(\mu)$ by the unitary map of Section 1, it follows easily from the form of the orthonormal basis for $H^2(\mu)$, $\Psi_0(z) = 1$, $\Psi_n(z) = (z - 1)^{-n} z(z - 1/2) \cdots (z - (n - 1)/n)$, $n \geq 1$, [9, 15] that $\mathfrak{M}_k$ goes into the invariant subspace of M_z in $H^2(\mu)$ consisting of those functions in $H^2(\mu)$ of the form $B(z)g(z)$ for some $g(z)$ in $H^2(\mu)$, where B is the finite Blaschke product with zeros at $0, 1/2, 2/3, \dots, 1 - 1/k$. This fact, together with Beurling's theorem that every invariant subspace of the simple unilateral shift in H^2 is of the form BH^2 for some inner function B , leads us to study the invariant subspaces of M_z in $H^2(\mu)$ which are generated by inner functions. Our first objective is to determine when two such subspaces are identical.

Lemma 2. Let ν be a finite positive compactly supported Borel measure on the plane and suppose that U is an open set. Assume that ν has no point masses in U and that there exists $N > 0$ such that

$$|p(w)|^2 \leq N \int |p|^2 d\nu$$

for every polynomial p and every w in U . Then the operator $M_z - w$ acting on $H^2(\nu)$ is bounded below for each w in U .

Proof. Let $D(w, r)$ denote the open disk centered at w with radius r and choose any polynomial p . Let r be so small that $D(w, r) \subset U$ and $\delta \equiv 1 - N\nu(D(w, r)) > 0$. Then

$$\|(M_z - w)p\|^2 \geq \int_{|z-w| \geq r} |z-w|^2 |p(z)|^2 d\nu(z) \geq r^2 \left(\int |p|^2 d\nu - \int_{D(w, r)} |p|^2 d\nu \right).$$

Now $|p(z)|^2 \leq N \|p\|^2$ for z in $D(w, r)$ and consequently $\|(M_z - w)p\|^2 \geq r^2 \delta \|p\|^2$. This holds for any polynomial p and the polynomials are dense in $H^2(\nu)$, hence $M_z - w$ is bounded below.

Definition. Let $H^\circ(\mu)$ denote $H^2(\mu) \cap L^\circ(\mu)$.

Recall that the unitary operator $U: H^2(\mu) \rightarrow \mathcal{H}$ takes every polynomial p onto its restriction to D . Since the polynomials are dense in $H^2(\mu)$ and norm-convergence in $H^2(\mu)$ implies pointwise convergence μ -a.e. in $\bar{D}$ for a subsequence, while norm-convergence in $\mathcal{H}$ implies pointwise convergence in D , it follows that for every f in $H^2(\mu)$, $(Uf)(z) = f(z)$ μ -a.e. in D . Since Uf is analytic in D , we can redefine f on a set of μ -measure zero so that f is analytic in D . In what follows we shall assume that every f in $H^2(\mu)$ has been redefined in this way. Then if $|z| < 1$, $|f(z)| \leq \|K_z\| \|f\|$ where K_z is the kernel function for $\mathcal{H}$, since this inequality holds in $\mathcal{H}$. We will also need the result by Yoshino [17] that a space $H^2(\nu)$ contains bf whenever it contains f , for any b in $H^\circ(\nu)$.

Lemma 3. $H^\circ(\mu) = H^\circ$.

Proof. Let f belong to H° . Then f is defined in D and Lebesgue-a.e. on C . Since the restriction of μ to C is absolutely continuous, f belongs to $L^\circ(\mu)$. Furthermore, the sequence $\{p_n\}$ of Fejer sums of f is a uniformly bounded sequence of polynomials which converges to f pointwise everywhere in D and a.e. on C . Thus $\{p_n\}$ converges to f pointwise μ -a.e. on $\bar{D}$. By the dominated convergence theorem, $\lim \int |p_n - f|^2 d\mu = 0$, as $n \rightarrow \infty$, hence f belongs to $H^2(\mu) \cap L^\circ(\mu)$.

To prove the reverse inclusion, let b belong to $H^\circ(\mu)$. For any f in $H^2(\mu)$, bf belongs to $H^2(\mu)$ and,

$$|b(z)f(z)| \leq \|K_z\| \|bf\| \leq \|K_z\| \|b\|_\infty \|f\|$$

for all $|z| < 1$. Hence

$$||K_z|| = \sup |f(z)|/|f| \leq ||K_z|| \, ||b||_\infty/|b(z)| ,$$

where the supremum is over all nonzero f , and so $|b(z)| \leq ||b||_\infty$ for $|z| < 1$. Thus b is a bounded analytic function in D . Let $\hat{b}$ be the unique function in H^∞ which agrees with b in D . Then, by the first half of the proof, $b - \hat{b}$ belongs to $H^2(\mu)$ and vanishes in D . Hence $U(b - \hat{b})$ is the zero element of $\mathcal{H}$ and therefore b and $\hat{b}$ are the same element of $H^\infty(\mu)$. Hence $b = \hat{b}$ is in H^∞ . Thus $H^\infty(\mu)$ is a subset of H^∞ , which completes the proof.

Let B denote an arbitrary inner function. Then $B = QS$, where

$$(1) \quad Q(z) = cz^m \prod_1^\infty \frac{\bar{z}_i}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z} ,$$

$$(2) \quad S(z) = \exp \left(- \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(\theta) \right) ,$$

$|c| = 1$, m is a nonnegative integer, $\{z_i\}$ are the nonzero zeros of B , and ν is a positive finite singular Borel measure on the unit circle C . The supports of Q , S , and B are defined by

$$\begin{aligned} \text{supp } Q &= C \cap (\text{closure of } \{z_i\}), \\ \text{supp } S &= \text{closed support of } \nu \text{ on } C, \\ \text{supp } B &= \text{supp } Q \cup \text{supp } S. \end{aligned}$$

Let $|\lambda| = 1$. It is well-known that λ fails to belong to $\text{supp } B$ if and only if there exists $\epsilon > 0$ such that $|B(z)|$ is bounded away from zero on the set $\{z: |z| < 1 \text{ and } |z - \lambda| < \epsilon\}$.

