Enumeration of Liftings in the Non-Principal Case

LAWRENCE L. LARMORE

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1. Introduction. If X and Y are spaces, let $[X; Y]$ be the set of homotopy classes of maps $X \rightarrow Y$.

Let $\pi: E \rightarrow B$ be a fibration with fiber ΩC , where C is an H -space, and let $f: X \rightarrow B$ be a map, where X is a CW -complex. The problem we consider in this paper is enumeration of the inverse image of $[f]$, the class containing f , under $\pi_*: [X; E] \rightarrow [X; B]$.

The case where π is a principal fibration has been discussed by James and Thomas [3], Nomura [7], and [4] (the only case considered in the latter is where C is an Eilenberg–MacLane space.). In [3], oriented real n -bundles over an n -complex, and real n -bundles over an n -complex for odd n , are classified up to oriented-bundle, and bundle equivalence, respectively. In this paper (Thm. 4.1) we classify real n -bundles over an n -complex for even n .

Remark 1.2 below is a special case of Theorem 1 of [7], and is simply a restatement of Theorem 1.2 of [3], without the hypothesis that B be an H -space; also of 3.2 of [4] without the hypothesis that C be an Eilenberg–MacLane space. For instructional purposes, the proof is repeated here.

In the current paper, a large class of cases is considered in which π is not a principal fibration. The problem shall be represented by the following diagram:

$$(1-1) \quad \begin{array}{ccccc} \Omega C & \longrightarrow & E & & \\ & \nearrow & \downarrow \pi & & \\ X & \xrightarrow{f} & B & \xrightarrow{k} & C^* \\ & & \searrow \beta & \nearrow \gamma & \nearrow \epsilon \\ & & & & Y \end{array}$$

where β and γ are fibrations and $\gamma \cdot k = \beta$, ϵ is a section of γ , $\gamma^{-1} * = C$ (we take $*$ to be the base-point of any space), $\gamma^{-1}y$ is a H -space with identity ϵy for all $y \in Y$, where the multiplication is continuous on $(\gamma \times \gamma)^{-1} \Delta Y \subset C^* \times C^*$, Y is path-

connected, and π is the fibration over B induced by k in the category $\mathfrak{X}_Y$ (see [1], [5], [6]). Specifically, let $P_Y C^* = \{\sigma: I \rightarrow C^* \mid \gamma \circ \sigma \text{ is constant, } \sigma(0) \in \epsilon(Y)\}$, and let

$$E = \{(b, \sigma) \in B \times P_Y C^* \mid \sigma(0) = \epsilon \beta b, kb = \sigma(1)\}.$$

Let $\Omega_Y C^* = \{\sigma \in P_Y C^* \mid \sigma(1) = \sigma(0)\}$, and let $\Omega_Y: \Omega_Y C^*$ be the obvious fibration, which has a canonical section ϵ' , given by $\epsilon'(y)(t) = \epsilon(y)$ for all $y \in Y, t \in I$.

Let $[X; E]_f, [X; \Omega_Y C^*]_{\beta, f}$ be the sets of fiber-homotopy classes of liftings of f to E , and $\beta \cdot f$ to $\Omega_Y C^*$, respectively. Since each fiber of γ is a homotopy commutative H -space, $[X; \Omega_Y C^*]_{\beta, f}$ is an Abelian group in an obvious way. [5] Our main algebraic result is as follows (where the concepts used are defined in the next section):

Theorem 1.1. *Let the situation in diagram (1-1) be given. Then:*

(I) $[X; E]_f$ is an Abelian affine group in a natural way, and is (if non-empty) non-canonically isomorphic to $[X; \Omega_Y C^*]_{\beta, f}$. In fact, if $[X; E]_f$ is non-empty, its difference group is canonically isomorphic to $[X; \Omega_Y C^*]_{\beta, f}$.

(II) There is a left affine action $\mu = \mu_f, \pi$ of $\pi_1(B^X, f)$ on $[X; E]_f$, and $(\pi_*)^{-1}[f] \subset [X; E]$ is in canonical one-to-one correspondence with the set of orbits of $[X; E]_f$ under that action (where B^X is the space of maps $X \rightarrow B$ with the compact open topology.)

The proof is postponed until section 3.

In the special case that π is principal, a far simpler statement, Remark 1.2, holds. Diagram (1-1) may be considered applicable in this case, by simply assuming that Y is a single point space, and thus $C^* = C$.

