The Distant Future

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§1. Introduction. Recall that via a Hilbert space isomorphism each stationary stochastic process, in the wide sense and without any continuity assumption, may be realized by selecting a suitable finite positive measure μ on the Bohr group G (the group dual to the real line $\mathbf{R}$ with the discrete topology) and letting the random variables be the characters $\{\chi_\lambda\}_{\lambda \in \mathbf{R}}$ of G regarded as functions in $L^2(\mu)$. The *distant future* of the process is the intersection $\bigwedge_{\tau \in \mathbf{R}} \mathfrak{M}_\tau$, where $\mathfrak{M}_\tau$ is the closed linear span of the random variables χ_λ with $\lambda > \tau$. In this paper we investigate the prediction problem of deciding in terms of μ when the distant future consists of the zero function alone. If the process is continuous, then well-known criteria are given by the classic results of Szegő [14] and Kolmogorov [8]. On the other hand, if μ is absolutely continuous with respect to Haar measure on G , then criteria may be found in the famous paper of Helson and Lowdenslager [7]. In the present paper, we solve the prediction problem without any supplementary hypothesis whatsoever.

To understand our solution, recall that $\mathbf{R}$, with the usual topology now, may be continuously imbedded in G as a dense one-parameter subgroup $\{e_t\}_{t \in \mathbf{R}}$ and, therefore, we may regard $\mathbf{R}$ as acting on G as a topological transformation group. The measure μ on G is called quasi-invariant in case its class of null sets is preserved by this action. If μ is quasi-invariant, then $\rho(t, x)$ will denote the derivative $d\mu_t/d\mu(x)$ where $\mu_t(M) = \mu(M - e_t)$ (M a Borel set). Our solution to the prediction problem is this: The distant future of the process consists of the zero function alone if and only if 1) μ is quasi-invariant and 2) there is a μ -null set N such that for each $x \notin N$, $\log \rho(t, x)$, as a function of t , belongs to $L^1(dt/1 + t^2)$.

Observe that the process is continuous precisely when μ is concentrated on the subgroup $\{e_t\}_{t \in \mathbf{R}}$ and, in this case, μ is quasi-invariant precisely when it is equivalent to linear Lebesgue measure dt on $\mathbf{R}$ transplanted to $\{e_t\}_{t \in \mathbf{R}}$. If $w = d\mu/dt$ is regarded as a function on $\mathbf{R}$, then $\rho(t, x) = w(s - t)/w(s)$, if $x = e_s$ and is zero otherwise. Hence the distant future vanishes when and only when $\log w(t)$ lies in $L^1(dt/1 + t^2)$. This is the result of Szegő as extended by

Kolmogorov. On the other hand, if μ is absolutely continuous with respect to Haar measure m on G , and if $w = d\mu/dm$, then μ is quasi-invariant if and only if $w \neq 0$ a.e.(m). In this case $\rho(t, x) = w(x - e_t)/w(x)$. Hence the distant future vanishes precisely when $\log w(x - e_t)$ lies in $L^1(dt/1 + t^2)$ a.e.(m). This is the result of Helson and Lowdenslager.

Our proof, presented in §4, is given in the context of function algebras and flows [10] and is rather different from Helson's and Lowdenslager's proof. Although different from their's, our proof rests on techniques which were first investigated by them and which have subsequently been developed further by Helson and Forelli. In sections two and three, we establish notation and present some known lemmas which have been modified to suit our purposes. Corollaries to and extensions of our solution to the prediction problem are presented in section five. In section six we establish a relation between our solution and the concept of a Szegő measure and apply our analysis to the representation theory of algebras of analytic functions associated with flows. When these results are specialized suitably, they yield the well-known results concerning the absolute continuity of the spectral measure for the minimal unitary dilation of a completely non-unitary contraction. Finally, in section seven, we investigate how likely it is that the process determined by a quasi-invariant measure has a zero distant future.

§2. Preliminaries. In this section we establish the notation and terminology which we shall use.

Throughout this paper we shall write γ for the measure on $\mathbf{R}$ whose derivative with respect to Lebesgue measure is the Cauchy density $1/\pi(1 + t^2)$ and we shall denote the Lebesgue and Hardy spaces based on γ by $L^p(\gamma)$ and $H^p(\gamma)$. The set of all functions φ in $L^\infty(\gamma)$ such that $e^{-i\lambda t}\varphi(t)$ belongs to $H^\infty(\gamma)$ will be denoted by $H^\infty[\lambda, \infty)$.

We fix once and for all a compact Hausdorff space X upon which it is assumed that $\mathbf{R}$ acts as a topological transformation group; and we write $x + t$ for the translate of x in X by t in $\mathbf{R}$. A function φ in $C(X)$, the space of continuous complex-valued functions on X , will be called *analytic* in case for each x in X the function of t , $\varphi(x + t)$, belongs to $H^\infty(\gamma)$. The space of all analytic functions on X , which is clearly a uniformly closed subalgebra of $C(X)$ containing the constants, will be denoted by $\mathfrak{A}$. We note that the functions in $\mathfrak{A}$ need not separate the points of X , but under very mild assumptions they do (cf. Theorem I of [12]). Observe that if X is the Bohr group, and if $\mathbf{R}$ acts on X in the fashion indicated in the introduction, then the algebra $\mathfrak{A}$ is precisely the closed subalgebra of $C(X)$ spanned by $\{\chi_\lambda\}_{\lambda \geq 0}$.

