Asymptotic Behavior and the Cauchy Problem for Ultrahyperbolic Operators

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1. Introduction. We consider the operator $L = -A_x + B_y$ where A_x and B_y are second order linear elliptic operators in $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, respectively. If both $m \geq 2, n \geq 2$ the operator L is ultrahyperbolic; while if either m or n is 1, L is hyperbolic. We shall be concerned with equations of the form

$$(1.1) \quad Lu = F(x, y, u, \nabla_x u, \nabla_y u)$$

in which F satisfies for all values of the arguments an inequality of the form

$$(1.2) \quad |F(x, y, z, a, b)| \leq \varphi_0(y) |z| + \varphi_1(y) |a| + \varphi_2(y) |b|.$$

When (1.2) holds the methods we employ yield information for the differential inequality

$$(1.3) \quad |Lu| \leq \varphi_0(y) |u| + \varphi_1(y) |\nabla_x u| + \varphi_2(y) |\nabla_y u|.$$

The asymptotic behavior of solutions of differential inequalities has been studied extensively. For operators in which B_y is of the first order and $n = 1$, the most comprehensive results are to be found in Agmon and Nirenberg [1]. If B_y is of second order and $n = 1$, lower bounds and estimates for asymptotic decay were obtained by Levine [2], Ogawa [5], and the authors [4, 7].

In Sections 2 and 3, we consider solutions $u = u(x, y)$ of (1.3) in a region $D \times \Gamma$ where D is a bounded domain in x -space and Γ is a positive cone in y -space, *i.e.*, closed under multiplication by positive scalars. We study the behavior as $|y| \rightarrow \infty$ of solutions which vanish on the boundary of $D \times \Gamma$. In the simplest case with A_x and B_y Laplace operators, $F = 0$, and $m = n$ it is clear that

$$(1.4) \quad u(x, y) = \sin kx_1 \sin kx_2 \cdots \sin kx_n \sin ky_1 \sin ky_2 \cdots \sin ky_n$$

is a solution of $\Delta_x u - \Delta_y u = 0$ which vanishes on the boundary of the domain D consisting of an n -dimensional cell of side π . The solution also vanishes on $D \times \partial\Gamma$. We observe that (1.4) is periodic and hence does not decay at all as $|y| \rightarrow \infty$.

Under fairly general hypotheses on the φ_i in (1.3), we establish the following results:

- a) Explicit lower bounds as $|y| \rightarrow \infty$ are given for functions which satisfy (1.3).
- b) Measured in a suitable energy norm, non-zero solutions cannot decay arbitrarily fast as $|y| \rightarrow \infty$. Consequently, non-zero solutions cannot have bounded support in $D \times \Gamma$.

We first establish a) and b) for the case $L = \Delta_y - \Delta_x$ and then extend the results to the operator $\Delta_y - A_x$. Similar results have been established for the operator $B_y - \Delta_x$ in [3]. The special ultrahyperbolic case with $n = 2$ has been treated earlier [8].

If all the φ_i are uniformly bounded, then the maximal rate of decay of solutions is not more rapid than $\exp(-\lambda |y|^2)$ for some value of λ . Under more restrictive hypotheses on the φ_i , the decay of solutions cannot be greater than that of some rational function. The techniques employed are refinements of those employed in [4, 7].

In Section 4, we consider the Cauchy problem for equations of the form

$$(1.5) \quad \Delta_y u - \Delta_x u = F(x, y, u).$$

The uniqueness of the solution of a variety of Cauchy problems is determined and, in each case, the result describes the full domain of dependence in which a solution can be determined from its initial data. A surface S of co-dimension 1 in $\mathbf{R}^{m+n}$ is a surface of uniqueness if a one-parameter family of surfaces $|y|^2 = |x|^2 + c$ intersect S in an appropriate series of compact surfaces K_ϵ . Thus for ultrahyperbolic operators we obtain the natural generalization of space-like and time-like surfaces which arise for hyperbolic equations. In fact, when m or n is 1, we obtain the standard uniqueness theorem for space-like surfaces by a new technique. In addition, we show the natural domain of determination for solutions of hyperbolic equations when data are prescribed on a time-like surface. If, for example, Cauchy data are given in (1.5) on a time-like closed finite cylinder, then the solution is determined in the entire cylinder. A uniqueness theorem of a different type for ultrahyperbolic operators in which solutions are determined in the entire space from information on a particular subspace was treated by Strichartz [9]. Earlier, a variety of uniqueness theorems for the ultrahyperbolic equation were established by Owens [6].

2. Exponential decay. Let L be an ultrahyperbolic operator and consider the equation

$$(2.1) \quad Lu = F(x, y, u, \nabla_x u, \nabla_y u)$$

where F satisfies

$$|F(x, y, z, a, b)| \leq \Phi_0 |z| + \Phi_1 |a| + \Phi_2 |b|$$

for all $z \in \mathbf{R}$, $a \in \mathbf{R}^m$, $b \in \mathbf{R}^n$, and Φ_i , $i = 0, 1, 2$, positive constants. Let D be a

bounded region in $\mathbf{R}^m$ and let Γ be a cone in $\mathbf{R}^n$. We consider solutions $u(x, y)$ of (2.1) in the domain $D \times \Gamma$ subject to the condition

$$(2.2) \quad u = 0 \quad \text{on} \quad \partial(D \times \Gamma).$$

We let $r^2 = y_1^2 + \cdots + y_n^2$ and define the sets

$$C(R) = \{y \in \Gamma : r = R\},$$

$$\Gamma(R) = \{y \in \Gamma : 1 < r < R\}, \quad \text{for} \quad 1 < R \leq \infty.$$

In $\mathbf{R}^m$ we introduce the inner product

$$(u, v) = (u, v)_D = \int_D u(x, y)v(x, y) dx$$

and the associated norm

$$\|v\| = \|v\|_D = (v, v)^{1/2}.$$

We shall measure the rate of decay of solutions of (2.1) by means of the "energy" integral

$$\mathcal{E}(u, R) = R^{1-n} \int_{C(R)} [||u||^2 + ||\nabla_x u||^2 + ||u_r||^2] d\sigma,$$

where $u_r = \sum_{k=1}^n r^{-1} y_k \partial u / \partial y_k$ is the radial derivative in $\mathbf{R}^n$. For brevity we shall employ the repeated indexes i and j to denote sums from 1 to m and k and l to denote sums from 1 to n .

