Special Limit Points for Fuchsian Groups and Automorphic Functions near the Limit Set

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1. Introduction and statement of results. A Fuchsian group G is a discrete group of bilinear transforms of the form

$$Vz = \frac{az + b}{cz + d}, \quad ad - bc = 1$$

where a, b, c, d and z are complex numbers and the elements of G preserve a disc (or half plane). We will consider only groups which preserve the unit disc (any Fuchsian group is conjugate to such a group) and we use Δ to denote the open unit disc. In this case the elements of G are of the form

$$(1) \quad Vz = \frac{az + \bar{c}}{cz + \bar{a}}, \quad |a|^2 - |c|^2 = 1.$$

If the transform V does not fix ∞ ($c \neq 0$) then the isometric circle of V is the circle $\{z: |cz + \bar{a}| = 1\}$ with center $-\bar{a}/c$ and radius $|c|^{-1}$.

The limit set of a Fuchsian group G is the set of points of accumulation of centers of isometric circles belonging to transforms in G (see [3], p. 42 for further details). Following Beardon and Maskit [1] we distinguish a class of limit points, called points of approximation, which are approximated very well by centers of isometric circles.

A limit point ξ is called a *point of approximation* if there is a constant $k > 0$ and a sequence $\{V_n\}$ of distinct elements of G such that, for all n ,

$$|\xi - c(V_n)| < k(r(V_n))^2$$

where $c(V_n)$ and $r(V_n)$ denote the center and radius respectively of the isometric circle of V_n .

The following result (Theorem A) gives some characterisations of points of approximation. We remark that this result is the planar version of a Theorem of Beardon and Maskit [1] which they proved for Kleinian groups in Euclidean 3-space.

Theorem A. *Let G be a Fuchsian group preserving Δ , let ξ be a limit point of G and let L denote the radius of Δ with end point ξ . The following statements are equivalent:*

- (i) ξ is a point of approximation.
- (ii) *There exists a sequence $\{V_n\}$ of distinct elements of G and a constant $k > 0$ such that*

$$|V_n(0) - V_n(\xi)| > k.$$

- (iii) *There exists a sequence $\{V_n\}$ of distinct elements of G and a constant ρ with $0 < \rho < 1$ such that*

$$V_n(L) \cap \{|z| < \rho\} \neq \emptyset$$

for each n .

Following Tsuji ([8] p. 510) we introduce a metric $[a, b]$ on Δ defined by

$$[a, b] = \left| \frac{a - b}{1 - \bar{a}b} \right|.$$

For $a \in \Delta$ and ρ satisfying $0 < \rho < 1$, we denote by $C(a, \rho)$ the disc

$$\{z: [z, a] \leq \rho\}.$$

Note that if V is a bilinear transform preserving Δ , then, for any $a, b \in \Delta$,

$$[V(a), V(b)] = [a, b]$$

from which

$$V(C(a, \rho)) = C(V(a), \rho).$$

We remark that if AB is a diameter of Δ then the union of all discs $C(a, \rho)$ where a lies on AB , is a lens shaped region bounded by two arcs of Euclidean circles through A and B (see [2] p. 23). We shall refer to this region as a lens on AB of opening ρ and denote it by $S(A, \rho)$.

To each limit point ξ of a Fuchsian group G we associate a set $\lambda(\xi)$ which is defined as follows: if $a \in \Delta$ and there is a sequence $\{V_n\}$ of distinct elements of G such that $\{V_n(a)\}$ converges to ξ and for any $\delta > 0$ $V_n(a)$ is contained in the lens $S(\xi, \delta)$ for all n large enough, then $a \in \lambda(\xi)$. Thus $\lambda(\xi)$ may be regarded as the set of points in Δ whose orbits approach ξ almost radially. Clearly if $a \in \lambda(\xi)$ then $V(a) \in \lambda(\xi)$ for any $V \in G$, thus $\lambda(\xi)$ is a G invariant subset of Δ .

In this paper we present a further characterisation of points of approximation based on the set $\lambda(\xi)$ (Theorem 1). We investigate $\lambda(\xi)$ for various limit points (Theorems 2 and 3) and we use this information to obtain some results about the behaviour of an automorphic function on radial approach to a limit point (Theorem 4).