Suppose $\{A_t\}_{t \in \mathbf{R}}$ is a uniformly bounded, strongly continuous representation of $\mathbf{R}$ on a Banach space $\mathfrak{X}$. For $x \in \mathfrak{X}$ and f in the group algebra $L^1(\mathbf{R})$, we define the convolution $x * f$ by the equation $x * f = \int_{\mathbf{R}} (A_t x) f(t) dt$. This notion of convolution converts $\mathfrak{X}$ into a module over $L^1(\mathbf{R})$ such that $\|x * f\|_{\mathfrak{X}} \leq (\sup \|A_t\|) \|x\|_{\mathfrak{X}} \|f\|_1$, and it follows that the set $J(x) = \{f \in L^1(\mathbf{R}) \mid x * f = 0\}$

is a closed ideal in $L^1(\mathbf{R})$. We shall call the hull of $J(x)$ the *spectrum* of x with respect to $\{A_t\}_{t \in \mathbf{R}}$ and we shall denote it by $sp_A(x)$. The basic facts about spectra may be found in [3].

If M is a subset of $\mathbf{R}$, we shall write C_M for the space $\{\varphi \in C(X) \mid sp_T(\varphi) \subseteq M\}$ where $\{T_t\}_{t \in \mathbf{R}}$ is the strongly continuous representation of $\mathbf{R}$ on $C(X)$ defined by the equation $(T_t\varphi)(x) = \varphi(x - t)$, $\varphi \in C(X)$. In Proposition 2.1 of [10] we indicated that $\mathfrak{A} = C_{[0, \infty)}$ and the proof outlined there shows that $C_{[\lambda, \infty)}$ is the set of all functions φ in $C(X)$ such that for each x in X , $\varphi(x + t)$ lies in $H^\infty[\lambda, \infty)$ as a function of t . An important fact used time and again throughout this paper is that the product of two functions, one from $C_{[s, \infty)}$ and the other from $C_{[t, \infty)}$, lies in $C_{[s+t, \infty)}$. This follows from the alternate description of $C_{[\lambda, \infty)}$ just given or from Forelli's important spectral-commutation theorem, Theorem 1 of [3].

All measures on X considered in this paper are finite, positive, and regular Borel. Just as on the Bohr group, a measure is called *quasi-invariant* in case its null sets are preserved under translation. If μ is a quasi-invariant measure on X , we shall write $\rho(t, x)$ for the derivative $(d\mu_t/d\mu)(x)$ where $\mu_t(M) = \mu(M - t)$ (M a Borel set) and we shall call it the ρ -function for μ . Actually, a little care must be taken in forming the ρ -function because it is defined only up to a μ -null set, one for each t . This ambiguity allows the possibility that suitable choices of it may not be measurable on the product space $\mathbf{R} \times X$. However, Propositions 1 and 1+ of [4] together with Theorem 2.6 of [1, Chapter II] show that the ρ -function may be chosen to be measurable on $\mathbf{R} \times X$. Henceforth, we assume that it is so chosen. Also, to simplify the exposition, we will often introduce the ρ -function of a quasi-invariant measure into the discussion without specifically stating that it is the ρ -function of the particular measure. This should cause no confusion since on the occasions when this is done, there will be only one quasi-invariant measure in sight.

If μ is a quasi-invariant measure on X , then the unitary representation $\{U_t\}_{t \in \mathbf{R}}$ of $\mathbf{R}$ on $L^2(\mu)$ defined by the formula $(U_tf)(x) = f(x - t)\rho^{1/2}(t, x)$, $f \in L^2(\mu)$, will be called the *permutation representation* of $\mathbf{R}$ on $L^2(\mu)$. Observe that by Proposition 1 of [4], it is strongly continuous. A Borel set M in X will be called *invariant* in case the characteristic function of M , χ_M , regarded as a multiplication operator on $L^2(\mu)$ commutes with the permutation representation of $\mathbf{R}$ on $L^2(\mu)$. This, of course, is the same as requiring that $\mu((M + t) \Delta M) = 0$ for all t . A quasi-invariant measure will be called *ergodic* in case the only invariant Borel subsets of X are null or have null complements.

When μ is a measure on X , we shall regard the functions in $L^\infty(\mu)$ as operators on $L^2(\mu)$ via multiplication, but we shall not introduce special notation to distinguish the roles which a function might be playing. The closure of a linear space $\mathfrak{N}$ in $L^2(\mu)$ will be denoted by $[\mathfrak{N}]_2$ but $[\mathfrak{A}]_2$ will usually be denoted by $H^2(\mu)$ for obvious reasons. The *distant future* in $L^2(\mu)$, the principal object of study in this paper, is defined to be $\bigwedge_{t \in \mathbf{R}} [C_{[t, \infty)}]_2$. (Here, $\bigwedge$ as well as $\bigvee$, when applied to subspaces of a Hilbert space, have their usual meaning.) Observe

that when X is the Bohr group what we are now calling the distant future coincides with the distant future described in the introduction.

§3. Lemmas. In this section we collect together several lemmas which we will need. Each one is at most a minor variation of a known result and so proofs will be abbreviated if given at all. For our first lemma, see Lemma 4 of [3].