The basic estimate for exponential decay with respect to the ultrahyperbolic operator $L = \Delta_y - \Delta_x$ is contained in the following inequality.

Theorem 1. *Let $u \in C^2$ on $D \times \Gamma$ and suppose that $L = \Delta_y - \Delta_x$. If u satisfies the boundary condition (2.2), then there are numbers $\lambda_1 > 0$, $\alpha > 1$, such that for $R > 1$, $\lambda > \lambda_1$,*

$$(2.3) \quad \begin{aligned} & \int_{\Gamma(R)} r^{3-\alpha-n} \exp(2\lambda r^\alpha) ||Lu||^2 dy + 6(\lambda\alpha)^3 R^{2\alpha-1} \exp(2\lambda R^\alpha) \mathcal{E}(u, R) \\ & \geq (\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} \exp(2\lambda r^\alpha) ||u||^2 dy \\ & \quad + \frac{1}{2}\lambda\alpha \int_{\Gamma(R)} r^{1-n} \exp(2\lambda r^\alpha) [||\nabla_x u||^2 + ||\nabla_y u||^2] dy + 2\lambda\alpha \mathcal{E}(u, 1). \end{aligned}$$

The value of λ_1 depends only on n and the value of u on $D \times C(1)$.

Proof. We define $w = u \exp(\lambda r^\alpha)$ and choose $\lambda > n - 1$. A computation yields

$$e^{2\lambda r^\alpha} Lu = Lw - 2\lambda\alpha r^{\alpha-1} w_r + \lambda\alpha r^{\alpha-2} (\lambda\alpha r^\alpha - \alpha - n + 2)w.$$

Therefore

$$e^{2\lambda r^\alpha} |Lu|^2 \geq (2\lambda\alpha r^{\alpha-1} w_r)^2 - 4\lambda\alpha r^{\alpha-1} w_r [Lw + \lambda\alpha r^{\alpha-2} (\lambda\alpha r^\alpha - \alpha - n + 2)w].$$

Multiplying the above expression by $r^{3-\alpha-n}$ and integrating over $D \times \Gamma(R)$, we obtain

$$(2.4) \quad \int_{\Gamma(R)} r^{3-\alpha-n} \exp(2\lambda r^\alpha) \|Lu\|^2 dy \geq I_1 + I_2 + I_3 + I_4$$

where

$$I_1 = (2\lambda\alpha)^2 \int_{\Gamma(R)} r^{\alpha+1-n} \|w_r\|^2 dy, \quad I_2 = -4\lambda\alpha \int_{\Gamma(R)} r^{2-n} (w_r, \Delta_s w) dy$$

$$I_3 = 4\lambda\alpha \int_{\Gamma(R)} r^{2-n} (w_r, \Delta_x w) dy,$$

$$I_4 = -(2\lambda\alpha)^2 \int_{\Gamma(R)} r^{\alpha-n} (\lambda\alpha r^\alpha - \alpha - n + 2) (w, w_r) dy.$$

Integrating by parts and taking (2.2) into account, we find

$$\begin{aligned} I_2 &= -4\lambda\alpha(n-1) \int_{\Gamma(R)} r^{1-n} \|w_r\|^2 dy + 2\lambda\alpha \int_{\Gamma(R)} r^{1-n} \|\nabla_s w\|^2 dy \\ &\quad - 2\lambda\alpha \int_1^R \frac{d}{dr} \int_{C(r)} r^{2-n} [2\|w_r\|^2 - \|\nabla_s w\|^2] d\sigma dr, \\ I_3 &= 2\lambda\alpha \int_{\Gamma(R)} r^{1-n} \|\nabla_x w\|^2 dy - 2\lambda\alpha \int_1^R \frac{d}{dr} \int_{C(r)} r^{2-n} \|\nabla_x w\|^2 d\sigma dr, \\ I_4 &= 2\lambda^2\alpha^2 \int_{\Gamma(R)} r^{\alpha-n-1} \{\lambda\alpha r^\alpha + (\alpha-1)(2\lambda\alpha r^\alpha - \alpha - n + 2)\} \|w\|^2 dy \\ &\quad - 2\lambda^2\alpha^2 \int_1^R \frac{d}{dr} \int_{C(r)} r^{\alpha-n} (\lambda\alpha r^\alpha - \alpha - n + 2) \|w\|^2 d\sigma dr. \end{aligned}$$

Since $\alpha > 1$, $\lambda \geq n-1$ and $r \geq 1$, it follows that the integrand in the first integral in I_4 is nonnegative. Combining all the integrals over $\Gamma(R)$ in I_1 through I_4 we see that their sum is nonnegative. Now we define

$$\begin{aligned} (2.5) \quad H(\rho) &= \rho^{2-n} \int_{C(\rho)} [2\|w_r\|^2 - \|\nabla_s w\|^2 + \|\nabla_x w\|^2] d\sigma \\ &\quad + \lambda\alpha\rho^{\alpha-n} (\lambda\alpha\rho^\alpha - \alpha - n + 2) \int_{C(\rho)} \|w\|^2 d\sigma. \end{aligned}$$

