Local Cone Structures on Varieties

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Introduction. In [1] it was shown that a real algebraic variety is locally a cone over a subvariety at an isolated singularity, and in [3], this result was generalized to give a cone theorem around any point. The object of this paper is to show that this local cone structure is unique in the sense that if a neighborhood of a point on a variety is expressed in two ways as a cone over a subvariety, then there is a deformation which carries the generators and base of the one cone into those of the other.

1. Statement of main theorem. Let $\mathfrak{V}$ be a bunch of real algebraic varieties in an open set of some Euclidean space satisfying the following conditions:
(1) If V_1 and V_2 are in $\mathfrak{V}$, so are all the components of $V_1 \cap V_2$.
(2) If V_1 is in $\mathfrak{V}$ and V_2 is the smallest real algebraic variety containing all singularities of V_1 , then all the components of V_2 are in $\mathfrak{V}$.

Condition (2) could also be stated by saying that the components of the singular subvariety of V_1 are in $\mathfrak{V}$, where it is understood that the singular subvariety is determined by the Jacobian conditions applied to generators of the ideal of V_1 . It has to be remembered that all the points of this subvariety are not necessarily singular in the local analytic sense.

Associated with the bunch $\mathfrak{V}$, a stratification can be defined as follows. Write Σ_k for the set of all varieties V_i in $\mathfrak{V}$ of dimension $\leq k$, and define

$$S_k = \Sigma_k - \Sigma_{k-1}.$$

Then S_k is to be the k -dimensional stratum of the required stratification. Note that the connected components of S_k are open subsets of varieties in $\mathfrak{V}$ and are at the same time real analytic manifolds.

The next step is to define an admissible curve family relative to the bunch $\mathfrak{V}$. Essentially the idea is to define a family having the property of the conclusion of Theorem 4.1 in [3].

Definition 1.1. Let $\mathfrak{V}$ be a bunch as above in the open set U of a Euclidean space. A family w of analytic curves is to be called admissible relative to $\mathfrak{V}$ if the following conditions hold. Let $|\mathfrak{V}|$ be the point set union of components of $\mathfrak{V}$.

- (1) There is exactly one member of w through each point of $U \cap |\mathfrak{V}|$.
- (2) If a member of w has a point in common with a variety of $\mathfrak{V}$, it lies entirely in that variety. In view of the condition satisfied by $\mathfrak{V}$, this means that each member of w lies in some stratum of the stratification associated with $\mathfrak{V}$.

(3) (Parametrization condition.) There is a polynomial $t = t(x)$ in the coordinates of the surrounding Euclidean space which can be taken as parameter on all the members of w .

(4) (Product condition.) Let W_0 be the set $t(x) = t_0$ (in the notation of condition (3)). This is a real algebraic subvariety of U . Map each point x of U on the pair $(p(x), t(x))$, where $p(x)$ is the point on W_0 on the curve of the family w through x . This map φ is to be a homeomorphism of U on $W_0 \times I$ (where I is to be an *open* interval of real numbers). Also, if V is a variety in $\mathfrak{U}$, then φ restricted to V is to be a homeomorphism of V on $(V \cap W_0) \times I$. This last sentence means compatibility relative to $\mathfrak{U}$ as in Theorem 4.1 of [3].

(5) (Analyticity condition.) Let X be a connected component of a stratum in the stratification associated with $\mathfrak{U}$ and let W_0 and φ be as in condition (4). Then X and $W_0 \cap X$ are real analytic manifolds and the restriction of φ to X is an analytic homeomorphism of X on $(X \cap W_0) \times I$.