Before stating our results we need some more notation. If S is a subset of Δ and G a Fuchsian group, then $G(S)$ denotes those points contained in S or one of its G -images.

A hyperbolic line (h -line) is the arc, in Δ , of a circle orthogonal to the unit circle.

Theorem 1. *Let G be a Fuchsian group and ξ a limit point for G .*

- (i) $\lambda(\xi)$ is empty if and only if ξ is not a point of approximation.
- (ii) If ξ is a point of approximation, then there exists an h -line M such that $\lambda(\xi)$ contains $G(M)$.

If ξ is a fixed point of a hyperbolic element of G then we can describe $\lambda(\xi)$ completely.

Theorem 2. *If G is a Fuchsian group and H is a hyperbolic element of G with fixed points ξ and η , then if M denotes the h -line with end points ξ and η we have*

$$\lambda(\xi) = \lambda(\eta) = G(M).$$

We consider also the transitive points of G . If G is a Fuchsian group of the first kind (the limit set of G is $\partial\Delta$), $\xi \in \partial\Delta$ and L is the radius of Δ with end point ξ , then G is said to be a transitive point if $G(L)$ is dense in Δ . We have the following result.

Theorem 3. *If G is a Fuchsian group of the first kind then ξ is a transitive point of G if and only if $\lambda(\xi) = \Delta$.*

P. J. Myrberg [6] has shown that if G is finitely generated and of the first kind then the set of intransitive points has measure zero and may be uncountable. However, no general classification of such points is known. In this connection the following easy consequence of Theorems 1, 2 and 3 is of some interest.

Corollary 1. *If G is a Fuchsian group of the first kind and ξ is fixed by a hyperbolic element of G , then ξ is a point of approximation and is intransitive.*

The fact that ξ is a point of approximation is a result of Beardon and Maskit [1] and may be deduced directly from Theorem A.

Let G be a Fuchsian group and we suppose from now on that G contains no element fixing the origin. Then D_0 , the Dirichlet fundamental polygon of G centered at the origin, may be defined (see [5], p. 146). We define an automorphic function for G to be a nonconstant function f with the following properties:

- (i) f is meromorphic in Δ ,
- (ii) f is invariant under G

and

- (iii) $f(z)$ tends to a finite or infinite limit as z tends to a parabolic vertex in D_0 .

Thus if P is a parabolic vertex of D_0 and z approaches P radially then $f(z)$ tends to a definite value. Denote the set of such values by $A(G, f)$. If $\xi \in \partial\Delta$ we define the radial cluster set of f at ξ , $R(f, \xi)$, as follows: if w is a complex number and there exists a sequence of points $\{z_n\}$ approaching ξ radially with $f(z_n) \rightarrow w$ as $n \rightarrow \infty$ then $w \in R(f, \xi)$.

It is known that if ξ is a transitive point for a finitely generated Fuchsian group of the first kind then $R(f, \xi)$ is the extended complex plane ([5] p. 331). In addition, Lehner ([5] p. 332) and Rosen [7] have results describing vertical cluster sets for intransitive points in the case of Modular and Hecke groups respectively in the upper half plane.

Our next result gives a more complete description of $R(f, \xi)$ in certain cases.

Theorem 4. *If G is a Fuchsian group, ξ is a limit point of G and f is an automorphic function for G , then*

- (i) $R(f, \xi)$ contains $f(\lambda(\xi))$.
- (ii) *If ξ and η are fixed by a hyperbolic element of G and M is the h -line with end points ξ and η ,*

$$R(f, \xi) = f(\lambda(\xi)) = f(M).$$

- (iii) *If G is finitely generated and of the first kind, then, for any limit point ξ , $R(f, \xi)$ is contained in $f(\lambda(\xi)) \cup A(G, f)$.*

From Theorems 1 and 4(i) we obtain

Corollary 2. *If G is a Fuchsian group, ξ is a point of approximation and f is an automorphic function for G , then $R(f, \xi)$ contains the f -image of an h -line.*