Then (2.4) yields the inequality

$$\begin{aligned} (2.6) \quad \int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^\alpha} \|Lu\|^2 dy &\geq 2\lambda\alpha \int_{\Gamma(R)} r^{1-n} [\|\nabla_x w\|^2 + \|\nabla_s w\|^2] dy \\ &\quad + 2(\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} \|w\|^2 dy \\ &\quad - 2\lambda\alpha H(R) + 2\lambda\alpha H(1). \end{aligned}$$

Taking into account the definition of w in terms of u , we have

$$|\nabla_y w|^2 = e^{2\lambda r^\alpha} \{ (\lambda \alpha r^{\alpha-1} u)^2 + 2\lambda \alpha r^{\alpha-1} u u_r + |\nabla_y u|^2 \}.$$

Using the inequality $|2\lambda \alpha r^{\alpha-1} u u_r| \leq \frac{3}{2} (\lambda \alpha r^{\alpha-1} u)^2 + (2/3) u_r^2$, we get

$$|\nabla_y w|^2 \geq e^{2\lambda r^\alpha} \{ \frac{1}{3} |\nabla_y u|^2 - \frac{1}{2} (\lambda \alpha r^{\alpha-1} u)^2 \}.$$

Thus from (2.6) we find

$$\begin{aligned} & \int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^\alpha} \|Lu\|^2 dy + 2\lambda \alpha H(R) \\ (2.7) \quad & \geq 2\lambda \alpha \int_{\Gamma(R)} r^{1-n} e^{2\lambda r^\alpha} [|\nabla_x u|^2 + |\nabla_y u|^2] dy \\ & + (\lambda \alpha)^3 \int_{\Gamma(R)} r^{2\alpha-1-n} e^{2\lambda r^\alpha} \|u\|^2 dy + 2\lambda \alpha H(1). \end{aligned}$$

From the definition of $H(\rho)$ given in (2.5), we have for, $R \geq 1$,

$$\begin{aligned} H(R) & \leq R^{2-n} \int_{C(R)} [\|w_r\|^2 + \|\nabla_x w\|^2 + \lambda^2 \alpha^2 r^{2\alpha-2} \|w\|^2] d\sigma \\ & \leq R^{2-n} e^{2\lambda R^\alpha} \int_{C(R)} [\|\lambda \alpha r^{\alpha-1} u + u_r\|^2 + \|\nabla_x u\|^2 + \|\lambda \alpha r^{\alpha-1} u\|^2] d\sigma. \end{aligned}$$

A computation yields the estimate

$$H(R) \leq 3(\lambda \alpha)^2 R^{2\alpha-1} e^{2\lambda R^\alpha} \mathcal{E}(u, R).$$

Without loss of generality we may assume $u \not\equiv 0$ for $r = 1$; otherwise we introduce a change in coordinates to guarantee this fact. Now, examining $H(1)$ and $\mathcal{E}(u, 1)$ it is easy to see that for sufficiently large λ , the inequality $H(1) \leq \mathcal{E}(u, 1)$ holds. Inequality (2.3) is a consequence.

Remark. The boundary condition (2.2) may be replaced by other conditions without change in the proof. Suppose that $\partial(D \times \Gamma)$ consists of two parts K_1 and K_2 such that $K_1 \cup K_2 = \partial(D \times \Gamma)$ and $K_1 \cap K_2 = \emptyset$. Then the conditions

$$\begin{aligned} (2.2a) \quad & u = 0 \quad \text{on } K_1 \\ & \frac{\partial u}{\partial n} = 0 \quad \text{on } K_2 \end{aligned}$$

yield the inequality (2.3).

Theorem 2. Suppose $u \in C^2$ on $D \times \Gamma$ and $L = \Delta_y - \Delta_x$. Let u satisfy (2.2) or (2.2a) and the inequality

$$\|Lu\|^2 \leq \Phi \{ \|u\|^2 + \|\nabla_x u\|^2 + \|u_r\|^2 \}$$

for all $y \in \Gamma$, where Φ is a constant. Then for sufficiently large λ the following inequality holds:

$$(2.8) \quad \lambda^3 R^3 \exp(2\lambda R^2) \varepsilon(u, R) \geq \int_{\Gamma(R)} r^{-n+3} e^{2\lambda r^2} \|u\|^2 dy + c_0 \lambda \varepsilon(u, 1)$$

where c_0 is a numerical constant.

Proof. We apply inequality (2.3) with $\alpha = 2$ to obtain

$$\begin{aligned} 48\lambda^3 R^3 e^{2\lambda R^2} \varepsilon(u, R) &\geq \int_{\Gamma(R)} \{8\lambda^3 - \Phi r^{-2}\} r^{-n+3} e^{2\lambda r^2} \|u\|^2 dy \\ &\quad + \int_{\Gamma(R)} \{\lambda - \Phi\} r^{1-n} e^{2\lambda r^2} \{|\nabla_x u|^2 + |\nabla_s u|^2\} dy \\ &\quad + 4\lambda \varepsilon(u, 1). \end{aligned}$$

We now choose λ so large that $8\lambda^3 > \Phi r^{-2} + 1$ and $\lambda > \Phi$ for all $r \geq 1$. Inequality (2.8) is the result.

With the aid of (2.8) we are able to establish the principal results on the asymptotic properties of solutions of (2.1), (2.2).