Remark 1.2. If $\pi: E \rightarrow B$ is a principal C -fibration, and if $(\pi_*)^{-1}[f]$ is not empty, then:

(I) $[X; E]_f$ is an Abelian affine group in a natural way, and there is a non-canonical affine isomorphism $\iota: [X; E]_f \rightarrow [X; \Omega C]$.

(II) There is a homomorphism $\phi_f: \pi_1(B^X, f) \rightarrow [X; \Omega C]$ such that liftings g_1, g_2 of f to E are homotopic if and only if $\iota[g_1] - \iota[g_2] \in \text{Im } \phi_f$. Thus $(\pi_*)^{-1}$ corresponds (non-canonically) to $\text{Coker } \phi_f$.

Proof. We may let $E = \{(b, \sigma) \in B \times PC \mid kb = \sigma(1)\}$, where $k: B \rightarrow C$ classifies π , and $PC = \{\sigma: I \rightarrow C \mid \sigma(0) = *\}$. Choose a specific lifting $g: X \rightarrow E$. For any $h: X \rightarrow \Omega C$, let $\iota^{-1}[h] = [\mu \cdot (g, h)]$, where μ is the unique map such that the following diagram is commutative, where $\nabla(\sigma, \tau) = \sigma \vee \tau$:

$$\begin{array}{ccccc} X & \xrightarrow{(g, h)} & E \times \Omega C & \xrightarrow{\mu} & E \\ & & \cap & & \cap \\ & & B \times PC \times \Omega C = B \times \Omega C \times PC & \xrightarrow{1 \times \nabla} & B \times PC \end{array}$$

Routinely, one may verify that ι is one-to-one and onto, and the induced Abelian affine structure on $[X; E]_f$ does not depend on the choice of g .

Now let $\alpha: X \times I \rightarrow B$ represent any $a \in \pi_1(B^X, f)$. Let $\alpha': X \times I \rightarrow E$ be a lifting of α such that $\alpha'(x, 0) = g(x)$ for all $x \in X$, and let $g_1(x) = \alpha'(x, 1)$ for all $x \in X$. Define $\phi_r a = [p_2 g_1 \vee (p_2 g)]$ where $p_2: E \rightarrow PC$ is projection. The rest of the proof is routine checking of details (including the fact that ϕ_r does not depend on the choice of g).

We remark that $\pi_1(B^X, f) \cong [X; \Omega B]$ if B is an H -space [3].

2. Affine groups. The concept of an affine group was introduced by Becker [1]. We give a more general definition; Becker's definition is then that of an Abelian affine group.

Let A be a set. An *affine structure* on A is a ternary operation $\tau: A \times A \times A \rightarrow A$ such that:

- (i) $\tau(\tau(a, b, c), d, e) = \tau(a, b, \tau(c, d, e))$ for all $a, b, c, d, e \in A$. (Associative law.)
- (ii) $\tau(a, b, b) = \tau(b, b, a) = a$ for all $a, b \in A$. (Cancellation law.) We say that τ is an *Abelian* affine structure on A if (i) and (ii) above hold, and if also:
- (iii) $\tau(a, b, c) = \tau(c, b, a)$ for all $a, b, c \in A$. (Commutative law.)

If A and B are affine groups (i.e., sets with affine structures), we say that $\phi: A \rightarrow B$ is an *affine morphism* if $\phi\tau(a, b, c) = \tau(\phi a, \phi b, \phi c)$ for all $a, b, c \in A$.

If $H \subset G$ are groups, and if $A \subset G$ is either a left or a right coset of H , then A has a natural affine structure as follows: for $a, b, c \in A$, let $\tau(a, b, c) = ab^{-1}c$. If H is an Abelian group, then A is an Abelian affine group. In fact, there is always a (non-canonical) affine isomorphism $\iota: A \rightarrow H$; simply let $\iota(x) = a^{-1}x$ for all $x \in A$, where $a \in A$ is held fixed.

In particular, every (Abelian) group is an (Abelian) affine group.