Lemma 4. *Suppose that B is an inner function and 1 does not belong to $\text{supp } B$. Then the operator $f \rightarrow Bf$ is bounded below on $H^2(\mu)$, and hence $BH^2(\mu)$ is a closed subspace of $H^2(\mu)$.*

Proof. We recall that the closed support of μ consists of the circles

$$C_n = \{z: |z - n/(n+1)| = 1/(n+1)\}, \quad n = 0, 1, 2, \dots$$

Since 1 does not belong to $\text{supp } B$, only finitely many zeros $z_1, z_2, \dots, z_n$ of B can lie on the circles C_n , $n = 1, 2, \dots$. Let $A(z) = \prod_{i=1}^n (z_i - z)/(1 - \bar{z}_i z)$ and $P = B/A$. Then P^{-1} is essentially bounded with respect to μ , hence the operator $f \rightarrow Pf$ is bounded below on $H^2(\mu)$. To complete the proof it suffices to show that the operator $f \rightarrow Af$ is also bounded below on $H^2(\mu)$. Fix one of the zeros of A , say z_k , and let U be an open set containing z_k whose closure is contained in D . Then for any polynomial p and any w in U , $|p(w)|^2 \leq N \int |p|^2 d\mu$ where $N = \sup_{z \in U} ||K_z||^2 < \infty$. It follows from Lemma 2 that $M_z - z_k$ is bounded below, and applying this for $k = 1, 2, \dots, n$ shows that the operator $f \rightarrow Af$ is bounded below as desired.

Let V denote the unitary map from H^2 onto $H^2(\mu)$ which takes T into M_z (see Section 1). We want to construct closed proper invariant subspaces for M_z on $H^2(\mu)$, and hence for T on H^2 , by using inner functions. A fundamental role is played by the inner functions

$$q_t(z) = \exp\left(-t \frac{1+z}{1-z}\right), \quad \text{for } t \geq 0.$$

The function in H^2 which is mapped by V onto $e^t q_t$ is $k_{1-a}(z) = [1 - (1-a)z]^{-1}$, where $t = -(\log a)/2$, $0 < a < 1$ [9]. L. Shulman has shown that each function $k_{1-a}(z)$ is cyclic for T [16]. It follows that q_t is cyclic for M_z for every $t > 0$; the next theorem shows that these are the only cyclic inner functions. For a proof of Shulman's result in a more general setting, see [10].

Theorem 3. *Let A and B be inner functions. A necessary and sufficient condition that the closure of $BH^2(\mu)$ contain the closure of $AH^2(\mu)$ is that $Aq_t = BF$ for some $t \geq 0$ and some inner function F .*

Proof. Suppose that $Aq_t = BF$ where F is inner and $t \geq 0$, and let f belong to the closure of $AH^2(\mu)$. Then $f = \lim Ar_n$ in the metric of $H^2(\mu)$ for some sequence $\{r_n\}$ of functions in $H^2(\mu)$. Since q_t is cyclic for M_z by the remark above, there is a sequence $\{p_n\}$ of polynomials such that $\|r_n - q_t p_n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$\begin{aligned} \|BFp_n - f\| &= \|Aq_t p_n - f\| \\ &\leq \|Aq_t p_n - Ar_n\| + \|Ar_n - f\| \\ &\leq \|q_t p_n - r_n\| + \|Ar_n - f\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence f belongs to the closure of $BH^2(\mu)$. Since f was arbitrary, the closure of $BH^2(\mu)$ contains the closure of $AH^2(\mu)$.

Now we introduce some notation. Let G be an inner function and suppose that $G = QS$ where Q is a Blaschke product and S is a singular inner function. For each $\epsilon > 0$, let $Q_\epsilon(z)$ be the Blaschke product formed from the zeros of G which lie in the set $\{z: |z| < 1 \text{ and } |z - 1| \geq \epsilon\}$, and let

$$S_\epsilon(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu_\epsilon(\theta)\right),$$

where ν_ϵ is the restriction of ν to $\{z: |z| = 1 \text{ and } |z - 1| > \epsilon\}$, and ν is associated with S as in (2). Let $G_\epsilon = Q_\epsilon S_\epsilon$. Clearly 1 does not belong to $\text{supp } G_\epsilon$ and the operator $f \rightarrow G_\epsilon f$ is bounded below on $H^2(\mu)$.

For the converse, assume that the closure of $BH^2(\mu)$ contains the closure of $AH^2(\mu)$, so that there is a sequence $\{p_n\}$ of polynomials such that $Bp_n \rightarrow A$ in the metric of $H^2(\mu)$ as $n \rightarrow \infty$. For each $\epsilon > 0$, define B_ϵ as in the preceding paragraph. Let R^ϵ be the inner function defined by $B = B_\epsilon R^\epsilon$. Since multiplication by B_ϵ is bounded below, the convergence of Bp_n implies that $R^\epsilon p_n$ is a

Cauchy sequence and hence converges to some g in $H^2(\mu)$. Then $Bp_n \rightarrow B.g$ so that $A = B.g$. We will show that g belongs to H^∞ . Since 1 does not belong to $\text{supp } B_\epsilon$, only a finite number of the zeros of B_ϵ lie on the circles C_n , $n = 1, 2, 3, \dots$. Let r be chosen in $(0, 1)$ so that all of these zeros lie inside the disk $|z| < r$. Then $1/B_\epsilon(z)$ is bounded on the set $(\bigcup_{n=1}^\infty C_n) \cap \{z: r < |z| < 1\}$. Since B_ϵ is inner, $|B_\epsilon| = 1$ a.e. on the circle C_0 . Thus $g = A/B_\epsilon$ is μ -essentially bounded on $\{z: r \leq |z| \leq 1\}$. On the other hand g is analytic and thus bounded in $|z| \leq r$. Thus g belongs to $L^\infty(\mu) \cap H^2(\mu) = H^\infty(\mu) = H^\infty$ by Lemma 3. Hence g is an inner function and so B_ϵ divides A in the sense of inner functions [8, Chapter 6]. Since ϵ is arbitrary, it follows from the representations (1) and (2) that Bq_t^{-1} divides A where $t = \sup \{s: q_s \text{ divides } B\}$. Therefore B divides Aq_t which completes the proof.