Theorem 3. Let $L = \Delta_v - \Delta_x$ and suppose that u is a C^2 solution of (2.1), (2.2) in $D \times \Gamma$. Then

i) there are constants K and μ independent of u such that

$$(2.9) \quad \varepsilon(u, R) \geq K R^{-3} \exp(-2\mu R^2) \varepsilon(u, 1)$$

for all $R > 1$.

ii) If $\varepsilon(u, R) = O(\exp(-\lambda R^2))$ for all $\lambda > 0$, it follows that $u \equiv 0$ in $D \times \Gamma(\infty)$.

iii) If $\varepsilon(u, R) = 0$ for some $R > 0$, then $\|u\|^2 = 0$ in $\Gamma(R)$. Consequently u cannot have bounded support in $D \times \Gamma(\infty)$ unless $u \equiv 0$.

Proof. Inequality (2.9) is a direct consequence of Theorem 2. Items (ii) and (iii) are immediate consequences of this inequality combined with (2.8).

Lower bounds for hyperbolic inequalities similar to that given in (2.9) were obtained by Ogawa [5] by means of logarithmic convexity.

We now establish results similar to those given in Theorem 3 for the operator

$$L = \Delta_v - A_x$$

where

$$A_x = \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

is a self-adjoint uniformly elliptic operator in $D \times \Gamma$.

We assume that $a_{ij} = a_{ij}(x, y)$ satisfy the additional condition

$$(2.10) \quad a_{ij} \xi_i \xi_j + 2r(a_{ij})_r \xi_i \xi_j \geq 0$$

for all $(x, y) \in D \times \Gamma$ and all $\xi \in \mathbf{R}^m$. Condition (2.10) is obviously satisfied when

$a_{ij} = a_{ij}(x)$. However, if $|r(a_{ij})_r| \leq m_0(2m)^{-1}$ for all $(x, y) \in D \times \Gamma$ where m_0 is the ellipticity constant of A_x , then a simple estimate shows that (2.10) holds. We now prove the analogue of Theorem 1.

Theorem 4. *Let $L = \Delta_y - A_x$ with A_x satisfying (2.10). Suppose that $u \in C^2$ in $D \times \Gamma$ and that u vanishes on $\partial(D \times \Gamma)$. Then there are positive constants k, K_0, K_1, λ_1 such that*

$$(2.11) \quad \begin{aligned} & \int_{\Gamma(R)} r^{3-\alpha-n} \exp(2\lambda r^\alpha) \|Lu\|^2 dy + K_1(\lambda\alpha)^3 R^{2\alpha-1} \exp(2\lambda R^\alpha) \varepsilon(u, R) \\ & \geq k\lambda\alpha \int_{\Gamma(R)} r^{-n+1} \exp(2\lambda r^\alpha) \{ \|\nabla_x u\|^2 + \|\nabla_y u\|^2 \} dy \\ & \quad + (\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} \exp(2\lambda r^\alpha) \|u\|^2 dy + K_0 \varepsilon(u, 1) \end{aligned}$$

for $\alpha > 1, R > 1, \lambda > \lambda_1$. The quantities k, K_0, K_1 depend only on the coefficients in A_x and the value of λ_1 depends only on n and the value of u restricted to $R = 1$.

Proof. We make an appropriate modification of Theorem 1. Let $w = u \exp(\lambda r^\alpha)$. Then

$$e^{\lambda r^\alpha} Lu = Lw - 2\lambda\alpha r^{\alpha-1} w_r + \lambda\alpha r^{\alpha-2} (\lambda\alpha r^\alpha - \alpha - n + 2)w.$$

Hence

$$\begin{aligned} e^{2\lambda r^\alpha} |Lu|^2 & \geq -2(2\lambda\alpha r^{\alpha-1} w_r) [\Delta_y w - A_x w + \lambda\alpha r^{\alpha-2} (\lambda\alpha r^\alpha - \alpha - n + 2)w] \\ & \quad + (2\lambda\alpha r^{\alpha-1} w_r)^2. \end{aligned}$$

We multiply this inequality by $r^{-\alpha-n+3}$ and integrate over $D \times \Gamma(R)$. We obtain

$$(2.12) \quad \int_{\Gamma(R)} r^{-\alpha-n+3} e^{2\lambda r^\alpha} \|Lu\|^2 dy \geq \sum_{i=1}^4 J_i$$

where $J_i = I_i$, $i = 1, 2, 4$, with I_i defined as in the proof of Theorem 1 and

$$J_3 = 4\lambda\alpha \int_{\Gamma(R)} r^{-n+2} \int_D w_r A_x w dx dy.$$

Integrating by parts we find

$$\begin{aligned} J_3 & = 2\lambda\alpha \int_{\Gamma(R)} r^{-n+1} \int_D (w_{x_i} a_{ij} w_{x_j} + r a_{ij,r} w_{x_i} w_{x_j}) dx dy \\ & \quad - 2\lambda\alpha R^{-n+2} \int_{C(R)} \int_D w_{x_i} a_{ij} w_{x_j} dx d\sigma \\ & \quad + 2\lambda\alpha \int_{C(1)} \int_D w_{x_i} a_{ij} w_{x_j} dx d\sigma. \end{aligned}$$

Taking (2.10) into account we obtain

$$J_3 \geq \lambda \alpha m_0 \int_{\Gamma(R)} r^{-n+1} \|\nabla_x w\|^2 dy \\ - 2\lambda \alpha M_0 R^{-n+2} \int_{C(R)} \|\nabla_x w\|^2 d\sigma + 2\lambda \alpha m_0 \int_{C(1)} \|\nabla_x w\|^2 d\sigma,$$

where M_0 , m_0 are the upper and lower ellipticity constants of A_x . Inserting the bounds for $J_1, \dots, J_4$ into (2.12) and proceeding as in the proof of Theorem 1, we obtain (2.11).

