Singular Non-homogeneous Abstract Cauchy and Dirichlet Type Problems Related by a Generalized Stieltjes Transform

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1. Introduction. In a recent paper, J. W. Dettman [3] studied the relationship between the solutions of the following pair of initial-boundary value problems:

$$(1.1) \quad \begin{cases} (a) & v_{it}(x, t) = P(x, D)v(x, t), v(x, 0+) = \varphi(x), v_t(x, 0+) = 0, \\ (b) & w_{it}(x, t) + P(x, D)w(x, t) = 0, w(x, 0+) = \varphi(x) \end{cases}$$

where $P(x, D)$ is a strongly elliptic operator independent of t . He showed that if $v(x, t)$ satisfies (1.1a) along with some mild growth bounds, then the solution of (1.1b) could be obtained from this $v(x, t)$ by means of a Stieltjes transform. He exploited this relationship to discuss the existence of normal derivatives of $w(x, t)$ at $t = 0+$ when $P(x, D) = \Delta_n$, $n = 1, 3$. In this paper, we consider a pair of problems whose solution functions (and nonhomogeneous terms) are connected by a generalized Stieltjes transform.

To describe these problems, let X be a Banach space and let B be the infinitesimal generator of an equibounded C_0 group [2]. Let $A = B^2$ and assume that the domain of A , $\mathfrak{D}(A)$, is a dense subspace of X . Let $\mathfrak{F}(Y)$ denote a class of strongly continuous functions $\{f\}$ each member of which has domain $[0, \infty)$ and range Y ($Y \subseteq X$) and let $\|f(t)\|$ be the corresponding norm function associated with f at the point t . Then the pair of problems we consider are:

$$(1.2) \quad \begin{aligned} (a) \quad & u_{it}(t) - t^\alpha Au(t) = f(t), \quad t > 0, \quad \alpha \geq 0, \quad f \in \mathfrak{F}(\mathfrak{D}(A)), \\ & \|u(t)\| \xrightarrow[t \rightarrow 0+]{} 0, \quad \|u_t(t)\| \xrightarrow[t \rightarrow 0+]{} 0, \\ (b) \quad & v_{it}(t) + t^\beta Av(t) = g(t), \quad t > 0, \quad \beta \geq 0, \quad g \in \mathfrak{F}(\mathfrak{D}(A)), \\ & \|v(t)\| \xrightarrow[t \rightarrow 0+]{} 0. \end{aligned}$$

We shall be concerned with (i) establishing a relation between the solution pairs $u(t)$ and $v(t)$ while simultaneously obtaining a compatibility condition involving f and g in order that this relation be valid and (ii) using the semi-group solution of (1.2a) to construct a solution of (1.2b). As we shall see, the construction of $v(t)$ requires that $\int_0^\infty \|f\| < \infty$. Even though (1.2b) may involve an appropriate $g \in \mathfrak{F}(\mathfrak{D}(A))$, the relation connecting f to g can lead to an f which fails to be in $\mathfrak{F}(\mathfrak{D}(A))$. Under certain conditions, we can introduce an approximation condition involving the class of functions

$$(1.3) \quad \overline{\mathfrak{F}}(\mathfrak{D}(A)) = \left\{ f \in \mathfrak{F}(\mathfrak{D}(A)) : \int_0^\infty \|f\| < \infty \right\}.$$

We then define a "generalized" solution of (1.2b), for restricted t , as the limit of an approximating sequence of $v_n(t)$ corresponding to $f_n(t)$.

In section 2, we develop a transformation between the functions $u(t)$ and $v(t)$ by using the methods of related non-homogeneous equations [1]. This connection imposes the compatibility condition between $f(t)$ and $g(t)$ taking the form of a generalized Stieltjes transform. The "hyperbolic" problem (1.2a) will be treated in section 3 and conditions will be imposed on f to ensure that $v(t)$ can be obtained from this $u(t)$ in section 4. To do this, we must choose $\alpha > \beta$. Finally, we examine conditions on $g(t)$ in (1.2b) to determine when that problem is solvable by the relation connecting $u(t)$ to $v(t)$. The corresponding f , obtained by the inverse Stieltjes transform, must be in $\overline{\mathfrak{F}}(\mathfrak{D}(A))$. If $\int_0^1 \|f\| < \infty$ but $f \notin \mathfrak{F}(\mathfrak{D}(A))$, we introduce the approximation assumption referred to earlier.

One can easily incorporate initial-boundary conditions into (1.2) analogous to those in (1.1). However, this leads to considerations similar to those already encountered in [3].