Lemma 3.1. *Let $\{U_t\}_{t \in \mathbf{R}}$ be a strongly continuous unitary representation of $\mathbf{R}$ on a Hilbert space $\mathcal{H}$ and let E be its spectral measure. Then for each closed subset M in $\mathbf{R}$, $E(M)\mathcal{H} = \{f \mid \text{sp}_U(f) \subseteq M\}$. Also, $E(\{0\})\mathcal{H} = \{f \mid U_t f = f \text{ for all } t\}$.*

The following lemma follows from Theorem 2 in [3].

Lemma 3.2. *Let μ be a positive measure on X and suppose $\{\mathfrak{M}_t\}_{t \in \mathbf{R}}$ is a family of subspaces in $L^2(\mu)$ which satisfy the following conditions:*

- a) $\bigvee_{t \in \mathbf{R}} \mathfrak{M}_t = L^2(\mu)$, and $\bigwedge_{t \in \mathbf{R}} \mathfrak{M}_t = \{0\}$;
- b) if $t < s$, then $\mathfrak{M}_s \subseteq \mathfrak{M}_t$;
- c) $\mathfrak{M}_t = \bigwedge_{s < t} \mathfrak{M}_s$; and
- d) $C_{[s, \infty)} \mathfrak{M}_t \subseteq \mathfrak{M}_{s+t}$, for all $s, t \in \mathbf{R}$.

Then μ is quasi-invariant and there is a strongly continuous unitary representation $\{U_t\}_{t \in \mathbf{R}}$ of $\mathbf{R}$ on $L^2(\mu)$ such that the imprimitivity equation

$$(3.1) \quad U_t^* \varphi U_t = T_t \varphi, \quad \varphi \in C(X),$$

is satisfied and such that $\mathfrak{M}_t = E([t, \infty))L^2(\mu)$ for all t where E is the spectral measure of $\{U_t\}_{t \in \mathbf{R}}$.

Let μ be a quasi-invariant measure on X . A cocycle on X is a measurable function θ on $\mathbf{R} \times X$ such that

- a) $|\theta(t, x)| = 1$ a.e. (μ) for each t in $\mathbf{R}$;
- b) when regarded as a function from $\mathbf{R}$ into $L^2(\mu)$, θ is continuous; and
- c) for each pair $s, t \in \mathbf{R}$, $\theta(s + t, x) = \theta(s, x)\theta(t, x - s)$ a.e. (μ) .

The role of cocycles is displayed in the following lemma which is a paraphrase of a theorem of Mackey [9, Theorem 5.4].

Lemma 3.3. *Let μ be a quasi-invariant measure on X and let $\{U_t\}_{t \in \mathbf{R}}$ be a strongly continuous unitary representation of $\mathbf{R}$ on $L^2(\mu)$ which satisfies equation (3.1). Then there is a cocycle θ on X such that*

$$(3.2) \quad (U_t f)(x) = \theta(t, x)f(x - t)\rho^{1/2}(t, x), \quad f \in L^2(\mu), \quad t \in \mathbf{R}.$$

Conversely, given a cocycle on X , the family $\{U_t\}_{t \in \mathbf{R}}$ of unitary operators on $L^2(\mu)$ defined by equation (3.2) is a strongly continuous unitary representation of $\mathbf{R}$ on $L^2(\mu)$ which satisfies equation (3.1).

Lemma 3.4. *Let μ be a quasi-invariant measure on X , let θ be a cocycle on X , and let $\{U_t\}_{t \in \mathbf{R}}$ be the strongly continuous unitary representation of $\mathbf{R}$ on $L^2(\mu)$ defined by equation (3.2). Then for f in $L^2(\mu)$, $\text{sp}_U(f) \subseteq [0, \infty)$ if and only if there*

is a μ -null N such that for all $x \notin N$ the function of t , $\theta(-t, x)f(x+t)\rho^{1/2}(-t, x)$, belongs to $H^2(\gamma)$.

Proof. First observe that since $\{U_t\}_{t \in \mathbf{R}}$ is a strongly continuous unitary representation of $\mathbf{R}$ on $L^2(\mu)$, the following equation is valid for each f in $L^2(\mu)$: $\|f\|^2 = \int_{\mathbf{R}} \|U_{-t}f\|^2 d\gamma = \int_{\mathbf{R}} \int_X |f(x+t)|^2 \rho(-t, x) d\mu d\gamma$. Hence by Fubini's theorem, for each f in $L^2(\mu)$ there is an exceptional null set off of which $\theta(-t, x)f(x+t)\rho^{1/2}(-t, x)$, as a function of t , belongs to $L^2(\gamma)$. Thus the assertion in the lemma makes sense.

The argument proving equation (30) on page 50 in [3] may be easily modified and used to prove the assertion that for f in $L^2(\mu)$, $sp_U(f) \subseteq [0, \infty)$ if and only if for each g in $L^2(\mu)$ the function of t , $(U_{-t}f, g)$, belongs to $H^\infty(\gamma)$. At this point the proof is completed just as in the last paragraph of text on page 192 in [6].

Our final lemma is an analogue for flows of the Wold decomposition of an isometry (cf. §3 of [7] also).