The difference group. Let A be an Abelian affine group. Let $\mathbf{Z}A$ be the free Abelian group of all finite formal sums of elements of A , and let $z: \mathbf{Z}A \rightarrow \mathbf{Z}$ be the homomorphism such that $z(a) = 1$ for all $a \in A$. Let $R_\tau \subset \mathbf{Z}A$ be the subgroup generated by all elements of the form $a - b + c - \tau(a, b, c)$ and let $A^* = \mathbf{Z}A/R_\tau$. Since $z \mid R_\tau = 0$, we may let $z: A^* \rightarrow \mathbf{Z}$. For $i \in \mathbf{Z}$, let $A^i = z^{-1}\{i\} \subset A^*$, a coset of A^0 , and hence an Abelian affine group. We call A^0 the *difference group* of A . Note that $A^1 \cong A$ canonically; we shall henceforth make this identification. The following operations shall thus be permitted; these operations taking place implicitly in the group A^* : for $a, b, c \in A$ and $x \in A^0$, we have $a - b + c \in A$, $a + x \in A$, $a - b \in A^0$, etc.

If $\phi: A \rightarrow B$ is a morphism of Abelian affine groups, let $\phi^*: A^* \rightarrow B^*$, $\phi^0: A^0 \rightarrow B^0$, be the obvious induced homomorphisms.

Affine autormorphisms. Again, let A be an Abelian affine group. Let $\text{Afaut } A$ be the group (under composition) of all affine automorphisms of A . Consider the sequence:

$$(2-1) \quad 0 \longrightarrow A^0 \xrightleftharpoons[s_a]{\lambda} \text{Afaut } A \xrightleftharpoons[r_a]{\rho} \text{Aut } A^0 \longrightarrow 1$$

where $\lambda(x)(a) = a + x$ for all $x \in A^0$, $a \in A$, and $\rho\phi = \phi^0$ for $\phi \in \text{Afaut } A$. For

any $a \in A$, define s_a, r_a as follows: if $\psi \in \text{Aut } A^0$, $b \in A$, let $r_a(\psi)(b) = a + \psi(b - a)$; for $\phi \in \text{Afaut } A$, let $s_a\phi = \phi(a) - a$. We state without proof the following easily verifiable remark:

Remark 2.1.

- (I) λ is a monomorphism.
- (II) ρ is an epimorphism.
- (III) $\text{Ker } \rho = \text{Im } \lambda$.
- (IV) r_a is a monomorphism and $\rho \cdot r_a =$ the identity on $\text{Aut } A^0$ for all $a \in A$.
- (V) $s_a \cdot \lambda =$ the identity on A^0 for all $a \in A$ (but in general s_a is not a homomorphism).
- (VI) "Ker" $s_a = \text{Im } r_a$ for all $a \in A$.
- (VII) If $x \in A^0$ and $\phi \in \text{Afaut } A$, $\phi(\lambda x)\phi^{-1} = \lambda(\phi^0 x)$

From [2, p. 462], we then immediately have that $\text{Afaut } A \cong A^0 \times_1 \text{Aut } A^0$, the semi-direct product, where 1 is the identity on $\text{Aut } A^0$.

Let G be a group. We say that $\mu: G \times A \rightarrow A$ is a left (right) affine action of G on A if μ is a left (right) action on A as a set, and if $\mu(g, \cdot)$ is an affine automorphism of A for each $g \in G$. Equivalently (as the reader may trivially verify), a left (right) action of G on A may be defined to be a homomorphism (anti-homomorphism) $\mu: G \rightarrow \text{Afaut } A$. We shall use these definitions interchangeably.

Let $\mu_1, \mu_2: G \rightarrow \text{Afaut } A$ be left (or right) affine actions. We say that μ_1 and μ_2 are *translation-equivalent* if $\mu_2(g) = (\lambda x)\mu_1(g)(\lambda x)^{-1}$ for all $g \in G$, for some fixed $x \in A^0$.

Classification of affine actions. For this paragraph, fix an Abelian affine group A and a group G .

Let $\mu: G \rightarrow \text{Afaut } A$ be a left (or right) affine action. If $\Gamma \subset G$, let $\Delta_\mu^\Gamma: A^0 \rightarrow \prod_{g \in \Gamma} A_g^0$ (where $A_g^0 = A^0$ for each $g \in \Gamma$) be given by: $(\Delta_\mu^\Gamma x)_g = (\mu g)^0 x - x$ for all $x \in A^0$, $g \in \Gamma$. For any $a \in A$, let $s_a^\Gamma \mu \in \prod_{g \in \Gamma} A_g^0$ be given by: $(s_a^\Gamma \mu)_g = s_a(\mu g)$ for all $g \in \Gamma$. We now define $s^\Gamma \mu \in \text{Coker } \Delta_\mu^\Gamma$ to be the element represented by $s_a^\Gamma \mu$. If $\Gamma = \{g\}$, write $s^g \mu \in A^0/((\mu g)^0 - 1)A^0$.