Corollary 1. $\overline{AH^2(\mu)} = \overline{BH^2(\mu)}$ if and only if $q_s A = cq_t B$ for some constant c of modulus 1 and some $s, t \geq 0$.

Corollary 2. An inner function B is cyclic for M_s if and only if $B = cq_t$ for some $t \geq 0$ and $|c| = 1$.

The proofs of these corollaries are clear from the theorem.

Theorem 4. H^2 is contained in $H^2(\mu)$ and the inclusion map $i: H^2 \rightarrow H^2(\mu)$ is bounded.

Proof. The map $i: H^2 \rightarrow H^2(\mu)$ is densely defined because $H^2(\mu)$ contains every polynomial. We will show that i is bounded on polynomials, i.e., $\|p\|_{H^2(\mu)} \leq M \|p\|_{H^2}$ for some fixed $M > 0$ and every polynomial p . It follows from the relationship between the measures μ and τ (see Section 1) that for any polynomial p ,

$$\begin{aligned} \|p\|_{H^2(\mu)}^2 &= \int \left| p\left(\frac{2z-1}{2z+1}\right) \right|^2 d\tau(z) \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \left| p\left(\frac{2(n/2+iy)-1}{2(n/2+iy)+1}\right) \right|^2 |\Gamma((n+1)/2+iy)|^2 dy \\ &\leq \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\zeta_n}{n!} \int_{-\infty}^{\infty} \left| \frac{1}{1/2+(n/2+iy)} \right|^2 \left| p\left(\frac{2(n/2+iy)-1}{2(n/2+iy)+1}\right) \right|^2 dy, \end{aligned}$$

where

$$\zeta_n = \sup_{-\infty < y < \infty} |1/2 + n/2 + iy|^2 |\Gamma((n+1)/2 + iy)|^2.$$

Clearly, the function $(z + 1/2)^{-1} p((2z - 1)/(2z + 1))$ is in the Hardy space of the right half-plane [8], so

$$\begin{aligned}
& \sup_{0 \leq x < \infty} \int_{-\infty}^{\infty} \left| \frac{1}{1/2 + (x + iy)} p \left(\frac{2(x + iy) - 1}{2(x + iy) + 1} \right) \right|^2 dy \\
&= \int_{-\infty}^{\infty} \left| \frac{1}{1/2 + iy} p \left(\frac{2iy - 1}{2iy + 1} \right) \right|^2 dy \\
&= \int_{-\infty}^{\infty} \left| p \left(\frac{2iy - 1}{2iy + 1} \right) \right|^2 \frac{dy}{y^2 + 1/4} = \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi \|p\|_{H^2}^2.
\end{aligned}$$

Hence $\|p\|_{H^2(\mu)}^2 \leq (\sum_{n=0}^{\infty} \zeta_n/n!) \|p\|_{H^2}^2$. We must show that this last sum converges. For each $x > 0$, $|\Gamma(x + iy)|$ has a maximum at $y = 0$ and so

$$\begin{aligned}
|iy + (n + 1)/2|^2 |\Gamma(iy + (n + 1)/2)|^2 &= |\Gamma(iy + 1 + (n + 1)/2)|^2 \\
&\leq \Gamma((n + 3)/2)^2.
\end{aligned}$$

Therefore $\zeta_n = \Gamma((n + 3)/2)^2$. A consideration of the even and odd terms of the sum $\sum_{n=0}^{\infty} \Gamma((n + 3)/2)^2/n!$ separately and a use of the identity $\Gamma(n + 1/2) = \pi^{1/2} \Gamma(2n)/2^{2n-1} \Gamma(n)$, $n = 0, 1, 2, \dots$, will now show that the sum converges. Hence $\|p\|_{H^2(\mu)}^2 \leq M \|p\|_{H^2}^2$ for some $M > 0$ and all polynomials. Thus the inclusion $i: H^2 \rightarrow H^2(\mu)$ is bounded on polynomials and can therefore be uniquely extended to H^2 . That this extension is actually the inclusion map can be seen by taking limits in norm and then pointwise limits on subsequences (see the remarks preceding Lemma 3). This completes the proof.

If q belongs to H^∞ and Q is the operator from H^2 to $H^2(\mu)$ defined by $Qf = qf$, then $QU_+ = M_\#Q$, and Q is bounded by Theorem 4. On the other hand, if $X: H^2 \rightarrow H^2(\mu)$ is bounded and $XU_+ = M_\#X$, then there is a function q in H^∞ such that $Xf = qf$ for all f in H^2 (e.g., an argument like [6, p. 212] will suffice). Thus there exist intertwining operators between U_+ and $M_\#$ but none between $M_\#$ and U_+ .

Corollary 3. *Let $f = Bg$ where B is inner and g is an outer function in H^2 . Then the smallest invariant subspace of $M_\#$ which contains $i(f)$ is $\overline{BH^2(\mu)}$.*

Proof. Clearly, $i(f)$ belongs to $\overline{BH^2(\mu)}$ and hence the invariant subspace $\mathfrak{M}$ generated by $i(f)$ is contained in $\overline{BH^2(\mu)}$. On the other hand, there is a sequence $\{p_n\}$ of polynomials such that $p_n g \rightarrow 1$ in H^2 , since g is outer. Then $p_n f = B p_n g \rightarrow B$ in H^2 , as $n \rightarrow \infty$. Since the inclusion i is continuous, $p_n i(f) = i(p_n f) \rightarrow i(B)$ in $H^2(\mu)$, as $n \rightarrow \infty$, so B is in $\mathfrak{M}$. It follows easily that the closure of $BH^2(\mu)$ is contained in $\mathfrak{M}$, which completes the proof.