2. The relation between solution pairs. We now derive the relation connecting $u(t)$ and $v(t)$ by the methods of [1]. Under the change of variables $t = ((\alpha + 2)\tau/2)^{2/(\alpha+2)}$, the equation in (1.2a) becomes

$$(2.1) \quad U_{\tau\tau}(\tau) + \frac{\alpha}{(\alpha + 2)\tau} U_\tau(\tau) - AU(\tau) = f^*(\tau)$$

in which

$$(2.2) \quad \begin{cases} \text{(a)} & U(\tau) = u\left(\left(\frac{\alpha + 2}{2}\tau\right)^{2/(\alpha+2)}\right) \\ \text{(b)} & f^*(\tau) = (2/[(\alpha + 2)\tau]^{2\alpha/(\alpha+2)})f\left(\left(\frac{\alpha + 2}{2}\tau\right)^{2/(\alpha+2)}\right). \end{cases}$$

Similarly, with the change of variables $t = ((\beta + 2)\tau/2)^{2/(\beta+2)}$, the equation in (1.1b) transforms into

$$(2.3) \quad V_{\tau\tau}(\tau) + \frac{\beta}{(\beta + 2)\tau} V_\tau(\tau) + AV(\tau) = g^*(\tau)$$

where

$$(2.4) \quad \begin{cases} (a) & V(\tau) = v\left(\left(\frac{\beta+2}{2}\tau\right)^{2/(\beta+2)}\right) \\ (b) & g^*(\tau) = (2/[(\beta+2)\tau])^{2\beta/(\beta+2)} g\left(\left(\frac{\beta+2}{2}\tau\right)^{2/(\beta+2)}\right). \end{cases}$$

We take the equation

$$(2.5) \quad W_\tau(\tau) - AW(\tau) = h(\tau)$$

as one intermediate to (2.1) and (2.3).

With the change of variables $\xi = (1/4)\tau^2$ in (2.1) and $\tau = \xi$ in (2.5), those equations become, respectively,

$$(2.6) \quad \begin{cases} (a) & \left\{ \xi D_\xi \left(\xi D_\xi + \frac{a+1}{2} - 1 \right) - \xi A \right\} U(2\xi^{1/2}) = \xi g^*(2\xi^{1/2}) \\ (b) & \{ \xi D_\xi - \xi A \} W(\xi) = \xi h(\xi) \end{cases}$$

with $a = \alpha/(\alpha+2)$. By theorem (3.2) of [1], we find that

$$(2.7) \quad U(2\xi^{1/2}) = \Gamma\left(\frac{a+1}{2}\right) \xi^{1-(a+1)/2} \mathfrak{L}_s^{-1} \{ s^{-(a+1)/2} W(1/s) \}_{s \rightarrow \xi}$$

provided that

$$(2.8) \quad \xi f^*(2\xi^{1/2}) = \xi^{2-(1+a)/2} \Gamma\left(\frac{a+1}{2}\right) \mathfrak{L}_s^{-1} \{ s^{-(a+1)/2-1} s^{-1} h(1/s) \}_{s \rightarrow \xi}.$$

In these, $\mathfrak{L}_s^{-1} \{ \}_{s \rightarrow \xi}$ denotes the inverse Laplace transform with ξ the variable of the inverted function.

In a similar way, the changes of variables $\tau = \xi^{-1/2}$ in (2.3) and $\tau = 1/(4\xi)$ in (2.5) yield

$$(2.9) \quad \begin{cases} (a) & \left\{ \frac{1}{4}A + \xi(\xi D_\xi) \left(\xi D_\xi + \frac{1-b}{2} \right) \right\} V(\xi^{-1/2}) = \frac{1}{4}g^*(\xi^{-1/2}) \\ (b) & \{ \frac{1}{4}A + \xi(\xi D_\xi) \} W(1/4\xi) = -\frac{1}{4}h(1/4\xi) \end{cases}$$

with $b = \beta/(\beta+2)$. By theorem (3.1) or [1],

$$(2.10) \quad V(\xi^{-1/2}) = \frac{1}{\Gamma[(1-b)/2]} \int_0^\infty e^{-\sigma} \sigma^{-(b+1)/2} W(1/(4\sigma\xi)) d\sigma$$

provided that the non-homogeneous terms satisfy

$$(2.11) \quad g^*(\xi^{-1/2}) = \frac{-1}{\Gamma[(1-b)/2]} \int_0^\infty e^{-\sigma} \sigma^{-(b+1)/2} h(1/(4\sigma\xi)) d\sigma.$$

We obtain the desired relation between $u(t)$ and $v(t)$ (as well as the one between $f(t)$ and $g(t)$) by eliminating $W(t)$ between (2.7) and (2.10) (or by eliminating

$h(t)$ between (2.8) and (2.11)). In fact, if we solve (2.7) for $W(1/s)$, replace s by $4\sigma\xi$, and insert the expression for $W(1/(4\sigma\xi))$ in (2.10), we obtain, by Fubini's theorem

$$(2.12) \quad V(\xi^{-1/2}) = \frac{(4\xi)^{(a+1)/2} \Gamma[(a-b)/2 + 1]}{\Gamma[(a+1)/2] \Gamma[(1-b)/2]} \int_0^\infty \frac{\lambda^{(a-1)/2} U(2\lambda^{1/2})}{(1+4\xi\lambda)^{(a-b)/2+1}} d\lambda$$

provided the growth of $||U(2\lambda^{1/2})||$ is suitably restricted. The relationship we seek follows from this by returning to the original variables. If we do this, we finally obtain

$$(2.13) \quad V(t) = C(\alpha, \beta) t \int_0^\infty \frac{\sigma^\alpha u(\sigma) d\sigma}{\left[\left(\frac{2}{\beta+2} \right)^2 t^{\beta+2} + \left(\frac{2}{\alpha+2} \right)^2 \sigma^{\alpha+2} \right]} \frac{\alpha - \beta}{(\alpha+2)(\beta+2)} + 1$$

where $C(\alpha, \beta)$ is a constant defined by

$$(2.14) \quad C(\alpha, \beta) = \frac{2^{2[(\alpha+1)/(\alpha+2)+1/(\beta+2)]} \Gamma[(\alpha-\beta)/(\alpha+2)(\beta+2)+1]}{\Gamma[(\alpha+1)/(\alpha+2)] \Gamma[1/(\beta+2)] (\alpha+2)^{\alpha/(\alpha+2)} (\beta+2)^{2/(\beta+2)}}.$$