With the aid of Theorem 4 we obtain results on asymptotic behavior for solutions of the inequality

$$\|Lu\|^2 \leq \Phi (\|u\|^2 + \|\nabla_x u\|^2 + \|\nabla_y u\|^2)$$

which are completely analogous to those given in Theorem 3.

3. Polynomial decay. In this section we show that if the right side of the equation

$$(3.1) \quad Lu = F(x, y, u, \nabla_x u, \nabla_y u)$$

satisfies a more restrictive condition than the one imposed in Section 2, then there are more severe limitations on the rate of decay of solutions of (3.1). We first suppose that $Lu = \Delta_y - \Delta_x$ and that

$$(3.2) \quad \|Lu\|^2 \leq \Phi_0^2 r^{-2} \|u\|^2 + \Phi_1^2 r^{-2} \|\nabla_x u\|^2$$

for all $y \in \Gamma$. As before, we impose the boundary condition

$$(3.3) \quad u = 0 \quad \text{on} \quad \partial(D \times \Gamma).$$

Under these hypotheses we shall show that the maximum rate of decay of solutions of (3.1), (3.3) is polynomial rather than exponential.

Since D is bounded, condition (3.3) implies there is a constant k depending only on the size of D such that

$$(3.4) \quad \|u\|^2 \leq k \|\nabla_x u\|^2.$$

Theorem 5. *Let $L = \Delta_y - \Delta_x$ and suppose that $u \in C^2(D \times \Gamma)$ and that u satisfies (3.3). Then there is a number α_0 such that for $\alpha > \alpha_0$ the following inequality holds:*

$$(3.5) \quad \int_{\Gamma(R)} r^{-n+2\alpha+4} \|Lu\|^2 dy + 6\alpha^2(\alpha+2)R^{2\alpha+2} \mathcal{E}(u, R) \\ \geq 4\alpha \int_{\Gamma(R)} r^{-n+2\alpha+2} \|\nabla_x u\|^2 dy + \alpha \mathcal{E}(u, 1).$$

The value of α_0 depends only on n and the value of u for $r = 1$.

Proof. We choose $w = r^\alpha u$ where $\alpha > n - 2$. Then, as in the proof of Theorem 1,

$$r^{2\alpha} |Lu|^2 \geq -2(2\alpha r^{-1} w_r)(Lw + \alpha(\alpha - n + 2)r^{-2}w) + (2\alpha r^{-1} w_r)^2.$$

We multiply this inequality by r^{-n+4} and integrate over $D \times \Gamma(R)$. We find

$$(3.6) \quad \int_{\Gamma(R)} r^{2\alpha-n+4} ||Lu||^2 dy \geq T_1 + T_2 + T_3 + T_4$$

where

$$T_1 = 4\alpha^2 \int_{\Gamma(R)} r^{-n+2} ||w_r||^2 dy, \quad T_2 = -4\alpha \int_{\Gamma(R)} r^{-n+3} \int_D w_r \Delta_x w dx dy,$$

$$T_3 = 4\alpha \int_{\Gamma(R)} r^{-n+3} \int_D w_r \Delta_x w dx dy,$$

$$T_4 = -4\alpha^2(\alpha - n + 2) \int_{\Gamma(R)} r^{-n+1} \int_D ww_r dx dy.$$

Integrating by parts and taking (3.3) into account, we obtain

$$\begin{aligned} T_2 &= -2\alpha \int_{C(R)} r^{-n+3} \{2 ||w_r||^2 - ||\nabla_x w||^2\} d\sigma \\ &\quad + 2\alpha \int_{C(1)} \{2 ||w_r||^2 - ||\nabla_x w||^2\} d\sigma \\ &\quad - 4\alpha(n - 2) \int_{\Gamma(R)} r^{-n+2} ||w_r||^2 dy, \\ T_3 &= -2\alpha \int_{C(R)} r^{-n+3} ||\nabla_x w||^2 d\sigma + 2\alpha \int_{C(1)} ||\nabla_x w||^2 d\sigma \\ &\quad + 4\alpha \int_{\Gamma(R)} r^{-n+2} ||\nabla_x w||^2 dy, \\ T_4 &= -2\alpha^2(\alpha - n + 2) \int_{C(R)} r^{-n+1} ||w||^2 d\sigma + 2\alpha^2(\alpha - n + 2) \int_{C(1)} ||w||^2 d\sigma. \end{aligned}$$

Since $\alpha > n - 2$, it follows that

$$\begin{aligned} T_1 + T_2 &\geq -2\alpha R^{-n+3} \int_{C(R)} \{2 ||w_r||^2 - ||\nabla_x w||^2\} d\sigma \\ &\quad + 2\alpha \int_{C(1)} \{2 ||w_r||^2 - ||\nabla_x w||^2\} d\sigma. \end{aligned}$$

We define

$$\begin{aligned} Q(\rho) &= \rho^{-n+3} \\ &\quad \cdot \int_{C(\rho)} \{||\nabla_x w||^2 + 2 ||w_r||^2 - ||\nabla_x w||^2 + \alpha(\alpha - n + 2)\rho^{-2} ||w||^2\} d\sigma. \end{aligned}$$

Then inequality (3.6) takes the form

$$(3.7) \quad \int_{\Gamma(R)} r^{2\alpha-n+4} ||Lu||^2 dy + 2\alpha Q(R) \geq 4\alpha \int_{\Gamma(R)} r^{2\alpha-n+2} ||\nabla_x u||^2 dy + 2\alpha Q(1).$$

Clearly,

$$Q(R) \leq R^{2\alpha-n+3} \int_{C(R)} \{ ||\nabla_x u||^2 + ||\alpha r^{-1}u + u_r||^2 + \alpha(\alpha + 2)r^{-2} ||u||^2 \} d\sigma.$$

Hence

$$Q(R) \leq 3\alpha(\alpha + 2)R^{2\alpha+2}\varepsilon(u, R).$$

To estimate $Q(1)$, we may assume without loss of generality that $u \not\equiv 0$ for $r = 1$. Therefore if α is sufficiently large,

$$Q(1) \geq \varepsilon(u, 1),$$

the size of α depending on the value of u for $r = 1$. Inequality (3.6) results when the estimates for $Q(R)$ and $Q(1)$ are inserted in (3.7).