Lemma 2.2. *Let $\mu: G \rightarrow \text{Afaut } A$ be a left (or right) affine action, where A is an Abelian affine group, and G is a group. Let $\Gamma \subset G$. Then:*

- (I) $s^\Gamma \mu \in \text{Coker } \Delta_\mu^\Gamma$ is well-defined.
- (II) If $\Lambda \subset \Gamma$, the projection $\text{Coker } \Delta_\mu^\Gamma \rightarrow \text{Coker } \Delta_\mu^\Lambda$ takes $s^\Gamma \mu$ to $s^\Lambda \mu$.
- (III) If Γ generates G , and if $\mu': G \rightarrow \text{Afaut } A$ is another left (or right) affine action, then μ' is translation-equivalent to μ if and only if $\rho \cdot \mu' = \rho \cdot \mu$ and $s^\Gamma \mu' = s^\Gamma \mu$.

Proof (I). Let $a, a' \in A$. By checking on each coordinate, we verify that $s_{a'}^\Gamma \mu - s_a^\Gamma \mu = \Delta_\mu^\Gamma(a' - a)$.

(II) Trivial.

(III) If μ' is translation-equivalent to μ , choose $x \in A^0$ such that $\mu'g =$

$(\lambda x)(\mu g)(\lambda x)^{-1}$ for all $g \in G$. Clearly, $\rho \cdot \mu' = \rho \cdot \mu$, since $(\lambda x)^0$ is trivial. Choose $a \in A$. For any $g \in \Gamma$, we have:

$$\begin{aligned} (s_a^\Gamma \mu')_o - (s_a^\Gamma \mu)_o &= s_a(\mu'g) - s_a(\mu g) = (\mu'g)a - a - ((\mu g)a - a) \\ &= (\mu'g)a - (\mu g)a = (\mu g)(a - x) + x - (\mu g)a \\ &= (\mu g)a - (\rho \mu g)x + x - (\mu g)a = -(\Delta_\mu^\Gamma)_o x. \end{aligned}$$

Conversely, suppose $\rho \cdot \mu' = \rho \cdot \mu$ and $s_a^\Gamma \mu' = s_a^\Gamma \mu - \Delta_\mu^\Gamma x$ for some $a \in A$, $x \in A^0$. For any $g \in \Gamma$, $s_a(\mu'g) = s_a(\mu g) - (\rho \mu g)x + x$. Thus, for any $b \in A$:

$$\begin{aligned} (\mu'g)b &= (\mu'g)a + (\rho \mu g)(b - a) = a + s_a(\mu'g) + (\rho \mu g)(b - a) \\ &= a + s_a(\mu g) - (\rho \mu g)x + x + (\rho \mu g)(b - a) \\ &= (\mu g)a + (\rho \mu g)(b - a - x) + x = (\mu g)(b - x) + x = (\lambda x)(\mu g)(\lambda x)^{-1}b. \end{aligned}$$

We have $\mu'g = (\lambda x)(\mu g)(\lambda x)^{-1}$ for all $g \in \Gamma$: since Γ generates G , the same equation holds for all $g \in G$, and we are done.

The following technical lemma will be used in the applications:

Lemma 2.3. *Let $\mu: G \rightarrow \text{Afaut } A$ be a left (or right) affine action, where G is a group, and A is an Abelian affine group. Let $\mathcal{G}$ be a family of subsets of G such that G is generated by the union of the members of $\mathcal{G}$, and such that if $g, g' \in G$ belong to distinct members of $\mathcal{G}$, $\text{Ker } ((\mu g)^0 - 1) + \text{Ker } ((\mu g')^0 - 1) = A^0$. Then an affine action $\mu': G \rightarrow \text{Afaut } A$ is translation equivalent to μ if and only if $\rho \cdot \mu' = \rho \cdot \mu$ and $s^\Gamma \mu = s^\Gamma \mu'$ for all $\Gamma \in \mathcal{G}$.*

We omit the proof, a routine generalization of the proof of Thm. 2.2 (III).