Now suppose G is finitely generated of the first kind, ξ is not fixed by a parabolic transform of G and L denotes the radius of Δ with end point ξ . Then Hedlund ([4] p. 538) has shown that there exists a sequence $\{V_n\}$ of distinct elements of G and a constant ρ , $0 < \rho < 1$, such that $V_n(L) \cap \{|z| < \rho\} \neq \emptyset$ for each n . It follows from Theorem A that ξ is a point of approximation. Thus from Theorems 1 and 4 we have

Corollary 3. *Let G be a finitely generated Fuchsian group of the first kind and f is an automorphic function for G ; then*

- (i) *f has a radial limit at ξ if and only if ξ is a parabolic vertex,*
- (ii) *if ξ is not a parabolic vertex $R(f, \xi)$ contains the f -image of an h -line.*

The proofs of the theorems are given in the following sections.

2. Proof of Theorem 1. We define an h -ray to be an arc AB of a circle orthogonal to $\partial\Delta$ with $A \in \partial\Delta$ and $B \in \Delta$. Let M be an h -ray, G a Fuchsian group and $a \in \Delta$. We say $G(M)$ approximates a if there exists a sequence $\{V_n\}$ of distinct transforms of G such that, for any $\rho > 0$, $V_n(M)$ meets $C(a, \rho)$ for n large enough.

We have the following result.

Lemma 1. *If G is a Fuchsian group, $\xi \in \partial\Delta$ and M is any h -ray with end point ξ , then $a \in \lambda(\xi)$ if and only if $G(M)$ approximates a .*

Proof. It is clearly sufficient to consider only the case where M is the radius to ξ . Now $a \in \lambda(\xi)$ if and only if there exists a sequence $\{V_n\}$ of distinct elements of G such that $\{V_n(a)\}$ converges to ξ and, for every $\rho > 0$, $V_n(a) \in S(\xi, \rho)$ for n large enough. Thus, from the definition of $S(\xi, \rho)$, $[V_n(a), M] \leq \rho$ for n large enough and this is equivalent to $[a, V_n^{-1}(M)] \leq \rho$. Thus $V_n^{-1}(M)$ meets $C(a, \rho)$ for n large enough and so $G(M)$ approximates a .

To prove Theorem 1 we first suppose $\lambda(\xi)$ is not empty, say $a \in \lambda(\xi)$. If L denotes the radius to ξ then by Lemma 1 $G(L)$ approximates a and ξ is a point of approximation by Theorem A (iii).

To complete the proof we show that if ξ is a point of approximation then $\lambda(\xi)$ contains an h -line. If ξ is a point of approximation then by Theorem A, taking a subsequence if necessary, we may suppose there exists β, γ on $\partial\Delta$, $\beta \neq \gamma$, with $V_n(0) \rightarrow \beta$ and $V_n(\xi) \rightarrow \gamma$. Thus if N is the h -line with end points β and γ , N is approximated by images of the radius to ξ . It follows from Lemma 1 that $N \subseteq \lambda(\xi)$ as required.

3. Proofs of Theorems 2 and 3. If H is a hyperbolic element of the Fuchsian group G with fixed points ξ and η and M is the h -line with end points ξ and η then for each integer n , H^n fixes M and so every point of M is approximated by $G(M)$. Thus, by Lemma 1, $M \subseteq \lambda(\xi)$ and then $G(M) \subseteq \lambda(\xi)$.

Now suppose $a \in \lambda(\xi)$, then by Lemma 1 $G(M)$ approximates a . Choose $\rho > 0$ and there exists a sequence $\{V_n\}$ of distinct elements of G such that $V_n^{-1}(M)$ intersects $C(a, \rho)$ and so M intersects $C(V_n(a), \rho)$. Thus

$$(2) \quad [V_n(a), M] \leq \rho.$$

Now choose a sequence of points $\{z_n\}$ such that

$$z_n \in M \cap C(V_n(a), \rho).$$

Since $z_n \in M$, we can select S_n to be a power of H so that $S_n(z_n)$ lies in a compact subset of M ; then $S_n V_n(a)$ lies in a compact subset of Δ . Thus on a subsequence $S_n V_n = T$, a fixed element of G . We now have

$$\begin{aligned} [T(a), M] &= [S_n V_n(a), M] \\ &= [S_n V_n(a), S_n(M)] \\ &= [V_n(a), M] \\ &\leq \rho \end{aligned}$$

from (2). Since this inequality is true for any $\rho > 0$, we have $T(a) \in M$ and $a \in G(M)$ as required.