Corollaries 1 and 3 together imply that $i(f)$ is cyclic for $M_\#$ if and only if $f = q_t g$ for some $t \geq 0$ and some outer function g in H^2 .

Theorem 5. *Let $\mathfrak{M}$ be an invariant subspace of M_z in $H^2(\mu)$. The following conditions are equivalent:*

- i) $\mathfrak{M} \cap i(H^2)$ is dense in $\mathfrak{M}$.
- ii) $\mathfrak{M} = \overline{BH^2(\mu)}$ for some inner function B .
- iii) $\mathfrak{M} \cap L^\infty(\mu)$ is dense in $\mathfrak{M}$.

Proof. Suppose first that $\mathfrak{M} \cap i(H^2)$ is dense in $\mathfrak{M}$. Since i is continuous, $i^{-1}(\mathfrak{M})$ is a closed invariant subspace of U_+ in H^2 , so $i^{-1}(\mathfrak{M}) = BH^2$, for some inner function B , by Beurling's theorem. Thus

$$\mathfrak{M} = \overline{\mathfrak{M} \cap i(H^2)} = \overline{i(i^{-1}(\mathfrak{M}))} = \overline{i(BH^2)} = \overline{BH^2(\mu)}.$$

Also, iii) implies i) because $\mathfrak{M} \cap L^\infty(\mu) = \mathfrak{M} \cap H^\infty(\mu) = \mathfrak{M} \cap i(H^\infty)$, by Lemma 3. Clearly ii) implies iii).

Corollary 4. *Let $\{B_\alpha\}$ be any family of inner functions. Then*

$$\vee_\alpha \overline{B_\alpha H^2(\mu)} = \overline{BH^2(\mu)}$$

where $\vee$ denotes "closed span" and B is the greatest common divisor of the functions B_α .

This corollary can be established by a straightforward argument using Theorems 3 and 5. A similar argument will show that the largest invariant subspace $\mathfrak{N}$ of $\bigcap_\alpha \overline{B_\alpha H^2(\mu)}$ such that $\mathfrak{N} \cap L^\infty(\mu)$ is dense in $\mathfrak{N}$ must have the form $\overline{DH^2(\mu)}$ where D is the least common multiple of the B_α 's. However, it is not always the case that $\mathfrak{N} = \bigcap_\alpha \overline{B_\alpha H^2(\mu)}$. For example, if $\{z_i\}$ is a non-Blaschke sequence in D such that the subspace $\mathfrak{M} = \{f \text{ in } H^2(\mu) : f(z_i) = 0 \text{ for } i = 1, 2, \dots\}$ is nonzero (see Section 4), then $\mathfrak{M} = \bigcap_{n \geq 1} \overline{B_n H^2(\mu)}$ where B_n is the Blaschke product with zeros $z_1, \dots, z_n$, but $\mathfrak{M}$ contains no nonzero elements of $L^\infty(\mu)$. The subspace $\mathfrak{N}$ is the zero subspace and the family $\{B_n : n \geq 1\}$ has the least common multiple 0, yet $\bigcap_{n \geq 1} \overline{B_n H^2(\mu)}$ contains nonzero functions.

We have not been able to determine whether $\bigcap_\alpha \overline{B_\alpha H^2(\mu)} = \overline{CH^2(\mu)}$ whenever the family $\{B_\alpha\}$ has a nonzero (i.e., an inner) least common multiple C .

At this point we consider the spectral properties of some restrictions of M_z^* . Let B be an inner function. A well-known theorem of Moeller [12] determines the spectrum of the restriction $U_+^* \mid (BH^2)^\perp$ in terms of the zeros and singularities of B . An analogous result holds in $H^2(\mu)$.

Theorem 6. *Let B be an inner function and let $\mathfrak{M} = \overline{BH^2(\mu)}$.*

- i) *Let $|\lambda| < 1$. Then λ is in the spectrum of $M_z^* \mid \mathfrak{M}^\perp$ if and only if $B(\bar{\lambda}) = 0$, and in this case λ is an eigenvalue.*
- ii) *Let $|\lambda| = 1$. Write $B = B_0 q_t$ where $t \geq 0$ and q_s does not divide B_0 for any $s > 0$. Then λ is in the spectrum of $M_z^* \mid \mathfrak{M}^\perp$ if and only if $\bar{\lambda} \in \text{supp } B_0$.*

Proof. Since $\overline{BH^2(\mu)} = \overline{B_0H^2(\mu)}$ by Corollary 1, it suffices to prove the theorem under the assumption that q_s does not divide B for any $s > 0$, so that $B = B_0$. For the first half of the proof we employ an idea used by Fuhrmann [5] to study compressions of analytic Toeplitz operators. Suppose that $|\lambda| \leq 1$. Assume in addition that $B(\bar{\lambda}) \neq 0$ if $|\lambda| < 1$ and that $\bar{\lambda}$ is not in $\text{supp } B$ if $|\lambda| = 1$. Then $|z - \bar{\lambda}| + |B(z)|$ is bounded away from zero for $|z| < 1$. By the Carleson corona theorem [4] there exists u and v in H^∞ with $(z - \bar{\lambda})u + Bv = 1$. Now consider the map $\varphi \rightarrow W_\varphi$ from H^∞ into the bounded operators on $\mathfrak{H}^\perp$, where $W_\varphi f = P(\varphi f)$, $f \in \mathfrak{H}^\perp$, and P is the projection of $H^2(\mu)$ onto $\mathfrak{H}^\perp$. This map is an algebra homomorphism since $\varphi\mathfrak{H} \subset \mathfrak{H}$ for each φ in H^∞ , and thus $W_{(s-\bar{\lambda})}W_u + W_BW_v = I$. Since $W_B = 0$ and the operators $W_{(s-\bar{\lambda})}$ and W_u commute, we see that $W_{(s-\bar{\lambda})}^* = M_s^* | \mathfrak{H}^\perp - \lambda$ is invertible.