A similar elimination of $h(t)$ between (2.8) and (2.11) gives the compatibility condition

$$(2.15) \quad g(t) = -C(\alpha, \beta) t^2 \cdot \int_0^\infty \frac{f(\xi) d\xi}{\left[\left(\frac{2}{\beta+2} \right)^2 t^{\beta+2} + \left(\frac{2}{\alpha+2} \right)^2 \xi^{\alpha+2} \right]} \frac{\alpha - \beta}{(\alpha+2)(\beta+2)} + 1.$$

We observe that if $\alpha = \beta = 0$, then (2.13) corresponds to the connecting relation obtained in [3].

3. The solution of (1.2a). For the purpose of constructing the solution of (1.2a) using semi-group theory, we consider the associated ordinary differential equation

$$(3.1) \quad y''(t) - a^2 t^\alpha y(t) = 0, \quad a \text{ real.}$$

One can easily derive the solution pair

$$(3.2) \quad \begin{aligned} y_1(t, \alpha) &= \Gamma\left(\frac{\alpha+1}{\alpha+2}\right) \sum_{n=0}^{\infty} \frac{a^{2n} t^{n(\alpha+2)}}{(\alpha+2)^{2n} n! \Gamma\left(n + \frac{\alpha+1}{\alpha+2}\right)} \\ y_2(t, \alpha) &= \Gamma\left(\frac{\alpha+3}{\alpha+2}\right) \sum_{n=0}^{\infty} \frac{a^{2n} t^{n(\alpha+2)+1}}{(\alpha+2)^{2n} n! \Gamma\left(n + \frac{\alpha+3}{\alpha+2}\right)} \end{aligned}$$

with $y_1(0, \alpha) = 1$, $y_1^{(1)}(0, \alpha) = 0$ and $y_2(0, \alpha) = 0$, $y_2^{(1)}(0, \alpha) = 1$. Using elementary properties of the Beta function, one can verify the following relations:

$$\begin{aligned}
y_1(t, \alpha) &= \frac{1}{B(1/2, \alpha/(2\alpha + 4))} \\
&\cdot \int_0^1 (1 - \sigma)^{-(\alpha+4)/(2\alpha+4)} \sigma^{-1/2} y_1\left(\frac{2t^{(\alpha+2)/2} \sigma^{1/2}}{\alpha + 2}, 0\right) d\sigma, \quad \alpha > 0, \\
(3.3) \quad y_2(t, \alpha) &= \frac{t}{B(2/(\alpha + 2), (\alpha + 1)/(\alpha + 2))} \\
&\cdot \int_0^1 (1 - \sigma)^{-\alpha/(\alpha+2)} \sigma^{-1/(\alpha+2)} y_1(t\sigma^{1/(\alpha+2)}, \alpha) d\sigma, \quad \alpha \geq 0.
\end{aligned}$$

The first of these may be regarded as a shifting formula on the parameter α (from 0 to α) and the second, when coupled with the first, can be similarly regarded. Note that

$$y_1(t, 0) = \frac{1}{2}(e^{at} + e^{-at})$$

and

$$y_2(t, 0) = \frac{1}{2} \int_0^1 \sigma^{-1/2} y_1(t\sigma^{1/2}) d\sigma = \frac{1}{2} \int_0^t (e^{a\xi} + e^{-a\xi}) d\xi.$$

These forms for $y_1(t, 0)$ and $y_2(t, 0)$ are suggested by the form of the semi-group solution of (1.2a) when $f = 0$ and $\alpha = 0$ ([5], [6]).

Next, we observe that if $\varphi \in \mathfrak{D}(A)$, then the function

$$(3.4) \quad W_1(t) = \frac{1}{2}[T_B(t) + T_B(-t)]\varphi = 0_1(t)\varphi$$

satisfies

$$(3.5) \quad W''(t) - AW(t) = 0, \quad t > 0; \quad W(0+) = \varphi, \quad W^{(1)}(0+) = 0$$

where $T_B(t)$ denotes the group generated by B [5]. The $0_1(t)$ in (3.4) will be used throughout as a short hand notation for the operator acting on φ in (3.4). Similarly, the function

$$(3.6) \quad W_2(t) = \int_0^t 0_1(\xi)\varphi d\xi = \int_0^t W_1(\xi) d\xi$$

satisfies (3.5) with initial conditions there interchanged.

