Infinitesimal Convexity Implies Local Convexity

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Suppose that M is a Riemannian manifold and that L is an isometrically imbedded, closed, oriented hypersurface. We say that L is *infinitesimally convex* if the second fundamental form of L is positive semidefinite at every point. L is *locally convex* if for every $p \in L$ there is a neighborhood U of $0 \in L_p$ (the tangent space at p) such that $\exp U$ has no points of M on the "inside" of L . By the *inside* of L we mean the component of a tubular neighborhood with L deleted which corresponds to the negative multiples of the orienting normal vector field under $\exp$. The exponential maps are those of M .

It is an immediate consequence of one of the standard interpretations of the second fundamental form, as the Hessian of a normal coordinate function restricted to L [BC, p. 198], that local convexity implies infinitesimal convexity. When M has constant curvature, the converse has been proved by doCarmo and Warner [CW]. It was doCarmo who brought my attention to the fact that the general case is not as obvious as it sounds.

The differentiability requirements for our proof of the converse seem to be C^4 , since at one step a curvature estimate is used to obtain a uniform bound for some focal distances in a family of surfaces. This bound probably could be obtained more directly in terms of metric components, and hence the differentiability requirements would be less.

The theorem is local, so we may assume that M is a geodesically convex neighborhood of p and that the exponential map of the normal bundle of L is a diffeomorphism from some neighborhood of the 0-section to M . Furthermore, we may assume that the sectional curvatures are bounded above.

The idea of the proof is to reduce it to the case where $\dim M = 2$, by constructing surfaces which sweep out a neighborhood of p and such that if $\exp U$ were to cross to the inside of L , then it would have to do so within one of these surfaces. In order to succeed with this approach we must insure that the intersection of a surface with L inherits the infinitesimal convexity of L and that the geodesic in the surface tangent to that intersection at p is also a geodesic in M .

and hence lying in $\exp U$. The second fundamental form of the intersection will be the restriction of that of L provided that the surface meets L orthogonally, so the natural choice of surfaces are the surfaces generated by the geodesics normal to L which pass through a geodesic tangent to L at p .

We can describe these surfaces more carefully as follows. Let N_0 be the orienting unit normal vector field on L . Extend N_0 to a vector field N on all of M by requiring that the integral curves of N be geodesics normal to L . If γ is a geodesic tangent to L at p , then our assumptions imply that γ is transverse to N . Hence the flow of N applied to γ generates a 2-dimensional submanifold $S(\gamma)$. The intersection of two of these surfaces is the geodesic σ normal to L at p . The proof in the surface case uses a construction which is valid out to the focal points of σ . In order to apply the surface case to the $S(\gamma)$'s and be sure that the region on which the construction is valid fills a neighborhood of p , we must verify that there is a uniform lower bound on the distance to the focal points in $S(\gamma)$ of σ .

By the construction of $S(\gamma)$, through each point there is a geodesic of M which is included in $S(\gamma)$. It follows from Synge's theorem [BC, p. 194] that the Gaussian curvature of $S(\gamma)$ is no more than the sectional curvature of M . Hence, the curvatures of the $S(\gamma)$'s are uniformly bounded above, say by $b > 0$. The standard estimate for the distance to a focal point of a geodesic is that it is at least $\pi/2b^{1/2}$. However, this is the distance measured within the $S(\gamma)$'s. If $r : M \rightarrow R$ is the distance in M from σ , then the rate of change of r along a curve is the cosine of the angle between the curve and the shortest geodesic to σ . Applying the flow of N to the geodesics γ , we obtain a field of curves, each one in some $S(\gamma)$. The images of L under the flow of N are orthogonal to the integral curves of N , by Gauss' lemma [BC, p. 147]. Since the curves in the field are tangent at σ to the images of L , they leave σ orthogonally. Thus the cosines in question have limit 1 at σ and are obviously continuous elsewhere. On some neighborhood of p there will be a positive lower bound c for these cosines. This implies that the distances $r(\gamma) : S(\gamma) \rightarrow R$ in $S(\gamma)$ from σ are related to r by the uniform inequality $r \geq cr(\gamma)$. Thus the M -distance from σ to any focal point is at least $c\pi/2b^{1/2}$.

Now we give a proof of the theorem when $\dim M = 2$ and σ has no focal points. Let X be one of the unit vector fields tangent to L and let E be the field of unit vectors on M tangent to geodesics normal to σ , directed so that $E(p) = X(p)$. Let F be the unit field orthogonal to E for which $F(p) = N(p)$. Because the integral curves of E are geodesics, the covariant derivatives $D_E E = D_E F = 0$. The other covariant derivatives are determined by $f : M \rightarrow R$ such that $D_F E = fF$, $D_F F = -fE$. Let $u : M \rightarrow R$ be the function which is constant on integral curves of E and is the arclength parameter on σ with $u(p) = 0$. Thus the curve $u = 0$ is the geodesic tangent to L at p and our goal is to show that $u \leq 0$ on L . Since u can be taken as a coordinate function on M , it follows that $Fu > 0$.

Let $X = cE + sF$, and consider these as functions of the arclength parameter t on L . Then $N = -sE + cF$ on L , $c^2 + s^2 = 1$, hence $D_X N = -s'E + c'F -$

$s^2fF - csfE$ and $cc' + ss' = 0$. The second fundamental form of L is

$$\langle X, D_x N \rangle = -cs' + sc' - (s^3 + c^2s)f$$

$$= -\frac{1}{c}(s' + cfs) \geq 0.$$

Since $s(0) = 0$, the standard treatment of a linear differential inequality (using an integrating factor) gives $s \leq 0$ for $t \geq 0$. Then we have $Xu = cEu + sFu = sFu \leq 0$ so that $u \leq 0$ for $t \geq 0$. Reversing X, E , we get $u \leq 0$ for $t \leq 0$, so that $u \leq 0$ on L as desired.

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