Theorem 7. *Let $L = \Delta_y - \Delta_x$ and suppose that u is a C^2 solution of (3.2) in $D \times \Gamma$ satisfying the boundary condition (3.3). Then, for sufficiently large α ,*

$$(3.8) \quad 6\alpha^2(\alpha + 2)R^{2\alpha+2}\varepsilon(u, R) \geq \alpha \int_{\Gamma(R)} r^{2\alpha-n+2} ||\nabla_x u||^2 dy + \alpha \varepsilon(u, 1).$$

Proof. Starting with inequality (3.5), we insert the estimate for $||Lu||$ from (3.2). Then we eliminate the quantity $||u||^2$ by means of (3.4). The result is

$$\geq (4\alpha - \Phi_0^2 k - \Phi_1^2) \int_{\Gamma(R)} r^{2+2\alpha-n} ||\nabla_x u||^2 dy + \alpha \varepsilon(u, 1).$$

We choose α so large that $4\alpha - \Phi_0^2 k - \Phi_1^2 > \alpha$ and (3.8) follows.

The analogue to Theorem 3 for polynomial decay is given in the next result.

Theorem 8. *Let $L = \Delta_y - \Delta_x$ and suppose that u satisfies (3.2) and (3.3) in $D \times \Gamma$. Then*

i) *there are positive constants K and μ such that*

$$\varepsilon(u, R) \geq KR^{-\mu}\varepsilon(u, 1)$$

for $R > 1$;

ii) *if $\varepsilon(u, R) = O(R^{-\beta})$ as $R \rightarrow \infty$ for all $\beta > 0$, then $u \equiv 0$ in $D \times \Gamma(\infty)$;*

iii) *if u has bounded support in $D \times \Gamma$, then $u \equiv 0$.*

Remarks. We give an example which shows that in general no slower-than-polynomial decay can result even when the right side of (3.2) vanishes. Introduce

spherical coordinates $(r, \theta_1, \dots, \theta_{n-1})$ in $\mathbf{R}^n$. Then the equation $\Delta_y u - \Delta_x u = 0$ takes the form

$$(3.9) \quad u_{rr} + (n-1)r^{-1}u_r + r^{-2}S_\theta u - \Delta_x u = 0$$

where S_θ is the portion of the Laplace operator acting on the variables $\theta = (\theta_1, \dots, \theta_{n-1})$. We denote the region $D \times \Gamma$ by $I \times C \times \Gamma$ where $I = (0, \infty)$ and C is the region in $\mathbf{R}^{n-1}$ determined by θ . The boundary condition requires that

$$(3.10) \quad u(r, \theta, x) = 0 \quad \text{for } \theta \in \partial C \quad \text{or } x \in \partial D.$$

We choose a domain C and solve the eigenvalue problem

$$\begin{aligned} S_\theta g &= -s^2 g \quad \text{in } C \\ g &= 0 \quad \text{on } \partial C. \end{aligned}$$

Next we choose a domain D and solve the eigenvalue problem

$$\begin{aligned} \Delta_x h &= -\ell^2 h \quad \text{in } D \\ h &= 0 \quad \text{on } \partial D. \end{aligned}$$

A function $u = f(r)g(\theta)h(x)$ is a solution of (3.9), (3.10) if f is any solution of

$$(3.11) \quad \frac{d^2 f}{dr^2} + (n-1)r^{-1} \frac{df}{dr} + (\ell^2 - s^2 r^{-2})f = 0$$

for $r > 0$. Choosing such a solution, we find that

$$\varepsilon(u, R) = |f(R)|^2 \|g\|_C^2 (\|h\|_D^2 + \|\nabla_x h\|_D^2) + |f'(R)|^2 \|g\|_C^2 \|h\|_D^2.$$

Equation (3.11) is a Bessel equation and solutions can be found which decay as a power of R as $R \rightarrow \infty$.

The results of this section can be expanded to yield polynomial decay for operators of the form $L = \Delta_y - A_x$ where

$$A_x = \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left(a_{ij}(x, y) \frac{\partial}{\partial x_j} \right)$$

is an elliptic operator. The theorems and arguments follow those in Theorems 5 through 8 with changes similar to those given in Section 2. Condition (2.10) is replaced by the slightly altered requirement

$$a_{ij}\xi_i\xi_j + r(a_{ij})_r\xi_i\xi_j \geq 0$$

for all $\xi \in \mathbf{R}^m$ and $(x, y) \in D \times \Gamma$.

4. The Cauchy problem. In this section we consider the operator

$$Lu = \Delta_y u - \Delta_x u$$

in $\mathbf{R}^n \times \mathbf{R}^m$ with $n \geq 1, m \geq 1$. Thus L may be hyperbolic or ultrahyperbolic.

We establish a variety of uniqueness theorems in the large for the Cauchy problem for equations of the form

$$(4.1) \quad Lu = F(x, y, u).$$

The results are valid across certain hypersurfaces which, for hyperbolic operators, may be space-like or time-like. The methods we describe define maximum domains of uniqueness for these problems.