To prove the converse, suppose first that $|\lambda| < 1$ and $B(\bar{\lambda}) = 0$. Then $0 = f(\bar{\lambda})B(\bar{\lambda}) = \langle f, K_{\bar{\lambda}} \rangle$ for any f in $H^2(\mu)$; hence the eigenvector $K_{\bar{\lambda}}$ for M_s^* lies in $\mathfrak{H}^\perp$. The corresponding eigenvalue is λ .

It remains to consider the case where $|\lambda| = 1$. Assume that λ is not in the spectrum of $M_s^* | \mathfrak{H}^\perp$. Recall that $i : H^2 \rightarrow H^2(\mu)$ is the (bounded) inclusion map and $i^* : H^2(\mu) \rightarrow H^2$ is its adjoint. Now $i^*M_s^* = U_+^*i^*$ and $\mathfrak{H}^\perp$ is invariant for M_s^* , so $i^*(\mathfrak{H}^\perp)$ is invariant for U_+^* . Moreover, if g is in $\mathfrak{H}^\perp$ and f is in H^2 , $\langle i^*g, Bf \rangle = \langle g, i(Bf) \rangle = \langle g, Bi(f) \rangle = 0$ and consequently $i^*(\mathfrak{H}^\perp)$ is not dense in H^2 . Therefore the closure of $i^*(\mathfrak{H}^\perp)$ has the form $(\varphi H^2)^\perp$ for some inner φ with $(\varphi H^2)^\perp \subset (BH^2)^\perp$ and $B = \varphi\psi$ for some inner ψ . On the other hand, $0 = \langle i^*g, \varphi f \rangle = \langle g, \varphi i(f) \rangle$ for all g in $\mathfrak{H}^\perp$ and f in H^2 . Hence $\overline{BH^2(\mu)}^\perp = \mathfrak{H}^\perp \subset \varphi H^2(\mu)^\perp$, so $\varphi q_s = BF$ for some $s \geq 0$ and some inner F by Theorem 3. It follows from this, the fact that B and q_s are relatively prime and the equation $B = \varphi\psi$ that $B = \varphi$. Thus the closure of $i^*(\mathfrak{H}^\perp)$ coincides with $(BH^2)^\perp$.

Now denote $M_s^* | \mathfrak{H}^\perp$ by R . By assumption, $(I - zR)^{-1}$ is analytically continuable across an open arc Γ of C containing $\bar{\lambda}$. Now if $g \in \mathfrak{H}^\perp$ and $|z| < 1$,

$$\begin{aligned} \langle i^*g \rangle(z) &= \langle i^*g, k_z \rangle = \langle g, ik_z \rangle \\ &= \int (1 - z\bar{\xi})^{-1} g(\xi) d\mu(\xi) \\ &= \langle (I - zR)^{-1}g, 1 \rangle. \end{aligned}$$

Hence i^*g has an analytic continuation across Γ . Now the closed span of $\{U_+^{*n}i^*g\}_{n=0}^\infty$ is contained in $(BH^2)^\perp$ and is invariant under U_+^* and thus has the form $(\varphi_s H^2)^\perp$ where φ_s is inner and divides B . B is the least common multiple of $\{\varphi_s : g \in \mathfrak{H}^\perp\}$ since $\{i^*g : g \in \mathfrak{H}^\perp\}$ is dense in $(BH^2)^\perp$. By [11, Theorem 1] each φ_s is analytically continuable across Γ , so that $\Gamma \cap \text{supp } \varphi_s$ is empty. Moreover, there exists an open set V so that $\Gamma = V \cap C$ and V does not intersect $\{\bar{\lambda} : \lambda \text{ is in the spectrum of } R\}$. It follows that B (and hence each φ_s) cannot vanish on $V \cap D$. From this and the fact that B is the least common multiple of that φ_s 's, it is easy to see that $\Gamma \cap \text{supp } B$ is empty. Since $\bar{\lambda} \in \Gamma$, the proof is complete.

Corollary 5. *Let N be a closed subset of $|z| < 1$. Suppose that 1 is not an isolated point of N and that $N \cap D$ is a finite or countable set $\{z_n\}$ (possibly empty) with $\sum (1 - |z_n|) < \infty$. Then there exists an invariant subspace $\mathfrak{M}$ for M_z such that the spectrum of $M_z^* | \mathfrak{M}^\perp$ is exactly N .*

Proof. Denote $\{\bar{\lambda} : \lambda \in N\}$ by $\tilde{N}$. Let ν be a finite positive singular measure on C whose closed support is $\tilde{N} \cap C$. Let S be the singular inner function associated with ν and let Q be the Blaschke product whose zeros are $\{\bar{z}_n\}$. Set $B = QS$ and $\mathfrak{M} = \overline{BH^2(\mu)}$. If $B = B_0 q_i$ as in Theorem 6, the assumption that 1 is not an isolated point of N implies that $\text{supp } B = \text{supp } B_0$. The corollary now follows from Theorem 6.

4. Other invariant subspaces. A natural question is whether Theorem 5 describes every invariant subspace of C_0 ; we will see that it does not. If $\{z_n\}$ is any sequence of points in the unit disk, then the subspace of functions in $H^2(\mu)$ which vanish at every z_n is obviously an invariant subspace of M_z . If $\sum [1 - |z_n|] = \infty$, then this subspace can contain no nonzero bounded analytic function [8, p. 63] and hence is not of the type listed in Theorem 5 unless it is the zero subspace. In this section, we exhibit a class of such nonzero subspaces which arise from sequences $\{z_n\}$ which approach 1 radially with $\sum [1 - |z_n|] = \infty$. On the other hand, it is shown that if a nonzero function in $H^2(\mu)$ vanishes on a sequence $\{z_n\}$ which does not accumulate at 1, then $\sum [1 - |z_n|] < \infty$. In this respect functions in $H^2(\mu)$ behave like H^2 functions except near the point 1. Indeed, functions in $H^2(\mu)$ are log-integrable with respect to Lebesgue measure on any closed subarc of the unit circle which omits 1.