Definition 1.2. The sets $t(x) = \text{constant}$ (notation of condition (3) above) are real algebraic subvarieties of U . They will be called transversal varieties to the family w . Note that condition (5) of the previous definition ensures that within each stratum the transversal varieties are analytic manifolds and are in fact transversal to the curves w . Also, Theorem 4.1 of [3] can be reformulated as follows, using the above definitions:

Theorem 1.1. *Let $\mathfrak{U}$ be a bunch of real algebraic varieties satisfying conditions (1) and (2) at the beginning of this section and let H be a family of parallel hyperplanes. Then H contains a finite set of hyperplanes such that, if U is the open set between any two of them, there is an admissible family of curves relative to a bunch of $\mathfrak{U}' \supset \mathfrak{U}$ in U such that H is a family of transversal varieties.*

The introduction of the larger bunch $\mathfrak{U}'$ is due to the fact that, in the proof of this theorem in [3], various additional varieties (for example branch varieties for various projections) are introduced, so that the analyticity condition would not immediately hold over the strata associated with $\mathfrak{U}$. A convenient terminology here is to say that the admissible family is piecewise analytic on the strata associated with $\mathfrak{U}$. That is to say it satisfies, instead of the analyticity condition (5) of Definition 1.1, the following weaker condition:

(5)' (Piecewise analyticity.) The analyticity condition (5) holds with respect to the strata associated with a bunch $\mathfrak{U}'$ containing the given bunch $\mathfrak{U}$.

In fact a bit more can be said here. For, if the proof of Theorem 4.1 of [3] is examined, it is clear that, if one starts the induction with algebraic arcs, then the lifting and filling-in constructions of that proof will always yield algebraic arcs. To formulate this properly it is convenient to introduce a stronger form of condition (5)':

(5)'' (Piecewise algebraicity.) The admissible family w will be called piecewise algebraic with respect to $\mathfrak{U}$ if, on the strata associated with a larger bunch $\mathfrak{U}'$,

it satisfies the analyticity condition and if, in addition, it is an algebraic family of real algebraic curves on each connected compound of these strata.

Then in the statement of Theorem 1.1, the admissible curve family described can be said to be piecewise algebraic relative to $\mathfrak{V}$.

The notion of a cone structure around a point of a variety appeared in [3] as a special case of the conclusion of Theorem 4.1. In the terminology introduced here this notion can be formulated as follows:

Definition 1.3. Let P be a point in Euclidean space, U a neighborhood of P and $\mathfrak{V}$ a bunch of real algebraic varieties in U with the properties (1) and (2) of the beginning of this section. $\mathfrak{V}$ will be said to have a cone structure in U with vertex P if the following conditions hold:

- (1) There is an admissible family of curves w in $U - P$ relative to $\mathfrak{V}$ such that each one has P as a limit as the parameter t tends to 0.
- (2) The transversal varieties fill the whole set $U - P$.

Any one of the transversal varieties will be called the base of the cone structure, and the members of the family w will be called the generators. The cone structure will be called analytic, piecewise analytic or piecewise algebraic relative to $\mathfrak{v}$ according as the generators satisfy condition (5), (5)', or (5)'' relative to this bunch. Theorem 4.1 of [3] can now be reformulated as follows:

Theorem 1.2. *Let $\mathfrak{V}$ be a bunch of varieties through a point P . Then P has a neighborhood in which $\mathfrak{V}$ has a piecewise algebraic cone structure with vertex P .*

The main result to be obtained now is that this cone structure is essentially unique. Stated more precisely:

Theorem 1.3. *Let $\mathfrak{V}$ be a bunch of real algebraic varieties through P and let $\mathfrak{V}$ have two piecewise algebraic (or analytic) cone structures in neighborhoods U and U' of P . Then there is a homeomorphism $U \cap |\mathfrak{V}| \rightarrow U' \cap |\mathfrak{V}|$ isotopic to the identity of $U \cap |\mathfrak{V}|$ on itself, which carries the generators of the one cone structure onto those of the other and the transversal varieties of the one onto those of the other.*

Most of the remainder of this paper will consist of the proof of this theorem.

2. Preliminary steps. The first step is to formulate a smoothing lemma which enables piecewise analytic admissible families to be made analytic. So let $\mathfrak{V}$ be a bunch of varieties as in the last section and let w be an admissible family relative to $\mathfrak{V}$, and suppose that, rather than satisfy condition (5) of Definition 1., w is piecewise analytic relative to $\mathfrak{V}$. That is to say w is admissible with respect to a larger bunch $\mathfrak{V}' \supset \mathfrak{V}$ and is analytic relative to $\mathfrak{V}'$.