Let Ω be the region in $\mathbf{R}^m \times \mathbf{R}^n$ defined by

$$\Omega = \{(x, y) : |x|^2 < |y|^2 < |x|^2 + 1\}$$

where $|x|^2 = x_1^2 + \cdots + x_m^2$, $|y|^2 = y_1^2 + \cdots + y_n^2$. We further define

$$r^2 = |x|^2 - |y|^2 + 1, \quad s^2 = |y|^2 - |x|^2.$$

Note that the quantity r differs from its definition in the previous sections.

Theorem 9. *Let $L = \Delta_y - \Delta_x$ and suppose that w is a function with two derivatives and compact support K in Ω . Set $\gamma = 2 - m - n$. Then there is a number β_0 such that for $\beta > \beta_0$, the following inequality holds:*

$$(4.2) \quad \int_{\Omega} s^{\gamma} r^{\beta+2} e^{2r^{-\beta}} |Lw|^2 dx dy \geq 4\beta^2 \int_{\Omega} s^{\gamma} r^{-2\beta-6} e^{2r^{-\beta}} w^2 dx dy.$$

The value of β_0 depends on m, n and

$$r_+ = \max_K r, \quad s_- = \min_K s.$$

Proof. We set $v = w \exp(r^{-\beta})$ with $\beta > 0$. A computation yields

$$\begin{aligned} e^{r^{-\beta}} Lw &= Lv - 2\beta r^{-\beta-2} (y_k v_{y_k} + x_i v_{x_i}) \\ &\quad + \beta r^{-\beta-2} \{ \beta r^{-\beta-2} s^2 - (\beta + 2) r^{-2} s^2 - m - n \} v. \end{aligned}$$

Hence,

$$\begin{aligned} e^{2r^{-\beta}} |Lw|^2 &\geq 4\beta^2 r^{-2\beta-4} (y_k v_{y_k} + x_i v_{x_i})^2 \\ (4.3) \quad &- 4\beta r^{-\beta-2} (y_k v_{y_k} + x_i v_{x_i}) Lv \\ &- 4\beta^2 r^{-2\beta-4} (y_k v_{y_k} + x_i v_{x_i}) \{ (\beta r^{-\beta} - \beta - 2) r^{-2} s^2 - m - n \} v. \end{aligned}$$

We multiply (4.3) by $s^{\gamma} r^{\beta+2} (4\beta)^{-1}$ and integrate over Ω . The result is

$$(4\beta)^{-1} \int_{\Omega} s^{\gamma} r^{\beta+2} e^{2r^{-\beta}} |Lw|^2 dx dy \geq I_1 + I_2 + I_3$$

where

$$I_1 = \beta \int_{\Omega} s^{\gamma} r^{-\beta-2} (y_k v_{y_k} + x_i v_{x_i})^2 dx dy,$$

$$I_2 = - \int_{\Omega} s^{\gamma} (y_k v_{y_k} + x_i v_{x_i}) Lv dx dy,$$

$$I_3 = -\beta \int_{\Omega} s^{\gamma} r^{-\beta-2} \{ (\beta r^{-\beta} - \beta - 2) r^{-2} s^2 - m - n \} (y_k v_{y_k} + x_i v_{x_i}) v dx dy.$$

We integrate by parts in I_2 and obtain

$$I_2 = \gamma \int_{\Omega} s^{\gamma-2} (y_k v_{y_k} + x_i v_{x_i})^2 dx dy.$$

For sufficiently large β it is clear that

$$I_1 + I_2 = \int_{\Omega} \{\beta r^{-\beta-2} + \gamma s^{-2}\} s^{\gamma} \{y_k v_{y_k} + x_i v_{x_i}\}^2 dx dy$$

is nonnegative. In fact, we merely require $\beta \geq -\gamma(s_-)^{-2}$. As for I_3 , we write

$$I_3 = I_{31} + I_{32} + I_{33}$$

ordered according to the powers of r in the integrand of I_3 . After integrating by parts we find

$$\begin{aligned} I_{31} &= -\beta^2 \int_{\Omega} r^{-2\beta-4} s^{\gamma+2} (y_k v_{y_k} + x_i v_{x_i}) v dx dy \\ &= \beta^2 \int_{\Omega} r^{-2\beta-6} s^{\gamma+2} \{1 + \beta s^2\} v^2 dx dy. \\ I_{32} &= \beta(\beta + 2) \int_{\Omega} r^{-\beta-4} s^{\gamma+2} (y_k v_{y_k} + x_i v_{x_i}) v dx dy \\ &= -\frac{1}{2}\beta(\beta + 2) \int_{\Omega} r^{-\beta-6} s^{\gamma+2} \{4 + \beta s^2\} v^2 dx dy. \\ I_{33} &= \beta(m + n) \int_{\Omega} r^{-\beta-2} s^{\gamma} (y_k v_{y_k} + x_i v_{x_i}) v dx dy \\ &= -\frac{1}{2}\beta(m + n) \int_{\Omega} r^{-\beta-4} s^{\gamma} \{2 + \beta s^2\} v^2 dx dy. \end{aligned}$$

Therefore,

$$I_3 = \beta \int_{\Omega} r^{-2\beta-6} s^{\gamma} v^2 dx dy$$

where

$$\varphi = \beta s^2 (1 + \beta s^2) - \frac{1}{2}(\beta + 2) r^{\beta} s^2 (4 + \beta s^2) - \frac{1}{2}(m + n) r^{\beta+2} (2 + \beta s^2).$$

Since $s_- > 0$ and $r_+ < 1$ in K , we find

$$\varphi \geq \beta s_-^2 (1 + \beta s_-^2) - \frac{1}{2} r_+^{\beta} \{(\beta + 2)(\beta + 4) + (m + n)(\beta + 2)\}.$$

Hence $\varphi \rightarrow +\infty$ as $\beta \rightarrow +\infty$. Thus for β sufficiently large we are assured that $\varphi \geq 1$ in K . Therefore

$$I_3 \geq \beta \int_{\Omega} r^{-2\beta-6} s^{\gamma} v^2 dx dy.$$

Combining the inequalities for $I_1 + I_2$ and I_3 , we obtain (4.2).