Example. If $0 < |1 - a| < 1$, then $a^{z/(1-z)}$ belongs to $H^2(\mu)$ (see [9, p. 218]) and hence so does $f(a, z) = a^{z/(1-z)} - \bar{a}^{\bar{z}/(1-\bar{z})}$. Let $a = re^{i\theta}$. If z is real, then so is $w = z/(1 - z)$, and $f(a, z) = 2ir^w \sin(\theta w)$ vanishes when $w = n\pi/\theta$, i.e., at $z = w/(1 + w) = 1 - (n\pi\theta^{-1} + 1)^{-1}$. For any real number $\gamma > 2$, $a = re^{i\theta}$ can be chosen so that $\gamma = \pi/\theta$. Then $f(a, z)$ is a function in $H^2(\mu)$ which vanishes at $z_n = 1 - (n\gamma + 1)^{-1}$ and $\sum [1 - |z_n|] = \infty$. Thus, the functions in $H^2(\mu)$ which vanish at the non-Blaschke sequence $\{1 - (n\gamma + 1)^{-1}\}_{n=0}^\infty$ form a nonzero invariant subspace of M_z whenever $\gamma > 2$. Conversely, we will see that for $\gamma \leq 2$, the corresponding space is $\{0\}$. Roughly speaking, the main result of this section is that a sequence $\{z_n\}$ which converges radially to 1 determines a nonzero invariant subspace if and only if it converges "faster than $\{1 - (2n + 1)^{-1}\}$ ". The precise statement and proof depend on a result by Fuchs on the zeros of functions analytic and of exponential type in a half-plane.

Lemma 5. (Fuchs, [1, p. 157]). *Let $\{\lambda_n\}$ be a sequence of positive numbers with $(\lambda_{n+1} - \lambda_n) > \delta > 0$, $n = 1, 2, \dots$, and such that*

$$(3) \quad \sum_{\lambda_n < r} \lambda_n^{-1} - a \log r$$

is unbounded above as $r \rightarrow \infty$ for some $a \geq 0$. If f is an analytic function of exponential type πa in the half-plane $\operatorname{Re} z \geq 0$ which vanishes at every λ_n , then f vanishes identically. Conversely, if $\sum_{\lambda_n < r} \lambda_n^{-1} - a \log r$ is bounded above as $r \rightarrow \infty$, then there is an analytic function f of exponential type πa in the half-plane $\operatorname{Re} z \geq 0$ which vanishes at each λ_n without vanishing identically.

Theorem 7. Let $\{z_n\}$ be a sequence of numbers with $0 < z_n < 1$, $(z_{n+1} - z_n) \geq \delta(1 - z_n)(1 - z_{n+1})$ for some $\delta > 0$, and consider the condition

$$(4) \quad \sum_{z_n < r} (1 - z_n) - a \log \frac{1}{1 - r}$$

is unbounded above as $r \rightarrow 1$. If (4) holds with $a \geq 1/2$ and f is in $H^2(\mu)$ and vanishes at each z_n , then f vanishes identically. Conversely, if (4) fails for some $a < 1/2$, then there is a nonzero f in $H^2(\mu)$ which vanishes at each z_n .

Proof. If f is in $H^2(\mu)$ and vanishes at each z_n , then $g(z) = f(z/(1+z))$ is analytic and of exponential type at most $\pi/2$ in the half-plane $\operatorname{Re} z \geq -1/2 + \epsilon$, for any $\epsilon > 0$ [9, pp. 217, 224] and $g(\lambda_n) = 0$, $n = 1, 2, \dots$, where $\lambda_n = z_n/(1 - z_n)$. If $\{z_n\}$ satisfies (4) with $a \geq 1/2$, $\{\lambda_n\}$ satisfies (3) with $a \geq 1/2$ and hence g and f vanish identically by Lemma 5.

To prove the converse, suppose (4) does not hold for some $a < 1/2$ and some sequence $\{z_n\}$ as above. Then (3) does not hold with this value of a for either of the sequences $\{\lambda_n\}$ or $\{w_n\}$, where $\lambda_n = z_n/(1 - z_n)$ and $w_n = \lambda_n + 3/2$. We note that $\lambda_n > -1/2$ and $w_n > 1$. By Lemma 5, there is a function $h(z)$, analytic and of exponential type $\pi a < \pi/2$ for $\operatorname{Re} z \geq 0$ which vanishes at each w_n without vanishing identically. Then $g(z) = h(z + 3/2)$ is analytic for $\operatorname{Re} z \geq -3/2$ and vanishes at each λ_n without vanishing identically. Choose b with $\pi a < b < \pi/2$. There is a constant $D > 0$ such that

$$(5) \quad |g(z)| \leq D e^{b|z|}, \quad \operatorname{Re} z > -3/2.$$

Choose d with $b < d < \pi/2$ and note that there is a θ_0 , $0 < \theta_0 < \pi/2$, such that $(\theta - d \sin \theta) \sin \theta + \log(2 \cos \theta) \cos \theta \geq 0$ whenever $\theta_0 \leq |\theta| \leq \pi/2$. If we set $\ell(\theta) = \theta \sin \theta + \log(2 \cos \theta) \cos \theta$, then $d(1 - \cos \theta) \leq d \sin^2 \theta \leq \ell(\theta)$ when $\theta_0 \leq |\theta| \leq \pi/2$. In addition, $\cos \theta$ is bounded away from zero and $\ell(\theta)$ is bounded when $0 \leq |\theta| \leq \theta_0$, so if α is large enough,

$$(6) \quad d(1 - \cos \theta) - \alpha \cos \theta \leq \ell(\theta), \quad -\pi/2 \leq \theta \leq \pi/2.$$