Corollary 2.4. *Let $\mu: G \rightarrow \text{Afaut } A$ be a left (or right) affine action, where G is a group and A is an Abelian affine group. Let $S \subset G$ be a generating set of $\text{Ker } (\rho \cdot \mu)$, and let $\Gamma \subset G$ be any set such that $S \cup \Gamma$ generates G . Then another left (right) affine action $\mu': G \rightarrow \text{Afaut } A$ is translation-equivalent to μ if and only if $\rho \cdot \mu' = \rho \cdot \mu$, $s^\Gamma \mu = s^\Gamma \mu'$, and $s^o \mu = s^o \mu'$ for all $g \in S$.*

3. Classification of liftings. In this section, we return to the situation depicted in diagram (1-1).

Proof of Theorem 1.1 (I): Let $g_1, g_2, g_3: X \rightarrow E$ be liftings of f . Recall $E \subset B \times P_Y C^*$, and let $p_2: E \rightarrow P_Y C^*$ be the projection. Let $g_4 = g_1 \# g_2 \# g_3: X \rightarrow E$ be the unique lifting of f such that $p_2 g_4 x = p_2 g_1 x \vee (p_2 g_2 x)^{-1} \vee p_2 g_3 x$ for all $x \in X$, and let $\tau(a_1, a_2, a_3) = a_4$, where $a_i \in [X; E]_f$ is represented by g_i . For any fixed lifting g of f to E , let $\iota_g: [X; \Omega_Y C^*]_{\beta, f} \rightarrow [X; E]_f$ be given by $\iota_g[h] = [h \# g]$, where $h \# g$ is the unique lifting of f such that $p_2(h \# g)x = h \vee p_2 g x$ for all $x \in X$. Routinely, one may verify that τ is a well-defined Abelian affine structure on $[X; E]_f$, and that ι_g is an affine isomorphism which sends 0 to $[g]$. Further, if $g': X \rightarrow E$ is any other lifting of f , $\iota_g x = \iota_g(x + \iota_g^{-1}[g'])$ for all $x \in [X; \Omega_Y C^*]_{\beta, f}$. It follows that if $[X; E]_f$ is not empty, its difference group may be identified

with $[X; \Omega_r C^*]_{\beta, f}$, matching $\iota_g x = [g]$ with x , for any x, g . We leave the details to the reader. (See [1], [5], [6])

(II): If $a \in \pi_1(B^X, f)$, let $\alpha: X \times I \rightarrow B$ represent a . Consider the diagram:

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ X & \xrightarrow[i_1]{i_0} & X \times I \xrightarrow{\alpha} B \end{array}$$

where $i_0, i_1: X \rightarrow X \times I$ are inclusions along $0, 1 \in I$, respectively; thus $\alpha \cdot i_0 = \alpha \cdot i_1 = f$. We have induced affine homomorphisms $i_0^*, i_1^*: [X \times I; E]_a \rightarrow [X; E]_f$, which are bijective by the homotopy lifting property and the covering homotopy extension property. Let $\mu = \mu_{f, \pi}$ be given by $\mu(a) = i_0^* \cdot (i_1^*)^{-1} \in \text{Aaut } [X; E]_f$. Trivially, μ is a left action.

Directly from the above definition of μ , it is clear that if $\mu(a, [g_1]) = [g_0]$ for any liftings g_0, g_1 of f to E , then g_0 and g_1 are homotopic: conversely, if $g_t: X \rightarrow E, 0 \leq t \leq 1$, is a fiber-homotopy of liftings of f , then trivially $\mu(a, [g_1]) = [g_0]$ where a is represented by $\alpha: X \times I \rightarrow B, \alpha(x, t) = g_t(x)$; and we are done.

We now consider the problem of determining the action $\mu = \mu_{f, \pi}$ of $\pi_1(B^X, f)$ on $[X; E]_f$. In order to enumerate liftings of f to E up to homotopy, it is clearly necessary only to classify the action μ up to translation-equivalence. We have a diagram of groups and homomorphisms with exact rows:

$$(3-1) \quad \begin{array}{ccccccc} K & \subset & \pi_1(B^X, f) & \xrightarrow{\beta_*} & \pi_1(Y^X, \beta \cdot f) \\ \downarrow \nu & & \downarrow \mu & & \downarrow \mu' = \mu_{f, \pi} \\ 0 \longrightarrow [X; \Omega_r C^*]_{\beta, f} & \xrightarrow{\lambda} & \text{Aaut } [X; E]_f & \xrightarrow{\rho} & \text{Aut } [X; \Omega_r C^*]_{\beta, f} \longrightarrow 1 \end{array}$$

where $K = \text{Ker } \beta_*$. The right square can be shown to be commutative simply by chasing definitions, a unique map ν thus exists which makes the left square commutative. By 1.1, 2.2, and 2.4, we obtain:

Remark 3.1. Given diagram (1-1), and given the existence of at least one lifting of f to E , in order to enumerate $(\pi_*)^{-1}[f] \subset [X; E]$ it is sufficient to know:

(i) the groups $\pi_1(B^X, f)$, $\pi_1(Y^X, \beta \cdot f)$, $[X; \Omega_r C^*]_{\beta, f}$ and the homomorphisms β_*, μ' (of diagram (3-1));

(ii) $s^\Gamma \mu \in \text{Coker } \Delta_\mu^\Gamma$ for any set $\Gamma \subset \pi_1(B^X, f)$ such that $\beta_* \Gamma$ generates $\text{Im } \beta_* \subset \pi_1(Y^X, \beta \cdot f)$;

(iii) $s^\sigma = \nu g \in [X; \Omega_r C^*]_{\beta, f}$ for each $g \in S$, where S is any generating set of K .

We remark that in the case that π is a principal fibration (cf. 1.2), $K = \pi_1(B^X, f)$ and $\nu = \phi_f$.

For purposes of this paper, we shall assume that the information (i) of Remark 3.1 is known. For any $\Gamma \subset \pi_1(B^X, f)$, $s^\Gamma \mu$ may be computed by using the following lemma.

Lemma 3.2. *Let diagram (1-1) be given, let $\Gamma \subset \pi_1(B^X, f)$, and suppose that $[X; E]_\Gamma \neq \emptyset$. Let $\bigvee_\Gamma S^1$ be a wedge of circles, one for each $a \in \Gamma$, and let $F_\Gamma: X \times \bigvee_\Gamma S^1 \rightarrow B$ be a map such that $F_\Gamma|_{(X \times S^1)_a}$ classifies a , for each $a \in \Gamma$. Let $g: X \rightarrow E$ be a lifting of f , and let $\theta: (X \times \bigvee_\Gamma S^1, X) \rightarrow (C^*, \epsilon(Y))$ be a map, which is rel Y homotopic to $k \cdot F_\Gamma$ such that the homotopy is given by $p_2 g$ along X . Then $S(s_{[\epsilon]}^\Gamma \mu) = [\theta]$, where*

$$s: [X \times \bigvee_\Gamma S^1, X; C^*]_{F_\Gamma} \xrightarrow{\cong} \prod_{a \in \Gamma} [X; \Omega_Y C^*]_{\beta, f}$$

is the suspension isomorphism.

Proof. After all the definitions, the proof is simply a matter of checking maps.

Note that the arbitrary choice of $[g]$ introduces the correct indeterminacy of $[\theta]$, i.e., if $[g]$ is altered by $x \in [X; \Omega_Y C^*]_{\beta, f}$, $[\theta]$ is altered by $\Delta_\mu^\Gamma x$.

In the applications in the next section, B will be an H -space. We state a useful computational lemma applicable in that case, which was originally given by James and Thomas [3, Thm. 1.3].

Lemma 3.3. (I) *If $f: X \rightarrow B$, B an H -space, then $\pi_1(B^X, f) \cong [X; \Omega B]$, where the homotopy class of a map $\theta: X \rightarrow \Omega B$ corresponds to the homotopy class represented by the composition*

$$F_{f, \theta}: X \times S^1 \xrightarrow{\Delta \times 1} X \times X \times S^1 \xrightarrow{1 \times c} X \times \Sigma X \xrightarrow{f \times \theta} B \times B \xrightarrow{m} B$$

where Δ is the diagonal map, c collapses $X \vee S^1$, ${}^1\theta$ is the adjoint of θ , and m is the multiplication of B .

(II) *Let $\sigma: H^*(B) \rightarrow H^*(\Omega B)$ be the looping suspension, of degree -1 (for any coefficients). Let $a \in H^*(B)$ such that $m^*a = \sum_{1 \leq i \leq \pi} b_i \otimes c_i$ (any suitable coefficients). Let $\theta: X \rightarrow \Omega B$. Then $F_{f, \theta}^*a = f^*a \otimes 1 + (\sum_{1 \leq i \leq n} f^*b_i \otimes \theta^*\sigma c_i) \otimes e^1$, where $e^1 \in H^1(S^1; \mathbb{Z})$ is the fundamental class.*