Lemma 3.5. *Let μ be a measure on X , and let $\mathfrak{D}$ be the distant future in $L^2(\mu)$. Then there is a Borel set $M \subseteq X$ such that $\mathfrak{D} = \chi_M L^2(\mu)$. Moreover, the distant future in $L^2((1 - \chi_M)\mu)$ consists of the zero vector alone while in $L^2(\chi_M \mu)$ the distant future is everything.*

Proof. We attend only to the first assertion since the second is clear. To show that $\mathfrak{D}$ has the indicated form, it suffices to show that $\mathfrak{D}$ is invariant under $C(X)$, and for this it suffices, by Lemma 3 of [3], to show that $\varphi\mathfrak{D} \subseteq \mathfrak{D}$ for each φ in $C(X)$ such that $sp_T(\varphi)$ is compact. But each such φ is contained in $C_{[s, \infty)}$ for some s and so for each $t \in \mathbf{R}$, $\varphi[C_{[t, \infty)}]_2 \subseteq [C_{[s+t, \infty)}]_2$. This implies that $\varphi\mathfrak{D} \subseteq \mathfrak{D}$.

§4. The main theorem. In this section we present our solution to the prediction problem.

Theorem I. *Let μ be a positive measure on X . Then the distant future in $L^2(\mu)$ is the zero subspace if and only if 1) μ is quasi-invariant and 2) $\log \rho(t, x)$ belongs to $L^1(\gamma)$ as a function of t for all x outside a μ -null set.*

Proof. Suppose the distant future in $L^2(\mu)$ vanishes and for t in $\mathbf{R}$ set $\mathfrak{M}_t = \bigwedge_{s < t} [C_{[s, \infty)}]_2$. Observe that $\bigvee_{t \in \mathbf{R}} \mathfrak{M}_t = L^2(\mu)$; if $s < t$, $\mathfrak{M}_t \subseteq \mathfrak{M}_s$; for each t , $\mathfrak{M}_t = \bigwedge_{s < t} \mathfrak{M}_s$; and $C_{[s, \infty)} \mathfrak{M}_t \subseteq \mathfrak{M}_{s+t}$ for all $s, t \in \mathbf{R}$. Since $\bigwedge_{t \in \mathbf{R}} \mathfrak{M}_t = \{0\}$ by hypothesis, Lemma 3.2 implies that μ is quasi-invariant. Also, Lemma 3.2 and 3.3 imply that there is a cocycle θ such that if $\{U_t\}_{t \in \mathbf{R}}$ is the representation of $\mathbf{R}$ on $L^2(\mu)$ it determines via equation (3.2), then $\mathfrak{M}_0 = \{f \in L^2(\mu) \mid sp_U(f) \subseteq [0, \infty)\}$. Since the constant function 1 lies in $\mathfrak{M}_0$, Lemma 3.4 implies that for almost all x , $\theta(-t, x)\rho^{1/2}(-t, x)$ belongs to $H^2(\gamma)$ as a function of t . This, of course, implies that $\log \rho(t, x)$ lies in $L^1(\gamma)$ as a function of t for almost all x .

For the converse, suppose that μ satisfies the conditions 1) and 2). By Lemma 3.5, there is a Borel set $M \subseteq X$ such that the distant future in $L^2(\mu)$ is $\chi_M L^2(\mu)$

while the distant future in $L^2((1 - \chi_M)\mu)$ vanishes. By what was just shown, $(1 - \chi_M)\mu$ is quasi-invariant and since μ is also quasi-invariant by hypothesis, Lemmas 6 and 7 of [3] imply that $\chi_M\mu$ is quasi-invariant. Moreover, it is easy to see that $\log(\chi_M\rho(t, x))$ belongs to $L^1(\gamma)$ as a function of t for all x outside a $\chi_M\mu$ -null set. Hence to show that $\mu(M) = 0$, we may assume that μ satisfies 1) and 2) and that the distant future in $L^2(\mu)$ is everything. We will reach a contradiction by showing that if μ is not zero, then 1) and 2) imply that the constant function 1 does not belong to $[C_{[\lambda, \infty)}]_2$ for each $\lambda > 0$.

First observe that the first part of the argument for Lemma 3.4 shows that for each x outside a null set the function of t , $\rho(-t, x)$, belongs to $L^1(\gamma)$. For each such x , we let $L^2(x)$ denote the L^2 -space based on the measure whose derivative with respect to γ is $\rho(-t, x)$, and for all other x , we let $L^2(x)$ denote the zero space. Fix $\lambda > 0$, and let $\delta(x)$ be the distance from the constant function 1 to the closure of $H^\infty[\lambda, \infty)$ in $L^2(x)$. Choose a sequence $\{\varphi_n\}_{n=1}^\infty$ in $H^\infty[\lambda, \infty)$ which is uniformly dense in the intersection of $H^\infty[\lambda, \infty)$ with the space of continuous functions on $\mathbf{R}$ which admit continuous extensions to the point at infinity. Then this sequence is dense in the closure of $H^\infty[\lambda, \infty)$ in $L^2(x)$ for each x in X and hence $\delta(x) = \inf \|1 - \varphi_n\|_{L^2(x)}$. By Fubini's theorem each term in this infimum is measurable, and so δ is measurable. On the other hand, hypothesis 2) implies that $\delta(x) > 0$ a.e. (μ) and so $\int_X \delta^2(x) d\mu > 0$. Now observe that for each $\varphi \in L^2(\mu)$ and each $t \in \mathbf{R}$, $\|1 - \varphi\|^2 = \int_X |1 - \varphi(x + t)|^2 \rho(-t, x) d\mu$. Hence if φ lies in $C_{[\lambda, \infty)}$, then, as was noted in section 2, $\varphi(x + t)$ lies in $H^\infty[\lambda, \infty)$ as a function of t for each x , and so the following inequality completes the proof:

$$\begin{aligned} \|1 - \varphi\|^2 &= \int_{\mathbf{R}} \left[\int_X |1 - \varphi(x + t)|^2 \rho(-t, x) d\mu \right] d\gamma \\ &= \int_X \left[\int_{\mathbf{R}} |1 - \varphi(x + t)|^2 \rho(-t, x) d\gamma \right] d\mu \geq \int_X \delta^2(x) d\mu > 0. \end{aligned}$$

§5. Corollaries and extensions. Our first corollary is the analogue for flows of the theorem of Helson and Lowdenslager discussed in the introduction. Since the proof outlined there works here as well, it will be omitted.

Corollary 5.1. *Let m be an invariant measure on X and let w be a non-negative measurable function on X . Then the distant future in $L^2(wdm)$ is the zero subspace if and only if the support of w is invariant and $\log w(x + t)$ lies in $L^1(\gamma)$ as a function of t for almost x in the support of w .*

Corollary 5.2. *If μ is an ergodic quasi-invariant measure on X , then either the distant future in $L^2(\mu)$ consists of the zero function alone, or else it is all of $L^2(\mu)$.*

Proof. By Lemma 3.4 there is a Borel set $M \subseteq X$ such that the distant future in $L^2(\chi_M\mu)$ is all of $L^2(\chi_M\mu)$ while the distant future in $L^2((1 - \chi_M)\mu)$ vanishes. By Theorem I, $(1 - \chi_M)\mu$ is quasi-invariant and so by Lemma 9

in [3], either $\mu(M) = 0$ or $\mu(X \sim M) = 0$. The first alternative implies that the distant future in $L^2(\mu)$ vanishes while the second implies that it is everything.

Let μ be a measure on X . Following Helson and Lowdenslager, we shall call $H^2(\mu) \ominus [C_{(0,\infty)}]_2$ the *innovation subspace* of $H^2(\mu)$.

Lemma 5.3. *Let μ be a measure on X . If f lies in $\{\bigwedge_{s<0} [C_{[s,\infty)}]_2\} \ominus [C_{(0,\infty)}]_2$ (in particular, if f lies in the innovation subspace of $H^2(\mu)$), then the measure $|f|^2 d\mu$ is invariant.*

Proof. Since the innovation subspace is orthogonal to the distant future, we may assume without loss of generality that the latter is the zero subspace. Therefore, as in the proof of Theorem I, we may form the strongly continuous unitary representation $\{U_t\}_{t \in \mathbb{R}}$ whose spectral measure E satisfies the equation $E([t, \infty))L^2(\mu) = \bigwedge_{s<t} [C_{[s,\infty)}]_2$, and we find that $[C_{(0,\infty)}]_2 = \bigvee_{r>0} \{C_{[r,\infty)}\}_2 = \bigvee_{r>0} E([r, \infty))L^2(\mu) = E((0, \infty))L^2(\mu)$. Hence $\{\bigwedge_{s<0} [C_{[s,\infty)}]_2\} \ominus [C_{(0,\infty)}]_2$ is the range of $E([0, \infty)) - E((0, \infty)) = E(\{0\})$ and so, by Lemma 3.1, consists of those vectors f in $L^2(\mu)$ such that $U_t f = f$ for all t . But by Lemmas 3.2 and 3.3, $\{U_t\}_{t \in \mathbb{R}}$ is determined via equation (3.2) by a suitable cocycle. Hence, if $U_t f = f$ for all t , then $|f(x)|^2 = |f(x-t)|^2 \rho(t, x)$ a.e. (μ) for each t ; and it follows that the measure $|f|^2 d\mu$ is invariant.

Corollary 5.4. *If a measure μ on X is singular with respect to each invariant measure, then the innovation subspace of $H^2(\mu)$ consists of the zero function alone.*

Remark 5.5. Forelli has shown us an example where, in the notation of Lemma 5.3, the innovation subspace is smaller than $\{\bigwedge_{s<0} [C_{[s,\infty)}]_2\} \ominus [C_{(0,\infty)}]_2$ but in this example the algebra $\mathfrak{A}$ of analytic functions on X does not separate the points of X . If we assume that $\mathfrak{A}$ separates the points of X , then we are unable to decide whether or not there may be proper inclusion. Observe that in the Bohr group setting this is the problem raised on page 183 of [7].

The following is an analogue for flows of the first part of Theorem 5 in [7].