Lemma 2.1. *With the above notations, there is an isotopy of the identity map of U on itself to a map of U onto itself carrying $\mathfrak{V}$ into itself and w into an admissible family relative to $\mathfrak{V}$ which is also analytic relative to $\mathfrak{V}$.*

The proof of this is simply a variation of the putting together of analytic manifolds with a common boundary to form an analytic manifold.

The next lemma is really an easy case of the main theorem in which the two admissible families in question have the same set of transversal varieties.

Lemma 2.2. *Let $\mathfrak{V}$ be a bunch of varieties in an open set U as in the last section, and let u and w be two admissible families relative to $\mathfrak{v}$ (analytic, or piecewise analytic) with the same set of transversal varieties. There then is an isotopic deformation of U into itself carrying $\mathfrak{V}$ and the transversal varieties into themselves and taking u into w .*

Proof. Note that the hypothesis means that the same parameter can be used on both u and w . Also the conclusion is stated rather informally rather than speaking of an isotopy of the identity map of U on itself. In the same spirit, the following steps describe the construction of the isotopy. Fix a transversal variety W_0 which will be taken as corresponding to the parameter value $t = 0$ on both families u and w . Then at stage t of the deformation the curve w_p of w through $p \in W_0$ is replaced by a curve which coincides with the curve W_p up to the point $p(t)$ on it with parameter value t , and from there on coincides with the curve of u through $p(t)$. Note that for $t = 0$ the curve described is the curve of u through p , and for some suitable positive value of t it is w_p . Also the process described is continuous both in p and in t . This gives the required deformation of u into w for the part of U for which the parameter is positive. A similar construction deals with the part for which the parameter is negative. Clearly the whole operation carries $\mathfrak{V}$ and the transversal varieties into themselves, and so the proof is completed.

Note that, in the deformation of u into w , the family of curves at each stage is more or less like an admissible family, except that the curves are only piecewise analytic. Either by adding a smoothing argument, or by using a different proof, depending on deforming the field of tangent vectors to u into these for w and solving the appropriate differential equations, they could be made analytic. But this refinement is not needed for the present discussion.

The main application of Lemma 2.2 in the present situation is to show the possibility of adjusting a given admissible family of curves so that it becomes admissible with respect to an enlarged bunch of varieties.

Lemma 2.3. *Let $\mathfrak{V}$ be a bunch of varieties as in Section 1 and let $\mathfrak{V}' \supset \mathfrak{V}$ be a bunch satisfying the same conditions. Let w be a family of admissible curves relative to $\mathfrak{V}$. Then there is an isotopy of the open set U containing $\mathfrak{V}$ (and $\mathfrak{V}'$) which carries $\mathfrak{V}$ into itself and carries w into a family which is admissible relative to $\mathfrak{V}'$.*

Proof. According to Theorem 1.1 there is a family w' admissible relative to $\mathfrak{V}'$, and of course also admissible relative to $\mathfrak{V}$. And according to Lemma 2.2 there is an isotopy of U into itself carrying w into w' . This completes the proof.

Note that the family w' obtained here is piecewise algebraic relative to $\mathfrak{U}$, and this property is independent of the given property of w (analytic, algebraic, or piecewise algebraic or analytic).

3. Proof of Theorem 1.3. The last section consisted of the easy steps in the proof of this theorem, and actually was concerned with admissible families in general. Now, in carrying out the main part of the proof, attention will be restricted to cone structures. So let $\mathfrak{U}$ be a given bunch of varieties through a point P , satisfying the conditions (1) and (2) at the beginning of Section 1, and let two cone structures of $\mathfrak{U}$ be given, with systems of generators u and w respectively. In a neighborhood of P , with P removed, u and w are admissible families relative to $\mathfrak{U}$. Now there exists a cone structure whose generators form a system u' piecewise algebraic relative to $\mathfrak{U}$, or, what is the same thing, algebraic relative to a bunch $\mathfrak{U}' \supset \mathfrak{U}$ (Theorem 6.2 of [3]). Use Lemma 2.2 to deform u into u' . Next, there is a deformation of w into a family of generators w' , for the second cone structure, admissible relative to $\mathfrak{U}'$ and piecewise algebraic relative to $\mathfrak{U}'$ (Lemma 2.3). Also the transversal varieties of the families u' and w' are the same as those of u and w respectively. An additional deformation applied to w' , using Lemma 2.1, will make this family analytic relative to $\mathfrak{U}'$. Summing up the operations so far carried out:

Lemma 3.1. *Given two cone structures for the bunch $\mathfrak{U}$, they may be isotopically deformed into cone structure for a larger bunch $\mathfrak{U}'$ such that the one set of generators forms an algebraic family relative to $\mathfrak{U}'$ while the other set forms an analytic family.*

For convenience the notation will be changed now. It will be assumed that the deformations described in this lemma have been carried out and that there are two cone structures now for the bunch $\mathfrak{U}$ with generator families u and w , the first algebraic and the second analytic relative to $\mathfrak{U}$.

Some terminology will now be introduced.

Definition 3.1. It will be assumed that the vertex P corresponds to parameter $t = 0$ on the family u . Then any direction corresponding to increasing values of t will be called upward and for decreasing values will be called downward. In particular, directions tangent to members of u will be called vertical, and directions tangent (in some stratum relative to $\mathfrak{U}$) to a transversal variety of u will be called horizontal.

The idea now is to show that the second cone structure, with the generator system w , can be deformed so that, sufficiently near P , the direction of decreasing parameter on w is downward. It will then be quite easy to deform the one cone structure into the other. This result will be proved by induction on the dimension of the strata associated with $\mathfrak{U}$.

Induction hypothesis. *Given a bunch $\mathfrak{U}$ of varieties through P and two cone structures for $\mathfrak{U}$ around P with generator families u and w , the latter being analytic relative to $\mathfrak{U}$, then the family w can be isotopically deformed so that it remains*

analytic relative to $\mathcal{V}$ and, sufficiently near P , the direction of decreasing parameter is downward on all strata of dimension less than r .

The statement just made is trivial for $r = 2$ since in effect the generators are unique on one-dimensional strata.

Note that the terms upward and downward depend on the transversal varieties of the first cone structure, and are unaffected by adjustments that may be made to the generators of that system. It will be seen presently that such adjustments may have to be made temporarily in applying the induction hypothesis.

Before actually carrying out the induction it is convenient to introduce two pieces of terminology which will be applied repeatedly in the proof. Suppose that on an analytic manifold two linearly independent analytic vector fields are given (that is to say, non-zero and linearly independent at each point). Then attach to each point a vector obtained from these fields by linear combination with non-zero coefficients. If the coefficients are analytic this will be a non-zero analytic vector field. Also if the coefficients depend on a parameter the result will be a one-parameter family of analytic vector fields, continuous or analytic, according to the given dependence on the parameter. In particular if the given vector fields are tangent fields to two families of curves and the appropriate set of differential equations is solved, a one parameter collection of families (containing the given ones) is obtained. This construction will be called the linear combination technique. There are immediately obvious variations of this that will be given the same name.

To describe the second piece of terminology, let V be an open set of some algebraic variety and let W be a subvariety. Suppose a one-parameter family of mappings φ_t is defined on some neighborhood U of W . The mappings may be continuous or analytic according to context, and so also may the dependence on the parameter. Assume that the neighborhood U is defined by some kind of distance function. For example, if f is the sum of the squares of equations defining W in V , U could be the set $f < \epsilon$ for some positive ϵ . Now define a new family of mappings ψ_t by setting $\psi_t(x) = \varphi_{t'}(x)$ where $t'(x, t)$ is defined to be equal to t inside a neighborhood U_1 of W and equal to something fixed (say 0) outside a neighborhood U_2 of W , with $U_1 \subset U_2 \subset U$. Now if, for example, φ_0 has an extension to all of V , this means that ψ_t can be extended to all of V for all t . Then the one-parameter family φ_t on U is replaced by a one-parameter family ψ_t which agrees with φ_t on a smaller neighborhood U_1 . This procedure will be called the neighborhood extension technique. Again there are several variations of this idea which will be given the same name.