4. Applications to enumeration of vector bundles. Let X be an n -dimensional CW -complex, $n \geq 2$, and let ξ be a real stable vector bundle over X , classified by $f: X \rightarrow BO$. Let the inclusion $BO(n) \subset BO$ be replaced by a fibration π (whose total space we also call $BO(n)$). Then real n -plane bundles over X , stably equivalent to ξ , are classified, up to bundle equivalence, by $(\pi_*)^{-1}[f] \subset [X; BO(n)]$. Since $\dim X = n$, $BO(n)$, for purposes of this problem, may be replaced by the first stage in the relative Postnikov tower for the map $BO(n) \rightarrow BO$, which is the total space of the principal $K(\mathbb{Z}_2, n)$ -bundle over BO classified by w_{n+1} if n is odd, and is the total space of the non-principal $K(\mathbb{Z}, n)$ -bundle over BO classified by $\beta[w_1]w_n \in H^{n+1}(BO; \mathbb{Z}(w_1))$, if n is even. The former case was solved in [3]; the latter case we consider here.

Generally, we shall let w_i denote the i th Stiefel-Whitney class of either ξ over X , or of the universal bundle over BO (it will always be clear which by the context). If $u \in H^1(W; \mathbb{Z}_2)$, W any space, we shall let $\mathbb{Z}[u]$ be the sheaf of

integers over W twisted by u (recall that $\mathbf{Z}_2 = \text{Aut } \mathbf{Z}$), and $\beta[u]$ denote the Bokstein homomorphism associated with the exact sequence of sheaves,

$$\mathbf{Z}[u] \xrightarrow{\times 2} \mathbf{Z}[u] \longrightarrow \mathbf{Z}_2.$$

We have, as a special case of diagram (1-1):

$$\begin{array}{ccccc} K(\mathbf{Z}, n) & \longrightarrow & E & & \\ & & \downarrow \pi & & \\ X & \xrightarrow{f} & BO & \xrightarrow{k} & K^T(\mathbf{Z}, n+1) \\ & & \searrow \beta \quad \gamma & \nearrow \epsilon & \\ & & P_\infty & = & K(\mathbf{Z}_2, 1) \end{array}$$

where $K^T(\mathbf{Z}, n+1) = K(\mathbf{Z}, n+1) \times S^\infty/\mathbf{Z}_2$, where the action is multiplication by -1 on $K(\mathbf{Z}, n+1)$ (taken as an involution with only the base-point fixed) and the antipodal map on S^∞ ; where γ is projection to $S^\infty/\mathbf{Z}_2 = P_\infty$, ϵ is the obvious section, β classifies w_1 , and k classifies $\beta[w_1]w_n$. E is a space which is n -equivalent to $BO(n)$, thus $(\pi_*)^{-1}[f]$ classifies n -bundles stably equivalent to ξ .

Let $O = \Omega BO$, the full orthogonal group. For any $i \geq 0$, let $h_i = \sigma w_{i+1} \in H^i(O, \mathbf{Z}_2)$. If $a \in [X; O]$, let $h_i a = \alpha^* h_i$ for $\alpha: X \rightarrow O$ representing a . To solve the enumeration problem, it is obviously sufficient to consider the case that X is connected.

Theorem 1.8 of [3] correctly classifies oriented n -bundles up to orientation preserving bundle equivalence over an n -complex, for n even, not simply bundle equivalence as stated. The same correction (first noticed by Robinson [8]) applies to theorem 1.10.

Theorem 4.1 (Main result). *Let a real stable vector bundle ξ over a connected n -complex X , n even, be classified by $f: X \rightarrow BO$. Fix $W_n \in H^n(X; \mathbf{Z}[w_1])$ such that $\rho W_n = w_n$, where ρ is reduction mod 2. Then the set of equivalence classes of n -plane bundles over X stably equivalent to ξ is in one-to-one correspondence with the orbits of $H^n(X; \mathbf{Z}[w_1])$ under the affine action,*

$$\mu: [X; O] \times H^n(X; \mathbf{Z}[w_1]) \rightarrow H^n(X; \mathbf{Z}[w_1])$$

where, for any $a \in [X; O]$ and $x \in H^n(X; \mathbf{Z}[w_1])$:

$$\mu(a, x) = \begin{cases} x + \beta[w_1] \sum_{1 \leq i < n} h_i a \otimes w_{n-i-1} & \text{if } h_0 a = 0 \\ W_n - x + \beta[w_1] \sum_{0 \leq i < n} h_i a \otimes w_{n-i-1} & \text{if } h_0 a = 1. \end{cases}$$