In terms of $0_1(t)$ and by the relations (3.3), we define operators $U_1(t)$ and $U_2(t)$ by

$$\begin{aligned}
U_1(t)\varphi &= \frac{1}{B[\frac{1}{2}, \alpha/(2\alpha + 4)]} \\
&\cdot \int_0^1 (1 - \sigma)^{-(\alpha+4)/(2\alpha+4)} \sigma^{-1/2} 0_1\left(\frac{2t^{(\alpha+2)/2} \sigma^{1/2}}{\alpha + 2}\right)\varphi d\sigma, \\
(3.7) \quad U_2(t)\varphi &= \frac{t}{B[2/(\alpha + 2), (\alpha + 1)/(\alpha + 2)]} \\
&\cdot \int_0^1 (1 - \sigma)^{-\alpha/(\alpha+2)} \sigma^{-1/(\alpha+2)} U_1(t\sigma^{1/(\alpha+2)})\varphi d\sigma.
\end{aligned}$$

Then $U_1(t)\varphi$ is a solution of (1.2a) with $f = 0$ and satisfies the conditions $[U_1(t)\varphi]_{t=0+} = \varphi$, $[D_t\{U_1(t)\varphi\}]_{t=0+} = 0$ while $U_2(t)\varphi$ satisfies (1.2a) with $f = 0$ and satisfies the complementary set of initial conditions. Thus, $U_1(t)$ and $U_2(t)$ are linear solution operators having domain $\mathfrak{D}(A)$.

Lemma 3.1. *Let B be the generator of an equibounded C_0 group in X and let $A = B^2$. If $\varphi \in \mathfrak{D}(A)$, then*

$$\|U_1(t)\varphi\| \leq M\|\varphi\| \quad \text{and} \quad \|U_2(t)\varphi\| \leq Mt\|\varphi\|$$

where

$$M = \|0_1(t)\|.$$

Proof. These follow trivially from (3.7) by taking the norms under the signs of integration and using the definition of the norm applied to $0_1(t)\varphi$.

Lemma 3.2. *Let $\varphi \in \mathfrak{D}(A)$ and let*

$$h(t) = [U_1(t)U_2^{(1)}(t) - U_1^{(1)}(t)U_2(t)]\varphi.$$

Then $h(t) = \varphi$.

Proof. Clearly, $h^{(1)}(t) = t^\alpha A[U_1(t)U_2(t) - U_2(t)U_1(t)]\varphi = 0$ and $h(t)$ is independent of t . Further $\lim_{t \rightarrow 0+} h(t) = \phi$ from the properties of $U_1(t)$ and $U_2(t)$. Hence, $h(t) = \varphi$.

Theorem 3.1. *Let $f(t) \in \mathfrak{F}(\mathfrak{D}(A))$. Then the function*

$$(3.8) \quad u(t) = \int_{0+}^t [U_1(\xi)U_2(t)f(\xi) - U_2(\xi)U_1(t)f(\xi)] d\xi$$

satisfies (1.2a).

Proof. Now

$$(3.9) \quad u^{(1)}(t) = \int_{0+}^t [U_1(\xi)U_2^{(1)}(t)f(\xi) - U_2(\xi)U_1^{(1)}(t)f(\xi)] d\xi$$

and

$$(3.10) \quad \begin{aligned} u^{(i)}(t) &= [U_1(t)U_2^{(i)}(t) - U_1^{(i)}(t)U_2(t)]f(t) \\ &+ \int_{0+}^t [U_1(\xi)U_2^{(i)}(t)f(\xi) - U_2(\xi)U_1^{(i)}(t)f(\xi)] d\xi. \end{aligned}$$

By lemma 3.2 and the fact that $U_i^{(i)}(t)f(\xi) = t^\alpha A U_i(t)f(\xi)$, $i = 1, 2$, we see the

$$u^{(i)}(t) = f(t) + \int_{0+}^t t^\alpha [U_1(\xi)U_2(t)Af(\xi) - U_2(\xi)U_1(t)Af(\xi)] d\xi.$$

Using the strong continuity, we can remove $t^\alpha A$ from under the sign of integration to get

$$u^{(i)}(t) = f(t) + t^\alpha A u(t).$$

Thus, the equation in (1.2a) is satisfied. With the change of variables $\xi = t\sigma$ in (3.9) and (3.10), it readily follows that

$$\lim_{t \rightarrow 0+} ||u(t)|| = 0 \quad \text{and} \quad \lim_{t \rightarrow 0+} ||u_t(t)|| = 0.$$

It is useful to illustrate (3.8) for a familiar example. Take $A = D_x^2$, $\alpha = 0$, $f(t) = F(x, t)$, and replace the operators $U_1(t)$ and $U_2(t)$ in (3.8) by $0_1(t)$ and $0_2(t) = \int_0^t 0_1(\xi) d\xi$. The group operator $T_B(t)$ in this case is just a translation operator.

Now

$$0_2(T)F(x, \xi) = \frac{1}{2} \int_0^T [F(x + \sigma, \xi) + F(x - \sigma, \xi)] d\sigma = \frac{1}{2} \int_{x-T}^{x+T} F(\eta, \xi) d\eta$$

and

$$0_1(S)0_2(T)F(x, \xi) = \frac{1}{4} \left\{ \int_{x+S-T}^{x+S+T} F(\eta, \xi) d\eta + \int_{x-S-T}^{x+T-S} F(\eta, \xi) d\eta \right\}.$$

From this and the properties of integrals, we obtain

$$[0_1(\xi)0_2(t) - 0_1(t)0_2(\xi)]F(x, \xi) = \frac{1}{2} \int_{x+\xi-t}^{x+t-\xi} F(\eta, \xi) d\eta.$$

Finally,

$$u(t) = \frac{1}{2} \int_0^t \left\{ \int_{x+\xi-t}^{x+t-\xi} F(\eta, \xi) d\eta \right\} d\xi$$

and this is the well-known formula for the solution of the nonhomogeneous wave equation with trivial initial conditions.