For example, the induction hypothesis says that the direction of the decreasing parameter on w can be made downward, near P , on the strata of dimension less than r . Thus on, and so also near, such strata both u and w are transversal to the transversal varieties of u , and so in a neighborhood of these strata an argument like that of Lemma 2.2 can be used to deform w into u near P . Then,

applying the neighborhood extension technique, the following statement is obtained:

Modified Induction Hypothesis. *There is an isotopic deformation of the family w so that, in a neighborhood of P , they coincide with the curves u on the strata of $\mathcal{V}$ of dimension less than r .*

So now, to get started with the proof of Theorem 1.3, let two cone structures be given for the bunch v with generator systems u , w , and families of transversal varieties μ , ω , respectively. Assume if necessary that Lemma 3.1 has been applied so that u is algebraic relative to $\mathcal{V}$ and w is analytic. Also assume that the Induction Hypothesis has been applied, so that, in a neighborhood U of P , the members of w , in the direction of decreasing parameter, are all downward on the strata of $\mathcal{V}$ of dimension less than r .

Some additional varieties are now to be included in the bunch $\mathcal{V}$. Since the system u is, by assumption, algebraic relative to $\mathcal{V}$ and since the transversal varieties ω are in any case algebraic, the points where members of u are tangent to members of ω lie on a collection of varieties, to be called branch varieties (of ω relative to u). Similarly there are the branch varieties of the branch varieties of these, and so on. Add all these varieties to $\mathcal{V}$ along with all intersections and singularities and call the resulting bunch $\mathcal{V}'$. It will of course satisfy the conditions (1) and (2) at the beginning of Section 1. Another way to look at this would be to write the equation giving the intersection of a curve of u with a member of ω as a polynomial in the parameter on u . Then to get $\mathcal{V}'$ the root distribution varieties (*cf.* terminology of [3]) are added to $\mathcal{V}$.

Apply Lemmas 2.1, 2.3 to make w admissible and analytic relative to $\mathcal{V}'$. Note that, by the induction hypothesis, the branch varieties do not meet (outside P) the strata of $\mathcal{V}$ of dimension less than r . Accordingly the adjustment of w to be admissible relative to $\mathcal{V}'$ can be made outside a neighborhood of these strata, and so the property that the direction of decreasing parameter on w is downward on these strata is not disturbed.

Now note that at a point of a branch variety the tangent to the appropriate member of w is transversal to ω , and so is certainly not vertical. It follows that in a sufficiently small neighborhood of the branch varieties it is possible to construct a vector field with the properties:

- (1) Each vector is tangent to the stratum of $\mathcal{V}$ containing its initial point.
- (2) The vectors projection vertically (*i.e.* along u) on tangent vectors to w in the branch variety.
- (3) The vectors are transversal to ω .

Hence, using this vector field near the branch varieties along with the tangents to the original w , the linear combination technique yields an isotopic deformation of w into a new generator family, still to be called w , which will still be transversal to ω , and will have the property that, sufficiently near the branch varieties, each curve of w lies vertically above or below a curve of w in the branch variety.

Also of course w still has the property that the direction of decreasing parameter is downward on strata of $\mathcal{U}$ of dimension less than r .

The next adjustment to w will be to make these curves vertical, either upward or downward, outside a neighborhood of certain of the branch varieties. As a preliminary to this take a point p not on any branch variety. At p the tangents to both kinds of generator, u and w , are both transversal to ω . If the two tangents, both taken in the direction of decreasing parameter, are on the same side of the variety of ω through p , then p will be called a downward point. If they lie on opposite sides, p will be called an upward point. Thus the branch varieties divide the whole space around P into downward and upward regions consisting, respectively, entirely of downward points and upward points.