Theorem II. *Let μ be an ergodic quasi-invariant measure on X . Then the innovation subspace of $H^2(\mu)$ is non-zero if and only if 1) μ is equivalent to an invariant measure m on X and 2) if $w = d\mu/dm$, then $\log w \in L^1(m)$.*

Proof. Suppose μ satisfies 1) and 2). Then, of course, m is ergodic. If m is normalized to have total mass 1, then by Theorem I of [10] (and its proof), m is a representing measure for a point in the maximal ideal space of $\mathfrak{A}$, $\mathfrak{A}$ is a weak-* Dirichlet algebra in $L^\infty(m)$, and $\int \varphi dm = 0$ for all φ in $C_{(0,\infty)}$. Hence by Szegő's theorem for weak-* Dirichlet algebras [13, Theorem 2.5.5], $\|1 - \varphi\| \geq \inf \{\|1 - \varphi\| \mid \varphi \in \mathfrak{A}, \int_X \varphi dm = 0\} = \exp \{\int_X \log w dm\} > 0$ for all φ in $C_{(0,\infty)}$, and this shows that the innovation subspace is non-zero. For the converse, suppose that the innovation subspace is non-zero and let f be a non-zero function in it. By Lemma 5.3, the measure $dm = |f|^2 d\mu$ is invariant, and by Lemma 9 of [3] it is ergodic. Hence if it is normalized to have total mass 1, then $\mathfrak{A}$ is a

weak-* Dirichlet algebra in $L^\infty(m)$ as before, and recalling the fact proved in the proof of Theorem I in [10] that $\{\varphi \in \mathfrak{A} \mid \int \varphi dm = 0\}$ is the intersection of the weak-* closure of $C_{(0,\infty)}$ with $\mathfrak{A}$, we find from Szegő's theorem again that $0 < \inf \left\{ \int |1 - \varphi|^2 d\mu \mid \varphi \in C_{(0,\infty)} \right\} = \exp \left\{ \int \log (1/|f|^2) dm \right\}$ which completes the proof.

§6. Szegő measures and representations. A measure μ on X is called a *Szegő measure* (with respect to $\mathfrak{A}$) in case the inclusion $\chi_M L^2(\mu) \subseteq H^2(\mu)$ (M a Borel set) implies that $\mu(M) = 0$. In [2] Foias and Suciú introduced the notion of a Szegő measure and illustrated its importance in the representation theory of function algebras. In this section we apply their analysis to investigate $\mathfrak{A}$. Observe that by Lemma 3.5, if μ is a Szegő measure on X , then the distant future in $L^2(\mu)$ is the zero subspace. The converse of this is not quite true; but it is, if a certain sort of trivial case is excluded.

Lemma 6.1. *Let μ be a quasi-invariant measure and let M be a Borel subset of X . Suppose there is a μ -null set N such that for each $x \notin N$, either $x + t$ belongs to M for all t outside a Lebesgue null set or else $x + t$ belongs to $X \sim M$ for all t outside a Lebesgue null set (the Lebesgue null set may vary with x). Then M is invariant.*

Proof. Since μ is quasi-invariant, the function of t , $\mu((M + t) \Delta M)$, is continuous by Proposition 1 of [4]. Hence to show this function is zero, it suffices to show that its integral with respect to γ is zero. But this follows from the hypothesis and Fubini's theorem:

$$\int_{\mathbf{R}} \mu((M + t) \Delta M) d\gamma = \int_{\mathbf{R}} \int_X \chi_{(M+t) \Delta M} d\mu d\gamma = \int_X \int_{\mathbf{R}} \chi_{(M+t) \Delta M} d\gamma d\mu = 0.$$

Lemma 6.2. *Let μ be a quasi-invariant measure on X . Then the following assertions are equivalent.*

- 1) *The permutation representation of $\mathbf{R}$ on $L^2(\mu)$ is the identity representation.*
- 2) *Every point in the closed support of μ is fixed.*
- 3) *The distant future in $L^2(\mu)$ consists of the zero vector alone and $H^2(\mu) = L^2(\mu)$.*

Proof. To see that 1) implies 2) suppose that the permutation representation $\{U_t\}_{t \in \mathbf{R}}$ of $\mathbf{R}$ on $L^2(\mu)$ is the identity representation. Since $\{U_t\}_{t \in \mathbf{R}}$ satisfies the imprimitivity equation, equation (3.1), we find that $\varphi = T_t \varphi$ a.e. (μ) for each t in $\mathbf{R}$ and each φ in $C(X)$. By regularity, we may conclude that $\varphi(x) = (T_t \varphi)(x)$ for all t in $\mathbf{R}$, all φ in $C(X)$, and every x in the closed support of μ . The assertion 2) follows. Since 2) clearly implies 3), we attend to the passage from 3) to 1). Suppose that 3) holds and, as in the proof of Theorem I, let $\{U_t\}_{t \in \mathbf{R}}$ be the strongly continuous unitary representation of $\mathbf{R}$ on $L^2(\mu)$ whose spectral measure E satisfies the equation $E([t, \infty))L^2(\mu) = \bigwedge_{s < t} [C_{[s, \infty)}]_2$. By Lemmas 3.2 and 3.3 there is a cocycle θ which determines $\{U_t\}_{t \in \mathbf{R}}$ via equation (3.2). By Lemma 3.4 and the fact that $L^2(\mu) = H^2(\mu) = E([0, \infty))L^2(\mu)$,

we find that for each f in $L^2(\mu)$, there is a μ -null set of x outside of which the function of t , $\theta(-t, x)f(x+t)\rho^{1/2}(-t, x)$, belongs to $H^\infty(\gamma)$. If $f = \chi_M$ for some Borel set M , then we may conclude that for each x outside a μ -null set either $x+t$ lies in M for almost all t , or $x+t$ lies in $X \sim M$ for almost all t . Hence, by Lemma 6.1, M is invariant. Since this is so for each Borel set, we may conclude that the permutation representation of $\mathbf{R}$ on $L^2(\mu)$ is the identity.