4. The function $v(t)$. We now select $\alpha > \beta$ and obtain conditions on $f(t)$ so that the solution $u(t)$ constructed in section 3 can be introduced into (2.13) to obtain $v(t)$.

From (2.13), we find

$$(4.1) \quad ||v(t)|| \leq C(\alpha, \beta)t$$

$$\int_0^\infty \frac{\sigma^\alpha ||u(\sigma)|| d\sigma}{\left[\left(\frac{2}{\beta+2} \right)^2 t^{\beta+2} + \left(\frac{2}{\alpha+2} \right)^2 \sigma^{\alpha+2} \right]} \frac{\alpha - \beta}{(\alpha+2)(\beta+2)} + 1.$$

Then the condition $\alpha > \beta$ shows that this integral converges if $||u(t)|| \leq Kt$ for some positive constant K .

Lemma 4.1. *Let B and A be as in lemma 3.1. Then the function $u(t)$ given by (3.8) satisfies the inequality*

$$(4.2) \quad ||u(t)|| \leq M^2 \left[t \int_0^t ||f(\xi)|| d\xi + \int_0^t \xi ||f(\xi)|| d\xi \right]$$

where $M = ||0_1(t)||$.

Proof. From (3.8) we have

$$\begin{aligned} ||u(t)|| &\leq \int_{0+}^t [||U_1(\xi)U_2(t)f(\xi)|| + ||U_2(\xi)U_1(t)f(\xi)||] d\xi \\ &\leq \int_{0+}^t [M ||U_2(t)f(\xi)|| + M\xi ||U_1(t)f(\xi)||] d\xi, \end{aligned}$$

this last following by an application of lemma 3.1. Repeating the use of lemma 3.1, we see that

$$\begin{aligned} ||u(t)|| &\leq \int_{0+}^t [M \cdot M t ||f(\xi)|| + M\xi \cdot M ||f(\xi)||] d\xi \\ &= M^2 \int_{0+}^t [t ||f(\xi)|| + \xi ||f(\xi)||] d\xi \end{aligned}$$

and this is the stated result.

Lemma 4.2. *Let B and A be as in lemma 3.1 and $u(t)$ as in (3.8). If $\int_0^\infty ||f(\xi)|| d\xi < \infty$, then $||u(t)|| \leq 2M^2 M^* t$ with $M^* = \int_0^\infty ||f(\xi)|| d\xi$.*

Proof. This follows immediately from lemma 4.1 by noting that

$$\int_0^t \xi ||f(\xi)|| d\xi \leq t \int_0^t ||f(\xi)|| d\xi \leq M^* t$$

We now have the main result of this section:

Theorem 4.1. *Let $f \in \overline{\mathfrak{F}}(\mathfrak{D}(A))$ with $\overline{\mathfrak{F}}(\mathfrak{D}(A))$ defined by (1.3). Then a solution of (1.2b) with $g(t)$ defined in terms of $f(t)$ by (2.15) is given by introducing the function $u(t)$ of (3.8) into (2.13).*

Proof. We will not systematically check to see if all conditions for a solution of (1.2b) are met since most of this has been incorporated into the method of related equations. We merely note that $||U(2\lambda^{1/2})||$ in (2.12) is bounded by $K^* \lambda^{1/(\alpha+2)}$ for some constant K^* and this suffices for applying Fubini's theorem to obtain (2.12). The reader is referred to [3] for a systematic checking of the correspondence relating solutions of (1.1a) and (1.1b).

We conclude this section with some results on the bounds on $||v(t)||$ and the dependence of $v(t)$ on $f(t)$. By inserting the bound on $u(t)$ of lemma 4.2 into (4.1), we find that the resulting integral can be evaluated directly by an anti-derivative. We have, in fact,

Lemma 4.3. *Assume that the conditions of theorem 4.1 hold. Then the corresponding $v(t)$ given by theorem 4.1 satisfies the inequality*

$$(4.3) \quad ||v(t)|| \leq \tilde{C}(\alpha, \beta) M^2 M^* t^{(\beta+2)/(\alpha+2)}$$

with

$$\tilde{C}(\alpha, \beta) = \frac{2(\alpha + 2)^{(\alpha+4)/(\alpha+2)}(\beta + 2)^{1-2/(\alpha+2)}\Gamma((\alpha - \beta)/(\alpha + 2)(\beta + 2) + 1)}{(\alpha - \beta)\Gamma[(\alpha + 1)/(\alpha + 2)]\Gamma[1/(\beta + 2)]}.$$