Now in a downward region, any linear combination of the tangent vectors to u and w , both in the direction of decreasing parameter, with non-negative coefficients will be a vector transversal to ω . The same holds in an upward region if the upward direction is used on u . Hence, using the linear combination technique, there is an isotopic deformation of w into a new family (still to be called w) which is the same as before in a neighborhood of the branch varieties (in fact there is no deformation made at all in this neighborhood) while, outside a larger neighborhood, the direction of decreasing parameter on the new w is vertically upward in the upward regions and vertically downward in the downward regions. Also clearly this second neighborhood can be chosen as in the previously made adjustment so that the curves of the new family w , in so far as they lie in this neighborhood, will project vertically (upward or downward) on curves of w in the branch varieties. Note that throughout all this deformation the family w remains transversal to ω .

Now apply the Induction Hypothesis to w relative to $\mathcal{U}'$. However, consider only the restriction of the asserted deformation to the branch variety. Thus, in the branch variety, w is isotopically deformed into a family on which the direction of the decreasing parameter is downward, sufficiently near P , on the strata of dimension less than r . But since the curves of w lie vertically above each other sufficiently close to the branch varieties, this deformation can be extended above and below the branch varieties and the resulting curves can be completed by adding upward or downward vertical segments. The final result at this stage is an isotopic deformation of the family w into a new family (still to be called w) with the following properties:

- (1) On the strata of $\mathcal{U}'$ of dimension $< r$ (and so also near them) the direction of decreasing parameter on w is downward.
- (2) Outside a neighborhood of the branch varieties the curves of w go vertically upward or downward (in the upward and downward regions, respectively).
- (3) (And for the next step this is the most important point) The parts of the curves of w other than the vertical segments described in (2) lie vertically above or below the curves of w in the branch variety.

Note that the curves of w no longer need be transversal to the original transversal varieties ω . So these varieties will play no further part in the argument.

Note also that, at any stage in this discussion, curves of w in the branch varieties never have vertical tangents. For if such a curve did have a vertical tangent at a point of a branch variety that point would lie in a lower dimensional stratum of $\mathfrak{U}'$ and so the whole curve in question would lie in that stratum.

At this point fix attention on an upward region U . There is a given tangent vector field to the family w , and the object is to deform it into a new field which will be downward on the strata of $\mathfrak{U}$ dimension r (the direction of the tangent field to w is taken as the direction of decreasing parameter). And the technique for doing this will be to construct such a vector field, making sure that it is never opposed in direction to the tangent field of w . The linear combination and neighborhood extension techniques will then give the required deformation.

Some notation is needed here to describe this operation. Let X and Y represent the upper and lower (in the obvious sense) boundaries of the upward region U . Let p be a point of U and consider the curve $w(p)$ of the family w through p . Going forward (direction of decreasing parameter) along $w(p)$, this curve eventually lies in a neighborhood of X and so lies vertically below a curve $w(p, X)$ of the family w in X . Denote by $S(p, X)$ the surface generated by curves of u through $w(p, X)$. Thus $S(p, X)$ is a vertical surface through $w(p, X)$. Note that it automatically contains the vertical segment (if any) forming part of the original curve $w(p)$. Similarly, going backwards along $w(p)$, it eventually lies near Y , and so vertically above a curve $w(p, Y)$ of the family w in Y , and so also in a vertical surface $S(p, Y)$ through $w(p, Y)$. $S(p, Y)$ also contains the vertical segment, if any, on $w(p)$. Note that, near any non-empty intersection of X and Y , which will always be contained in some lower dimensional stratum, the curves $w(p)$, $w(p, X)$ and $w(p, Y)$ will all lie near some curves of w in that stratum, and also $S(p, X)$ and $S(p, Y)$ will coincide.

Now at a point near X the two directions of tangent to the family w and vertical are linearly independent (this can be checked by examining the adjustments already made to w) and, of course, are tangent to the appropriate surface $S(p, X)$. So, by the linear combination technique, construct a vector field v which agrees with the tangent field to w on X while on or near strata of $\mathfrak{U}'$ of dimension r , which will be bounded by lower dimensional strata on X , this field is to be downward, but not vertical. Clearly this construction can be continued downwards to the vertical segments on the family w in U . The key point there is that still a well defined tangent plane to the appropriate $S(p, X)$ exists, spanned by the vertical direction and the tangent to $w(p, X)$, or, what is the same thing, its horizontal component (remembering that the tangent to w on X is never vertical). At a point p of a vertical segment of w in U call this horizontal component $h(p, X)$. Similarly construct a second vector field v agreeing with the tangents to $\mathfrak{U}$ on Y , downward around r -dimensional strata of $\mathfrak{U}'$, with a horizontal component $h(p, Y)$ at a point p of a vertical segment of w in U .