Proof. Since BO is an H -space, $\pi_1(BO^X, f)$ may be replaced by $[X; O]$. By 3.1, we need only consider $\mu(a, x)$ for $a \in [X; SO]$ and $\mu(\zeta, x)$, where ζ is represented by the composition

$$X \times S^1 \xrightarrow{p_2} S^1 \xrightarrow{i} BO,$$

where i represents the generator of $\pi_1(BO) = \mathbf{Z}_2$. The fact that the desired formula for $\mu(a, x)$ holds for $a \in [X; SO]$ follows immediately from lemmas 3.2, 3.3, and the Wu formula $m^*w_n = \sum_{0 \leq i \leq n} w_i \otimes w_{n-i}$. We are thus reduced to proving that, for all $x \in H^n(X; \mathbf{Z}[w_1])$, $\mu(\zeta, x) = W_n - x + \beta[w_1]w_{n-1}$. Now ζ maps to the generator of $\pi_1(BO) = \pi_1(P_\infty) = \mathbf{Z}_2$, which acts on the n^{th} homotopy group of the fiber of π by multiplication by -1 . Thus $\mu(\zeta, x) = -x + s$ where $s \in H^n(X; \mathbf{Z}[w_1])$ does not depend on the choice of x , and has indeterminacy $2H^n(X; \mathbf{Z}[w_1])$ (cf. lemma 2.2), i.e., is determined by its mod 2 reduction. Now $\rho\beta[w_1]w_n = w_{n+1}$, thus by 3.2 and 3.3, $\rho s = w_n$. Since $\rho\beta[w_1]w_{n-1} = 0$, $s = W_n + \beta[w_1]w_{n-1}$ up to indeterminacy.

Projective spaces. Let P_n be real projective n -space, $n \geq 2$ even. Now $[P_n; O] = \mathbf{Z}_2 + \mathbf{Z}_2$ with generators ζ and η , where η is represented by the adjoint of the Whitehead map $\Sigma P_n \rightarrow BSO: h_i \zeta = u^i$ for all $1 \leq i \leq n$, where $u \in H^1(P_n; \mathbf{Z}_2)$ is the generator.

Let ξ be the stable bundle kh , where h is the canonical line bundle over P_n . Let

$$w_i = w_i(\xi) = \binom{k}{i} u^i$$

for $0 \leq i \leq n$. We have:

$$H^n(P_n; \mathbf{Z}[w_1]) \cong \begin{cases} \mathbf{Z}_2 \text{ generated by } \beta u^{n-1} & \text{if } k \text{ even} \\ \mathbf{Z} \text{ generated by } T & \text{if } k \text{ odd} \end{cases}$$

where T is the top class, the Poincaré dual of the generator $1 \in H^0(P_n; \mathbf{Z})$. Note $\rho T = u^n$. Applying 4.1, we have that if k is even, for all $x \in H^n(P_n; \mathbf{Z})$:

$$\begin{aligned} \mu(\zeta, x) &= x + \binom{k}{n} u^{n-1} \\ \mu(\eta, x) &= x + \binom{k-1}{n-1} u^{n-1} \end{aligned}$$

while if k is odd, for all $x \in H^n(P_n; \mathbf{Z}[u])$:

$$\begin{aligned} \mu(\zeta, x) &= \begin{cases} -x & \text{if } \binom{k}{n} \text{ even} \\ T - x & \text{if } \binom{k}{n} \text{ odd} \end{cases} \\ \mu(\eta, x) &= x. \end{aligned}$$

If k is odd, alteration by T in $[P_n; BO(n)]_{kh}$ changes the Euler characteristic by 2. We thus have:

Remark 4.2. Let $\xi = kh$ over P_n , $n \geq 2$ even. (I) If k is even, there are two equivalence classes of n -plane bundles representing ξ if and only if $\binom{k}{n}$ and $\binom{k-1}{n-1}$

are both even; otherwise there is just one. (II) If k is odd, there are infinitely many n -plane bundles over P_n representing ξ . In fact, if κ is such a bundle, and $\chi(\kappa) \in \mathbf{Z}$ is its Euler characteristic, then κ is classified by the absolute value of $\chi(\kappa)$, which may be any even non-negative integer if $\binom{k}{n}$ is even, and any odd positive integer if $\binom{k}{n}$ is odd.

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California State College
Dominguez Hills

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