Theorem III. *A measure μ on X is a Szegő measure if and only if 1) the distant future in $L^2(\mu)$ is the zero subspace and 2) there is no invariant Borel set $M \subseteq X$ of positive measure such that the permutation representation of $\mathbf{R}$ on $L^2(\chi_M\mu)$ is the identity representation.*

Proof. We have already indicated that if μ is a Szegő measure then 1) holds; but 2) also holds by Lemma 6.2. For the converse, suppose that 1) and 2) hold. By Theorem 4 of [2] we may find a Borel set $M \subseteq X$ such that $(1 - \chi_M)\mu$ is a Szegő measure and $H^2(\chi_M\mu) = L^2(\chi_M\mu)$. Since the distant futures in $L^2(\mu)$ and $L^2((1 - \chi_M)\mu)$ vanish, μ and $(1 - \chi_M)\mu$ are quasi-invariant by Theorem I and so by Lemmas 6 and 7 of [3], M is invariant and $\chi_M\mu$ is quasi-invariant. By Proposition 2 of [5], the distant future in $L^2(\chi_M\mu)$ vanishes. Hence by hypothesis 2) and Lemma 6.2, we may conclude that $\mu(M) = 0$ and that μ is a Szegő measure.

The following corollary is an immediate consequence of Lemma 6.2 and Theorem III.

Corollary 6.3. *Suppose that there are no points in X which are fixed by the action of $\mathbf{R}$. Then a measure μ on X is a Szegő measure if and only if the distant future in $L^2(\mu)$ is the zero subspace.*

Let $\mathcal{H}$ be a (complex) Hilbert space and denote the algebra of bounded linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. A homomorphism σ of $\mathfrak{A}$ into $\mathcal{L}(\mathcal{H})$ such that $\|\sigma(\varphi)\|_{\mathcal{L}(\mathcal{H})} \leq \|\varphi\|_\infty$ and such that $\sigma(1)$ is the identity operator on $\mathcal{H}$ is called a (contractive) *representation* of $\mathfrak{A}$ on $\mathcal{H}$. A C^* -representation π of $C(X)$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ is called a *dilation* of σ in case $\sigma(\varphi) = P_{\mathcal{H}}\pi(\varphi)|_{\mathcal{H}}$, $\varphi \in \mathfrak{A}$, where $P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$ and where the vertical bar denotes restriction. The representation σ is called *pure* provided that there is no non-zero subspace $\mathfrak{M}$ of $\mathcal{H}$ which reduces σ such that the representation obtained by restricting σ to $\mathfrak{M}$ extends to a C^* -representation of $C(X)$. The dilation π is called *minimal* in case the smallest subspace of $\mathcal{K}$ containing $\mathcal{H}$ and reducing π is $\mathcal{H}$ itself.

While it is known that there are function algebras with representations which do not have dilations, we do not know whether or not every representation of the algebra $\mathfrak{A}$ of analytic functions associated with an arbitrary flow has a dilation. If $\mathfrak{A}$ is a Dirichlet algebra, as is the case when the flow is strictly ergodic [10, Theorem II], then each representation of $\mathfrak{A}$ has a dilation by Theorem 6 of [2].

Recall that every C^* -representation π of $C(X)$ is determined by a spectral

measure E on X via the equation $\pi(\varphi) = \int_X \varphi dE$, $\varphi \in C(X)$. As with scalar measures, we shall say that a spectral measure E is *quasi-invariant* in case its null sets are preserved under translation. We say that a scalar measure μ on X *determines the measure class* of E in case E and μ have the same null sets. Finally, if E is a spectral measure on X and if f is a vector in the space where E takes its values, then we shall write μ_f for the measure defined by the formula $\mu_f(M) = (E(M)f, f)$ where M is a Borel set in X .

The following theorem extends the well-known fact that spectral measure of the minimal unitary dilation of a completely non-unitary contraction is absolutely continuous; it also extends the results in [11], but the proofs are different.

Theorem IV. *Let σ be a pure representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$, suppose that σ has a minimal dilation π acting on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$, and let E be the spectral measure for π . Then a) for each non-zero vector f in $\mathfrak{H}$, μ_f is quasi-invariant; b) E is quasi-invariant; and c) if there is a scalar measure μ which determines the measure class of E (so that by b) μ is quasi-invariant) and if for $f \in \mathfrak{H}$, $F_f = d\mu_f/d\mu$, then for each non-zero vector f in $\mathfrak{H}$, μ_f is supported on the set of those x such that $\log(F_f(x - t)\rho(t, x))$ belongs to $L^1(\gamma)$ as a function of t .*

Proof. a) By Theorem 7 in [2], μ_f is a Szegő measure on X and so by Theorems I and III, μ_f is quasi-invariant.

b) If M is a Borel set such that $E(M) \neq 0$, then by the minimality of π , there is a vector f in $\mathfrak{H}$ such that $\mu_f(M) = (E(M)f, f) \neq 0$. Since μ_f is quasi-invariant, $(E(M + t)f, f) \neq 0$ for all t , and so E is quasi-invariant.

c) Since μ_f is quasi-invariant by a), the ρ -function of μ_f is $F_f(x - t)\rho(t, x)/F_f(x)$. Since μ_f is a Szegő measure, Theorems I and III yield the result.