By a direct application of this lemma, we can easily prove

Theorem 4.2. *Let $f_1(t)$ and $f_2(t)$ satisfy the conditions of theorem 4.1 with*

$$\int_0^\infty ||f_1(\xi) - f_2(\xi)|| d\xi \leq \frac{\epsilon}{M^2 \tilde{C}(\alpha, \beta) T^{(\beta+2)/(\alpha+2)}}$$

for $\epsilon > 0$ and $T > 0$. Then the functions $v_1(t)$ and $v_2(t)$ corresponding to $f_1(t)$ and $f_2(t)$ satisfy

$$||v_1(t) - v_2(t)|| \leq \epsilon \quad \text{for } 0 \leq t \leq T.$$

5. The compatibility condition. Suppose we are given the problem (1.2b). If we wish to solve this by setting up the intermediate problem (1.2a) and then use (2.13), the first question encountered is the one of determining the function $f(t)$ in (1.2a). With $\mathfrak{F}(\mathfrak{D}(A))$ as defined in (1.3), our previous work shows that we can construct a function if $f \in \mathfrak{F}(\mathfrak{D}(A))$. We need to obtain conditions on $g(t)$ or some closely associated function to ensure that $f \in \mathfrak{F}(\mathfrak{D}(A))$.

With the changes of variables $y = (2/(\alpha + 2))\xi^{\alpha+2}$ and $x = (2/(\beta + 2))t^{\beta+2}$, the connecting relation (2.15) becomes

$$(5.1) \quad G(x) = \int_0^\infty \frac{F(y) dy}{(x + y)^{1+\nu}}$$

where

$$(5.2) \quad \begin{cases} (a) & G(x) = -\frac{1}{K_2} x^{-2/(\beta+2)} g\left(\left(\frac{\beta+2}{2}\right)^{2/(\beta+2)} x^{1/(\beta+2)}\right) \\ (b) & F(y) = y^{-(\alpha+1)/\alpha+2} f\left[\left(\frac{\alpha+2}{2}\right)^{2/(\alpha+2)} y^{1/(\alpha+2)}\right] \end{cases}$$

and

$$(5.3) \quad \begin{cases} \nu = (\alpha - \beta)/[(\alpha + 2)(\beta + 2)] \\ K_2 = \frac{1}{(\alpha + 2)} \left(\frac{\alpha + 2}{2}\right)^{2/(\alpha+2)} \left(\frac{\beta + 2}{2}\right)^{4/(\beta+2)} C(\alpha, \beta). \end{cases}$$

This relation is just a generalized Stieltjes transform between $F(x)$ and $G(x)$. We reduce this to the standard Stieltjes transform by an application of the Weyl transform (see [4], p. 213 and p. 181). Define

$$(5.4) \quad \mathfrak{G}(x) = \Gamma(\nu + 1) W_{\nu-1}(G, x) = \nu \int_x^\infty G(y) (y - x)^{\nu-1} dy.$$

The corresponding inversion formula is then

$$(5.5) \quad G(x) = -\frac{x^{1-\nu}}{\nu B(\nu, 1-\nu)} D_x \int_x^\infty \frac{x^\nu \mathfrak{G}(y)}{y(y-x)^\nu} dy$$

From the expression for $G(x)$ in (5.2a) and the value of ν , it is easy to check the exponents in the integrand of (5.4) to see that $\mathfrak{G}(x)$ exists if $\|g(x)\|$ is bounded. Then by applying the transform $\Gamma(\nu + 1)W_{\nu-1}$ to both members of (5.1), that relation reduces to

$$(5.6) \quad \mathfrak{G}(x) = \int_0^\infty \frac{F(y) dy}{x + y},$$

the standard Stieltjes transform.

According to Widder, (5.6) is invertible by a function $F(x)$ with $\int_0^\infty \|F(x)\| dx < \infty$ iff $\mathfrak{G}(x)$ satisfies the conditions D for the Stieltjes transform ([7], p. 374). Let the operators $L_{K,t}$ be defined by

$$(5.7) \quad \begin{cases} L_{0,t}[\mathfrak{G}(x)] = \mathfrak{G}(t), & L_{1,t}[\mathfrak{G}(x)] = D_t[t\mathfrak{G}(t)] \\ L_{K,t}[\mathfrak{G}(x)] = \frac{(-1)^{K-1}}{K!(K-2)!} D_t^{2K-1}[t^K \mathfrak{G}(t)], & K = 2, 3, \dots \end{cases}$$

Then $\mathfrak{G}(x)$ satisfies the conditions D if $\mathfrak{G}(x)$ has derivatives of all orders in $0 < x < \infty$ such that

$$(5.8) \quad \begin{aligned} (a) \quad & \int_0^\infty \|L_{K,t}[\mathfrak{G}(x)]\| dt < \infty, \quad K = 1, 2, \dots \\ (b) \quad & \lim_{K, t \rightarrow \infty} \int_0^\infty \|L_{K,t}[\mathfrak{G}(x)] - L_{t,t}[\mathfrak{G}(x)]\| dt \rightarrow 0 \\ (c) \quad & \lim_{x \rightarrow 0+} x\mathfrak{G}(x) = 0. \end{aligned}$$

If $\mathfrak{G}(x)$ satisfies these conditions, it is easy to show that this is equivalent to $\int_0^\infty \|f(t)\| dt < \infty$. We need only use (5.2) and a simple change of variables. From such a $\mathfrak{G}(x)$, one can recover the corresponding $G(x)$ by (5.5) and, hence, $g(t)$ by (5.2a).