Thus if $c = \alpha + d$, we have by (5) and (6)

$$\begin{aligned} |e^{-c(-3/2 + re^{i\theta})} g(-3/2 + re^{i\theta})| &\leq E e^{(b-d)r} e^{r[d(1 - \cos \theta) + (d-c) \cos \theta]} \\ &\leq EN(1+r)^{-2} e^{r\ell(\theta)}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \end{aligned}$$

where $E = D \exp((3/2)(b+c))$ and $N = \sup_{r>0} (1+r)^2 e^{(b-d)r} < \infty$. It follows from [14, p. 131] that $e^{-cz}g(z)$ has a Newton series expansion

$$e^{-cz}g(z) = \sum_0^{\infty} a_n \psi_n(z)$$

convergent in $\operatorname{Re} z > -3/2$. Since the abscissa of convergence is less than -1 , the abscissa of absolute convergence is negative. It follows that $\sum |a_n| < \infty$ [14, p. 115] and hence $\sum |a_n|^2 < \infty$. Therefore, the Newton series also converges to $\exp(-cz)g(z)$ in the metric of the Hilbert space $\mathcal{K}$ which has the Newton polynomials for an orthonormal basis. Since $\mathcal{K}$ consists of all functions u of the form $u(z) = f(z/(1+z))$ for some f in $H^2(\mu)$ [9, p. 217], it follows that $f(z) = \exp(-cz/(1-z))u(z/(1-z))$ belongs to $H^2(\mu)$, vanishes at each z_n , and is not identically zero. This completes the proof.

We note that if $0 < \epsilon < 2$, then condition (4) holds for the sequence

$$z_n = 1 - (1 + (2 - \epsilon)n)^{-1} \quad \text{with} \quad a \geq 1/2,$$

but not for the sequence

$$z_n = 1 - (1 + (2 + \epsilon)n)^{-1} \quad \text{with} \quad a < 1/2.$$

For any countable subset A of the open unit disk let $\mathfrak{M}_A$ be the closed subspace consisting of all functions in $H^2(\mu)$ which are identically zero on A . Clearly $\mathfrak{M}_A$ is invariant for M_z .

Corollary 6. *There exist countable subsets A and B of $|z| < 1$ such that $\mathfrak{M}_A$ and $\mathfrak{M}_B$ are non-zero but $\mathfrak{M}_A \cap \mathfrak{M}_B = \mathfrak{M}_{A \cup B} = \{0\}$.*

Proof. Let $(\pi/4) < \theta < (\pi/2)$, $z_n = 1 - (n\pi\theta^{-1} + 1)^{-1}$, $w_n = 1 - ((n - 1/2)\pi\theta^{-1} + 1)^{-1}$, $n = 1, 2, \dots$ and define $A = \{z_n\}$, $B = \{w_n\}$. By the example at the beginning of this section, since $\theta < \pi/2$, the functions in $H^2(\mu)$ which vanish at each z_n form a nonzero invariant subspace $\mathfrak{M}_A$ for M_z . For the same reason, the functions in $H^2(\mu)$ which vanish at each w_n form a nonzero invariant subspace $\mathfrak{M}_B$ for M_z . However, by applying Theorem 7 to the union of the sequences $\{z_n\}$ and $\{w_n\}$ and using the fact that $\pi/4 < \theta$, one easily sees that $\mathfrak{M}_A \cap \mathfrak{M}_B = \mathfrak{M}_{A \cup B} = \{0\}$.

Corollary 7. *Let $\{\lambda_n\}$ be a sequence of positive numbers such that $(\lambda_{n+1} - \lambda_n) \geq \delta > 0$, $n = 1, 2, \dots$, and consider the condition*

$$(5) \quad \sum_{\lambda_n < r} \lambda_n^{-1} - a \log r$$

is unbounded above as $r \rightarrow \infty$. If (5) holds for some $a \geq 1/2$, then the functions $\{(1-z)^{\lambda_n}\}$ span a dense subset of H^2 . Conversely, if (5) fails for some $a < 1/2$ then the functions $\{(1-z)^{\lambda_n}\}$ do not span a dense subset of H^2 .

Proof. Under the unitary equivalence between $I - C_0$ in H^2 and $F(z) \rightarrow zF(z)$ in $\mathcal{H}$, the function $(1-z)^{\lambda_n}$ in H^2 corresponds to the kernel function in $\mathcal{H}$ at

$z_n = \lambda_n/(1 + \lambda_n)$ (see Section 1). Therefore the functions $\{(1 - z)^{\lambda_n}\}$ are linearly dense in H^2 if and only if the kernel functions at z_n are linearly dense in $\mathcal{H}$. This last condition holds if and only if there is no F in $\mathcal{H}$ (or $H^2(\mu)$) which vanishes at each z_n without vanishing identically. The result now follows from Theorem 7.

We now turn to the behavior of $H^2(\mu)$ functions away from the point $z = 1$. For $0 < r < 1$, let Ω_r denote the region $D \cap \{z: \operatorname{Re} z < r\}$. An examination of the kernel function for $H^2(\mu)$ may convince the reader that there is a constant $C > 0$ (depending on r) so that $|f(z)| \leq C(1 - |z|^2)^{-1/2} \|f\|$ when f is in $H^2(\mu)$ and $z \in \bar{\Omega}_r$ (an explicit estimate can be found in [15]). Let f belong to $H^2(\mu)$ and choose a sequence $\{p_n\}$ of polynomials tending to f in norm. The boundary $\partial\Omega_r$ of Ω_r is the union of an arc Γ and a line segment L . We can conclude from the above inequality that

$$\int_L |f(z) - p_n(z)| |dz| \leq \|f - p_n\| C \int_L (1 - |z|^2)^{-1/2} |dz|.$$

The integral on the right hand side is finite, so that

$$\int_L |f(z) - p_n(z)| |dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, the restriction of μ to $\bar{\Omega}_r \cap C$ dominates $\delta |dz|$, where δ is some positive number and $|dz|$ is arclength on $\bar{\Omega}_r \cap C$, so $\int_\Gamma |f(z) - p_n(z)|^2 |dz| \rightarrow 0$ as $n \rightarrow \infty$. Thus we have $\int_{\partial\Omega_r} |f(z) - p_n(z)| |dz| \rightarrow 0$ as $n \rightarrow \infty$. It follows easily that the restriction of f to Ω_r lies in the Hardy-type space $E^1(\Omega_r)$ (see [4, Theorem 10.4]). This suggests that f behaves locally like an H^2 function away from $z = 1$. We make this precise in:

Theorem 8. *i) Let f be a nonzero element of $H^2(\mu)$ which vanishes on the sequence $\{z_n\}$ in D . If 1 is not an accumulation point of $\{z_n\}$, then $\sum (1 - |z_n|) < \infty$.*

ii) Let f be a non-zero element of $H^2(\mu)$ and suppose $0 < a < b < 2\pi$. Then $\int_a^b \log |f(e^{i\theta})| d\theta > -\infty$.

iii) If f is in $H^2(\mu)$, then $f(z)$ has non-tangential boundary values $\tilde{f}(e^{i\theta})$ at almost every $e^{i\theta}$ in C (with respect to Lebesgue measure $d\theta$). Moreover $\tilde{f}(e^{i\theta}) = f(e^{i\theta})$ $d\theta$ -a.e.