The essential step now is to see that the horizontal fields $h(p, X)$ and $h(p, Y)$ can be deformed into each other, at least in a neighborhood of r -dimensional

strata of $\mathfrak{V}$. And this can be done by noting that these fields are determined respectively by the tangent fields to w in X and Y . If the family w in X is projected vertically on Y , two cone structures on Y are obtained and the modified form of the induction hypothesis implies that these can be deformed into each other in a neighborhood of $(r - 1)$ -dimensional strata of $\mathfrak{V}'$ on Y . Note that the projection of X and Y only gives a cone structure on part of a variety but since this corresponds to an interchange of roots of the equation of a variety carrying a cone structure, in fact the projection induces a cone structure on that variety, and so the application of the induction hypothesis is valid.

Return now to the construction of the vector field v . This has been constructed in neighborhoods of X and Y and extended to the part of U where w is vertical. Since an r -dimensional stratum of $\mathfrak{V}$, in so far as it is contained in U , will be bounded by $(r - 1)$ -dimensional strata of $\mathfrak{V}'$ in X and Y , the above remark about the horizontal fields $h(p, X)$ and $h(p, Y)$ implies that, around such an r -stratum, the two constructions of $\mathfrak{V}$ can be fitted together to give a field which is downward, but not vertically downward. Thus in a neighborhood of an r -dimensional stratum of $\mathfrak{V}$ in U , v will be linearly independent of the tangent field to w . A similar discussion can be carried out in any downward region, so that v , with the property just stated, is constructed throughout a neighborhood of P . Note that it coincides with the tangent field of w on strata of $\mathfrak{V}$ of dimension less than r .

Now apply the linear combination and neighborhood extension techniques to get a deformation of w in a neighborhood of P into a curve family tangent to v near the r -dimensional strata of $\mathfrak{V}$. This field is downward and so the induction step is completed. Since the family w is now deformed into a new family on which the direction of decreasing parameter is downwards near P , and so transversal to the transversal varieties of u , Theorem 1.3 follows immediately.

4. Some consequences of Theorem 1.3. Lemma 2.3 says that a cone structure can be deformed so as to be compatible with a larger bunch of varieties. This pre-supposed that the cone structure was given in a whole neighborhood of P (i.e. that the original bunch included the surrounding Euclidean space). On the other hand suppose $\mathfrak{V}$ is any bunch around P , not including the ambient space, and suppose a cone structure is given for $\mathfrak{V}$ around P , with generator system u . However, there is always a second cone structure, with generators w , say, for the bunch $\mathfrak{V}'$ obtained by adding the ambient space to $\mathfrak{V}$. And, by Theorem 1.3, u can be deformed into w on $|\mathfrak{V}|$ around P . Combining this with Lemma 2.3, the following extension theorem is obtained.

Theorem 4.1. *Let $\mathfrak{V}$ be a bunch of varieties around P and $\mathfrak{V}'$ a bunch containing $\mathfrak{V}$. Then any cone structure for $\mathfrak{V}$ can be deformed isotopically to a cone structure which can be extended to $\mathfrak{V}'$.*

Another application of Theorem 1.3 concerns a rather weak form of the Poincare problem for homotopy spheres. Let S be a homotopy sphere and let V be the cone over S with vertex P . Thus V automatically has a cone structure

whose transversal varieties are copies of S (no generality is lost by assuming S to be algebraic—cf [2]). On the other hand suppose it happens that V is a manifold (no singularity at P). Then V has a second cone structure whose transversal varieties are spheres. An application of Theorem 1.3 gives the following consequence.

Theorem 4.2. *If the cone over a homotopy sphere S is a manifold, S is a sphere.*

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