§7. A final remark. It is natural to ask how likely it is that a given quasi-invariant measure μ has the property that the distant future in $L^2(\mu)$ vanishes. The following proposition exhibits a sense in which it is very likely.

Proposition 7.1. *If ν is a non-zero quasi-invariant measure on X , then there is a measure μ equivalent to ν such that the distant future in $L^2(\mu)$ is the zero subspace.*

Proof. First we assert that there is a non-zero measure μ_1 which is absolutely continuous with respect to ν such that the distant future in $L^2(\mu_1)$ vanishes. Let $\{U_t\}_{t \in \mathbf{R}}$ be the permutation representation of $\mathbf{R}$ on $L^2(\nu)$, let f be a non-zero vector in $L^2(\nu)$ such that $sp_\nu(f) \subseteq [0, \infty)$, and let $d\mu_1 = |f|^2 d\nu$. (To see that such a vector exists, observe that since ρ is real, the argument in the second paragraph of page 48 in [3] shows that $sp_\nu(\bar{f}) = -sp_\nu(f)$ for all f in $L^2(\nu)$. Hence since ν is non-zero, there is a non-zero f in $L^2(\nu)$ such that $sp_\nu(f) \cap [0, \infty) \neq \emptyset$, and so, by Lemma 3.1, we may find a non-zero f so that $sp_\nu(f) \subseteq [0, \infty)$.) By Lemma 3.4, μ_1 is supported by a set M with the property that for each x in X , but outside a null set for ν , either $x + t$ lies in M for almost all t or $x + t$ lies in $X \sim M$ for almost all t . Hence, by Lemma 6.1, M is invariant,

and so μ_1 is quasi-invariant. Moreover, since the ρ -function, ρ_1 , for μ_1 is $|f(x-t)|^2 \rho(t, x)/|f(x)|^2$, it follows from Lemma 3.4 that $\log \rho_1(t, x)$ belongs to $L^1(\gamma)$ as a function of t for all x outside a μ_1 -null set. Hence by Theorem I, the distant future in $L^2(\mu_1)$ vanishes. The rest of the proof follows Forelli's argument in §2.4 of [5]. Select a maximal collection $\mathfrak{J}$ of non-zero functions in $L^2(\nu)$ with the properties that $sp_\nu(f) \subseteq [0, \infty)$ for all f in $\mathfrak{J}$ and that if f and f' belong to $\mathfrak{J}$, then $ff' = 0$. We just showed that $\mathfrak{J}$ is non-empty. Since ν is finite, $\mathfrak{J}$ is at most countable, and so we may normalize the elements of $\mathfrak{J}$ so that the series $\sum_{f \in \mathfrak{J}} f$ converges both in mean and pointwise to a function F . Set $d\mu = |F|^2 d\nu$. Since $sp_\nu(F) \subseteq [0, \infty)$, the argument given above for μ_1 shows that the distant future in $L^2(\mu)$ vanishes. Since the support of μ is invariant, so is its complement and unless its complement is a null set for ν , we may apply the above reasoning to produce a measure μ_2 singular with respect to μ but absolutely continuous with respect to ν such that the distant future in $L^2(\mu_2)$ also vanishes. Since this would contradict the maximality of $\mathfrak{J}$, we may conclude that μ and ν are equivalent.

REFERENCES

1. J. DOOB, *Stochastic processes*, John Wiley and Sons, New York, 1953.
2. C. FOIAS & I. SUCIU, *Szegő-measures and spectral theory in Hilbert spaces*, Rev. Roumaine Math. Pures Appl. **XI**, 2 (1966), 147–159.
3. F. FORELLI, *Analytic and quasi-invariant measures*, Acta Math. **118** (1967), 33–59.
4. ———, *Conjugate functions and flows*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 215–33.
5. ———, *What makes a positive measure the total variation measure of an analytic measure?*, J. London Math. Soc. (2) **2** (1970), 713–718.
6. T. GAMELIN, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, 1969.
7. H. HELSON & D. LOWDENSLAGER, *Prediction theory and Fourier series in several variables. II*, Acta Math. **106** (1961), 176–213.
8. A. KOLMOGOROFF, *Stationary sequences in Hilbert space*, Boll. Moskov. Gos. Univ. **2** (1941), 40 pp. (in Russian).
9. G. MACKEY, *Unitary representations of group extensions I*, Acta Math. **99** (1958), 265–311.
10. P. MUHLY, *Function algebras and flows I*, Acta Sci. Math. (Szeged) **35** (1973), 111–121.
11. ———, *Some remarks on the spectra of unitary dilations*, Studia Math. **XLIX** (1974), 139–147.
12. P. MUHLY, *Function algebras and flows IV*.
13. T. SRINIVASAN, & J.-K. WANG, *Weak-* Dirichlet algebras*, Proc. of an International Symposium held at Tulane University, F. T. Birtel, ed., 1965, 216–249.
14. G. SZEGÖ, *Beiträge zur Theorie der Toeplitzschen Formen (Erste Mitteilung)*, Math. Z. **6** (1920), 167–202.

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