Proof. Let $\{z_n\}$ be a sequence in D whose closure does not contain 1. Choose $r < 1$ so that $\{z: \operatorname{Re} z < r\}$ contains the closure of $\{z_n\}$. Let $\Gamma = C \cap \{z: \operatorname{Re} z < r\}$. Suppose that φ is a conformal mapping of D onto Ω_r (as above) and let $\Gamma' = \varphi^{-1}(\Gamma)$, where we consider φ as extended to a homeomorphism of $\bar{D}$ onto $\bar{\Omega}_r$. Let $w_n = \varphi^{-1}(z_n)$, $n = 1, 2, 3, \dots$. Suppose $f \neq 0$ is in $H^2(\mu)$ and $f(z_n) = 0$, $n = 1, 2, 3, \dots$. The restriction of f to Ω_r is in $E^1(\Omega_r)$, so $f(\varphi(z))\varphi'(z)$ is in H^1 of the disk [4, p. 169]. Moreover this last function vanishes on $\{w_n\}$, so $\sum (1 - |w_n|) < \infty$. Now, by the reflection principle φ and φ^{-1} are analytically

continuable across Γ' and Γ respectively. It follows that φ satisfies a Lipschitz condition on any closed subset of the disk $\bar{D}$ which does not intersect the arc complementary to Γ' . Let A denote the positive integers n such that $w_n/|w_n|$ is in Γ' . Then A contains all but finitely many positive integers and, for some $M > 0$,

$$\begin{aligned} \sum (1 - |z_n|) &\leq \sum \left| \varphi\left(\frac{w_n}{|w_n|}\right) - \varphi(w_n) \right| \\ &\leq M \sum \left| \frac{w_n}{|w_n|} - w_n \right| \\ &= M \sum (1 - |w_n|) < \infty, \end{aligned}$$

where the sums are taken over A .

The proof of ii) uses the fact that $|\varphi'(z)|$ is bounded and bounded away from zero on closed subarcs J of Γ' and the fact that

$$\int_J \log |f(\varphi(e^{ix}))\varphi'(e^{ix})| dx > -\infty$$

(since $f(\varphi(z))\varphi'(z)$ is in H^1).

Part iii) follows from the well-known results about non-tangential boundary values of H^1 functions and the fact that φ and φ^{-1} are conformal on Γ' and Γ respectively. We omit the details.

In contrast to the result in ii) above, suppose that $0 < |1 - a| < 1$. Then $f(z) = a^{z/(1-z)}$ belongs to $H^2(\mu)$ and for each $\epsilon > 0$,

$$\begin{aligned} \int_0^\epsilon \log |f(e^{i\varphi})| d\varphi &= -\infty \\ \int_{2\pi-\epsilon}^{2\pi} \log |f(e^{i\varphi})| d\varphi &= \infty. \end{aligned}$$

Thus the result in ii) fails if the arc contains the point 1.

Corollary 8. *If $f \neq 0$ is in $H^2(\mu)$, there exists $\delta > 0$ such that $\|M_z^n f\| \geq \delta$, $n = 1, 2, 3, \dots$.*

Proof. Let f be non-zero in $H^2(\mu)$. Then $\|M_z^n f\|^2 = \int_D |z^n f(z)|^2 d\mu(z) \geq \int_C |z^n f(z)|^2 d\mu(z) = \int_C |f|^2 d\mu$. Theorem 8 ii) implies that $|f| > 0$ μ -a.e. on C , so the last integral is positive.

By combining Corollary 8 with the result by Shields and Wallen that $(I - C_0^*)^n$ converges strongly to zero [15], we see that $I - C_0$ belongs to the class C_{10} of contractions introduced by Nagy and Foias [13].

Concluding remarks. The problem of finding all invariant subspaces of C_0 contains the problem of characterizing the zero sets of functions in $H^2(\mu)$. The proof of Theorem 7 indicates that this, in turn, is probably as difficult as characterizing the zero sets of analytic functions of exponential type at most $\pi/2$ in the right half-plane. We list two specific questions whose solution may shed light on the general problem.

Question 1. Let $\mathfrak{N}$ be an invariant subspace for M_z and let N denote the spectrum of $M_z^* \mid \mathfrak{N}^\perp$. Can any conditions be put on N which will insure that $\mathfrak{N} = \overline{BH^2(\mu)}$ for some inner function B ? One possibility is that $1 \notin N$. By Corollary 5, this condition is not necessary for $\mathfrak{N}$ to be $\overline{BH^2(\mu)}$. On the other hand, the only subspaces we know which are not of this form come from non-Blaschke sequences approaching 1, and in this case 1 is a limit point of eigenvalues for $M_z^* \mid \mathfrak{N}^\perp$.

Question 2. Does the map $B \rightarrow \overline{BH^2(\mu)}$ preserve both lattice operations in the restricted sense that

$$\overline{B_1 H^2(\mu)} \cap \overline{B_2 H^2(\mu)} = \overline{CH^2(\mu)}$$

where C is the least common multiple of the inner functions B_1 and B_2 ?

Added in proof. In a recent preprint, Charles Horowitz shows that the phenomenon described in Corollary 6 also occurs in the Bergman space $H^2(m)$, where m is area Lebesgue measure on the unit disk.

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