While there appears to be no easy way to transfer conditions D on $\mathfrak{G}(x)$ to ones on $g(x)$, we can obtain some necessary conditions. With the change of variables $y = x\sigma$, (5.4) becomes

$$(5.9) \quad \mathfrak{G}(x) = -\frac{\nu}{K_2} x^{-(\alpha+\beta+4)/(\alpha+2)(\beta+2)} \cdot \int_1^\infty g\left(\left[\frac{\beta+2}{2}\right]^{2/(\beta+2)} (x\sigma)^{1/(\beta+2)}\right) \sigma^{-2/(\beta+2)} (\sigma-1)^{\nu-1} d\sigma.$$

The condition (5.8c) clearly holds if $\|g(x)\|$ is bounded while the differentiability of $\mathfrak{G}(x)$ requires the infinite differentiability of g . The chain rule further shows that each of the integrals

$$\int_1^\infty g^{(k)}\left(\left[\frac{\beta+2}{2}\right]^{2/(\beta+2)} (x\sigma)^{1/(\beta+2)}\right) \sigma^{(k+2)/(\beta-2)} (\sigma-1)^{\nu-1} d\sigma$$

must exist for $0 < x < \infty$, $k = 0, 1, 2, \dots$. This is easily shown to be equivalent to the condition that $t^k \|g^{(k)}(t)\|$ be bounded as $t \rightarrow \infty$. If this $g(t)$ then defines a $\mathcal{G}(x)$ which satisfies the conditions D , we can obtain $F(t)$ by means of one of the following inversion formulas:

$$(5.10) \quad \begin{aligned} (a) \quad & \frac{1}{2}[F(t+) + F(t-)] = \lim_{k \rightarrow \infty} L_{k,t}[\mathcal{G}(x)] \\ (b) \quad & \frac{1}{2}[F(t+) + F(t-)] = \frac{1}{2\pi i} [\mathcal{G}(te^{-i\pi}) - \mathcal{G}(te^{i\pi})] \end{aligned}$$

([7], pp. 340 and 345). From this $F(t)$ we can recover the corresponding $f(t)$ by use of (5.2b). If this $f(t) \in \overline{\mathfrak{F}}(\mathfrak{D}(A))$, then we have the necessary function to go with (1.2a). We also note that for some situations involving particular real-valued functions $g(t)$, the transforms (5.1) or (5.6) can be inverted directly with tables (see [4], ch. XIV).

Even if $g(t)$ leads to an $f(t)$ with $\int_0^\infty \|f(\xi)\| d\xi < \infty$, this $f(t)$ may fail to be in $\overline{\mathfrak{F}}(\mathfrak{D}(A))$. In some cases, it is possible to define a generalized solution of (1.2b) for $0 \leq t \leq T$. We consider this question for restricted values of α and β since our approach depends upon having appropriate inequalities. The important case $\beta = 0$ will be included.

The idea behind the approach is to approximate such an $f(t)$ by a sequence of functions $\{f_n(t)\}$ in $\overline{\mathfrak{F}}(\mathfrak{D}(A))$, obtain the corresponding sequences $\{u_n(t)\}$ and $\{v_n(t)\}$, and then invoke the completeness of X to deduce a limiting function $v(t)$. Each $f_n(t)$ in the sequence should define a corresponding $g_n(t)$ such that $g_n(t)$ gets close to $g(t)$ as $f_n(t)$ gets close to $f(t)$. This last condition requires that we use norm estimates involving (2.15) and appropriate integrability conditions are needed to obtain good estimates. For this purpose, we have:

Definition 5.1. Let $f(t)$ correspond to $g(t)$ as above and let $\mu < 1$. A function $f^*(t) \in \overline{\mathfrak{F}}(\mathfrak{D}(A))$ is said to be an (ϵ, μ) approximate of $f(t)$ if

$$(5.11) \quad \begin{cases} (a) \quad \int_0^\infty \|f(t) - f^*(t)\| dt < \frac{1}{2}\epsilon \\ (b) \quad \int_0^1 t^{-\mu} \|f(t) - f^*(t)\| dt < \frac{1}{2}\epsilon. \end{cases}$$

Definition 5.2. Let $\{\epsilon_n\}$ be a sequence of positive numbers with $\epsilon_n \rightarrow 0$. A sequence $\{f_n(t)\}$, $f_n(t) \in \overline{\mathfrak{F}}(\mathfrak{D}(A))$, is called a μ -sequence if $f_n(t)$ is an (ϵ_n, μ) approximate of $f(t)$, $n = 1, 2, \dots$.

If the space X with operator admits such sequences, the results of sections 3 and 4 show how the $\{u_n(t)\}$ and $\{v_n(t)\}$ sequences can be constructed from the $\{f_n(t)\}$ sequence. In particular, theorem 4.2 shows that $\{v_n\}$ sequence is Cauchy for $0 \leq t \leq T$ for some $T > 0$ and, hence, converges to a function $v(t)$ from $[0, T]$ to X . It is easy to show that two μ -sequences corresponding to f lead to the same limit function $v(t)$.