The next result gives a simple application of inequality (4.2) to the Cauchy problem for Equation (4.1). To establish uniqueness theorems we assume that F in (4.1) is continuous and that $F(x, y, p)$ satisfies a uniform Lipschitz condition in p . Thus if v is the difference of two solutions, v satisfies the inequality

$$(4.4) \quad |Lv| \leq \phi |v|.$$

Theorem 10. *Let Y be a constant between 0 and 1. Suppose that u and $(\partial u / \partial n)$ vanish on the set $\{|y| = Y\} \cap \Omega$. If u satisfies (4.4) in the portion of Ω where $|y| \leq Y$, then $u \equiv 0$ there.*

Proof. We define u to be identically zero for $|y| \geq Y$. We wish to show that u vanishes throughout Ω . For sufficiently small $\epsilon > 0$, we define the smooth function χ between 0 and 1 by

$$\chi(r) = \begin{cases} 1 & \text{for } 0 < r \leq 1 - 2\epsilon \\ 0 & \text{for } 1 - \epsilon < r < 1. \end{cases}$$

Let $w = \chi(r)u$ in Ω . Then w has support in the set

$$K_\epsilon = \{(x, y) : (x, y) \in \Omega, |y| \leq Y, r \leq 1 - \epsilon\}.$$

It follows that

$$\begin{aligned} r_+ &= \max_{K_\epsilon} (r) = 1 - \epsilon \\ s_- &= \min_{K_\epsilon} (s) = (\epsilon(2 - \epsilon))^{1/2}. \end{aligned}$$

Then from Theorem 10 applied to w we obtain

$$(4.5) \quad \int_{K_\epsilon} s^\gamma r^{\beta+2} e^{2r-\beta} |Lw|^2 dx dy \geq 4\beta^2 \int_{K_\epsilon} s^\gamma r^{-2\beta-6} e^{2r-\beta} |w|^2 dx dy.$$

We set $R = 1 - 2\epsilon$. Then $K_\epsilon = K_+ \cup K_-$ where

$$K_+ = \{(x, y) : (x, y) \in K_\epsilon, R > r\}, K_- = \{(x, y) : (x, y) \in K_\epsilon, R \leq r\}.$$

Then $w = u$ in K_- and also

$$|Lw| = |Lu| \leq \phi |u| \quad \text{in } K_-.$$

Thus for sufficiently large β , we have

$$(4.6) \quad \int_{K_+} s^\gamma r^{\beta+2} e^{2r-\beta} |Lw|^2 dx dy \geq \int_{K_-} (4\beta^2 r^{-2\beta-6} - \phi^2 r^{\beta+2}) s^\gamma e^{2r-\beta} |u|^2 dx dy.$$

From (4.5), we obtain

$$e^{2R-\beta} \int_{K_+} s^\gamma |Lw|^2 dx dy \geq e^{2R-\beta} (4\beta^2 - \phi^2) \int_{K_-} s^\gamma |u|^2 dx dy.$$

Letting β tend to $+\infty$, we find $u \equiv 0$ in K_- . Since ϵ is arbitrary, we conclude that $u \equiv 0$ in K_0 , as required.

The following general uniqueness theorem is an immediate consequence of Theorem 10.

Theorem 11. *Let S be a surface in $\mathbf{R}^{n+m}$ of co-dimension 1, and select the coordinate system so that 0 is not on S . Assume that the characteristic surface $C : |y| = |x|$ intersects S so that the set K_0 of S cut off by C is compact. Suppose that the surfaces $|y|^2 = |x|^2 + c$, for $0 \leq c \leq A$, cut off from S a one parameter decreasing family of compact sets K_c with K_A of co-dimension greater than 1. Then the solution of the Cauchy problem for (4.1) with data given on K_0 is uniquely determined in the region of $\mathbf{R}^{n+m}$ bounded by S and C .*

Examples. When $m = 1$ Equation (4.1) is normally hyperbolic. Suppose that S is a space-like surface. Then Theorem 11 yields the customary uniqueness theorem for the Cauchy problem by means of a Carleman-type estimate rather than the classical energy integral method.

An interesting case occurs when L is normally hyperbolic and S is a time-like cylinder in $\mathbf{R}^{n+1}$. If Cauchy data are prescribed on the surface of S for $-a \leq y \leq a$, then Theorem 11 establishes the uniqueness of the Cauchy problem for the interior of the cylinder up to a characteristic cone with vertex within the cylinder. Then the fact that the solution vanishes on the surface of the cone implies that it vanishes inside the entire cylinder.

If $m > 1$, $n > 1$, then an appropriate uniqueness theorem for the Cauchy problem entails prescribing data on a cylindrical-type domain S , as indicated in the preceding example. It is under such circumstances that the characteristic surfaces will intersect S in a family of compact sets, and it appears that if properly posed problems are to be obtained for ultrahyperbolic operators, the domains to be considered should be of the type we have described.

By the methods employed in [4] and [7], Theorem 11 can be extended to solutions of $Lu = F(x, y, u, \nabla u)$ in the case $m = 1$, i.e., the normal hyperbolic equation.

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