To prove Theorem 3 we note that if ξ is transitive and L is the radius to ξ then $G(L)$ is dense in Δ by definition. Thus by Lemma 1 $\lambda(\xi) = \Delta$. Conversely, if $\lambda(\xi) = \Delta$ then by Lemma 1 $G(L)$ is dense in Δ and ξ is transitive.

4. Proof of Theorem 4. To prove part (i) we assume that $a \in \lambda(\xi)$ and we find a sequence $\{z_n\}$ tending to ξ radially such that $f(z_n) \rightarrow f(a)$ as $n \rightarrow \infty$.

If a is not a pole of f we choose a sequence $\{\rho_n\}$ of positive numbers such that for each n

$$\sup \{|f(z) - f(a)| : z \in C(a, \rho_n)\} < \frac{1}{n}.$$

Now by lemma 1, for each n we can find $V_n \in G$ such that $C(V_n(a), \rho_n)$ intersects L . Let z_n be a point of this intersection. Clearly $\{z_n\}$ tends to ξ radially and z_n is equivalent to a point in $C(a, \rho_n)$ so $|f(z_n) - f(a)| < 1/n$. Thus $f(a)$ is contained in $R(f, \xi)$.

If a is a pole of f we choose the sequence $\{\rho_n\}$ with the condition

$$\inf \{|f(z)| : z \in C(a, \rho_n)/a\} > n$$

and proceed as before.

Now suppose M is the h -line joining ξ and η , the fixed points of the hyperbolic element H . Then from Theorems 2 and 4 (i)

$$f(M) = f(\lambda(\xi)) \subseteq R(f, \xi).$$

Thus to prove part (ii) we have only to show $R(f, \xi) \subseteq f(M)$.

If $w \in R(f, \xi)$, there exists a sequence $\{z_n\}$ converging radially to ξ such that $f(z_n) \rightarrow w$ as $n \rightarrow \infty$. Now let $\{w_n\}$ be a sequence of points of M such that $[z_n, w_n] \rightarrow 0$ as $n \rightarrow \infty$.

Clearly we can select S_n to be a power of H such that $S_n(w_n)$ lies in a compact subset of M . Then there exists $z \in M$ such that $S_n(w_n) \rightarrow z$ on a subsequence, and since $[S_n(w_n), S_n(z_n)] = [w_n, z_n]$ we see that $S_n(z_n) \rightarrow z$. Thus $f(S_n(z_n)) \rightarrow f(z)$ and since $f(S_n(z_n)) = f(z_n)$ we observe that $w = f(z)$ where $z \in M$ as required.

To prove part (iii) we suppose $w \in R(f, \xi)$. If $w \in A(G, f)$ we are finished, so we suppose that $w \notin A(G, f)$. Since G is finitely generated and of the first kind it is known (see [5] p. 174) that if D_0' denotes D_0 together with a complete set of inequivalent boundary arcs, then the value w is attained by f at precisely k points $a_1, \dots, a_k$ of D_0' . Since $w \notin A(G, f)$ then no a_i is a parabolic vertex.

There is a sequence $\{z_n\}$ converging to ξ on the radius L with $f(z_n) \rightarrow w$ as $n \rightarrow \infty$. If z_n' denotes the equivalent of z_n in D_0' , then clearly the sequence $\{z_n'\}$ must accumulate at some a_i say a_1 . Thus on a subsequence $z_n' \rightarrow a_1$. This implies that images of L meet arbitrarily small neighborhoods of a_1 and by Lemma 1 $a_1 \in \lambda(\xi)$. Thus $w = f(a_1) \in f(\lambda(\xi))$ as required.

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