We now examine the question of the convergence of the sequence $\{g_n(t)\}$ of transforms of $\{f_n(t)\}$ to $g(t)$ for restricted α and β . If we use (2.15) to relate $g(t)$ to $f(t)$ and $g_n(t)$ to $f_n(t)$, we find that

$$(5.12) \quad ||g(t) - g_n(t)|| \leq C(\alpha, \beta)t^2 \cdot \int_0^\infty \frac{||f(\xi) - f_n(\xi)|| d\xi}{\left[\left(\frac{2}{\beta+2}\right)^2 t^{\beta+1} + \left(\frac{2}{\alpha+2}\right)^2 \xi^{\alpha+2}\right]} \frac{\alpha - \beta}{(\alpha+2)(\beta+2)} + 1.$$

From this, we estimate the closeness of $g_n(t)$ to $g(t)$ in the separate ranges $t \geq 1$ and $0 \leq t \leq 1$. We let C^* be a generic symbol denoting a constant depending upon α and β in the following discussion.

If $t \geq 1$, we have the estimate

$$(5.13) \quad \begin{aligned} ||g(t) - g_n(t)|| &\leq C(\alpha, \beta)t^2 \int_0^\infty \frac{||f(\xi) - f_n(\xi)|| d\xi}{\left[\left(\frac{2}{\beta+2}\right)^2 t^{\beta+2}\right]} \frac{\alpha - \beta}{(\alpha+2)(\beta+2)} + 1 \\ &= C^* t^{-(\alpha-\beta)/(\alpha+2)-\beta} \int_0^\infty ||f(\xi) - f_n(\xi)|| d\xi \leq \frac{1}{2} C^* \epsilon_n \end{aligned}$$

since $\alpha > \beta \geq 0$. Hence $g_n(t)$ converges to $g(t)$ if $t \geq 1$.

Next, suppose $0 \leq t \leq 1$. From the inequality $x + y \geq x^\lambda y^{1-\lambda}$, $0 < \lambda < 1$, $x \geq 0$, $y \geq 0$, we have

$$\left(\frac{2}{\beta+2}\right)^2 t^{\beta+2} + \left(\frac{2}{\alpha+2}\right)^2 \xi^{\alpha+2} \geq \left(\frac{2}{\beta+2}\right)^{2\lambda} \left(\frac{2}{\alpha+2}\right)^{2(1-\lambda)} t^{\lambda(\beta+2)} \xi^{(1-\lambda)(\alpha+2)}.$$

Then

$$(5.14) \quad \begin{aligned} &\left[\left(\frac{2}{\beta+2}\right)^2 t^{\beta+2} + \left(\frac{2}{\alpha+2}\right)^2 \xi^{\alpha+2}\right]^{\alpha-\beta/(\alpha+2)(\beta+2)+1} \\ &\geq C^* t^{\lambda[(\alpha-\beta)/(\alpha+2)+\beta+2]} \xi^{(1-\lambda)[(\alpha-\beta)/(\beta+2)+\alpha+2]}. \end{aligned}$$

If we insert this in (5.12), we get

$$(5.15) \quad ||g(t) - g_n(t)|| \leq C^* t^{2-\lambda[(\alpha-\beta)/(\alpha+2)+\beta+2]} \int_0^\infty \frac{||f(\xi) - f_n(\xi)|| d\xi}{\xi^{(1-\lambda)[(\alpha-\beta)/(\beta+2)+\alpha+2]}}.$$

The exponent of t is to be non-negative while the exponent of ξ in the denominator must be less than 1. These two conditions can be met if $1 - 1/[(\alpha - \beta)/(\beta + 2) + \alpha + 2] < \lambda \leq 2/[(\alpha - \beta)/(\alpha + 2) + \beta + 2]$. Such a choice for λ is possible if $\alpha < 2/(\beta + 1)$. When we use this and the condition $\alpha > \beta$, it is easy to see that we can select β in the range $0 \leq \beta < 1$ and then pick an appropriate value of $\alpha < 2$. Under these circumstances, (5.15) can be replaced by

$$(5.17) \quad ||g(t) - g_n(t)|| \leq C^* \left[\int_0^1 \frac{||f(\xi) - f_n(\xi)|| d\xi}{\xi^{(1-\lambda)[(\alpha-\beta)/(\beta+2)+\alpha+2]}} + \int_1^\infty ||f(\xi) - f_n(\xi)|| d\xi \right]$$

since $t \leq 1$. This inequality shows that if $f_n(t)$ is a μ -sequence with $\mu = (1 - \lambda)((\alpha - \beta)/(\beta + 2) + \alpha + 2)$, then

$$||g(t) - g_n(t)|| \leq C^* \left(\frac{\epsilon_n}{2} + \frac{\epsilon_n}{2} \right) = C^* \epsilon_n$$

and $g_n(t)$ converges to $g(t)$ if $0 \leq t \leq 1$.

If one were to admit μ -sequences with $\mu \geq 1$, then the proof of the convergence of $g_n(t)$ to $g(t)$ for $0 \leq t \leq 1$ could be simplified. In general, this assumption would appear to impose strong restrictions on the behavior of f and the f_n near $t = 0$. The continuity of $f(t)$ in a neighborhood of $t = 0$ should permit such a relaxation.

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