Factorization of Positive Operator Valued Functions on a Banach Space

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1. Introduction. The main purpose of this note is to extend the well-known results of R. G. Douglas [4] on factoring positive operator valued functions on a Hilbert space to positive operator valued functions on a Banach space. The problem of factoring positive operators on a Hilbert space plays an important role in the study of Hilbert space valued stationary processes (see, for example, [8], [9], [10]). There is a similar tie between the factorability of positive operator valued functions on a Banach space and Banach space valued stationary processes [1], [2], [3], [12].

When f is a positive operator valued function on a Hilbert space, $f^{1/2}$, the square root of f whose existence is known, plays a significant role. But when f is a positive operator valued function on a Banach space $\mathfrak{X}$ the existence of a square root in the ordinary way does not make sense. Nevertheless we can prove (cf. Lemma 3.1) the existence of a measurable operator valued function A on $\mathfrak{X}$ into some auxiliary Hilbert space which behaves almost like a square root in the sense that $f = A^*A$. The operator valued function A , called a quasi square root, enables us to extend to the Banach space case a lemma of Helson ([6], p. 117) and the main lemma of Douglas [4].

In §2 we set up necessary terminologies and state some known results. Section 3 includes the proof of existence of a quasi square root and two lemmas on the characterization of factorability of a positive operator on a Banach space. The results of §4 extend in a natural way most of the work of R. G. Douglas [4] to the Banach space case. In establishing these results we make use of our fundamental lemmas proved in §3 and Douglas' techniques employed in [4].

§2. Ancillary results. In this paper all Banach spaces and Hilbert spaces are assumed to be separable. Let $\mathcal{K}$ be a Hilbert space and let $L_2(\mathcal{K})$ denote the Hilbert space of functions defined on the unit circle with values in $\mathcal{K}$ that are weakly measurable and have square summable norm with respect to (w.r.t.) Lebesgue measure. The $L_2(\mathcal{K})$ inner product of two functions $g_1 = g_1(e^{i\theta})$ and

$g_2 = g_2(e^{i\theta})$ is given by $(1/2\pi) \int_c (g_1(e^{i\theta}), g_2(e^{i\theta})) d\theta$, where c denotes the interval $[0, 2\pi]$. The operator U defined to be multiplication by $e^{i\theta}$ on $L_2(\mathcal{K})$ is unitary and a shift. The subspace $H_2(\mathcal{K})$ consisting of those functions $g = g(e^{i\theta})$ in $L_2(\mathcal{K})$ for which $\int_c g(e^{i\theta}) e^{in\theta} d\theta = 0$ for $n > 0$ is an invariant subspace of U .

Let $\mathfrak{X}$ be a Banach space. $\mathfrak{X}^*$ denotes the space of all bounded conjugate linear functionals on $\mathfrak{X}$. For any two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, $B(\mathfrak{X}, \mathfrak{Y})$ denotes the class of all bounded linear operators on $\mathfrak{X}$ into $\mathfrak{Y}$. A weakly measurable $B(\mathfrak{X}, \mathcal{K})$ valued function $A = A(e^{i\theta})$ is said to be analytic if for each $x \in \mathfrak{X}$, $A(e^{i\theta})x \in H_2(\mathcal{K})$.

2.1 Definition. Let f be in $B(\mathfrak{X}, \mathfrak{X}^*)$. f is said to be nonnegative if for all $x \in \mathfrak{X}$, $(f(x))(x) \geq 0$. $B^+(\mathfrak{X}, \mathfrak{X}^*)$ denotes the class of all nonnegative operators on $\mathfrak{X}$ into $\mathfrak{X}^*$.

2.2 Definition. Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. We say that $f(e^{i\theta})$ is factorable if there exists a Hilbert space $\mathcal{K}$ and an analytic $B(\mathfrak{X}, \mathcal{K})$ valued function $A = A(e^{i\theta})$ such that

$$f(e^{i\theta}) = A^*(e^{i\theta})A(e^{i\theta}) \text{ a.e.,}$$

in the sense that $(f(e^{i\theta})x)(y) = (A(e^{i\theta})x, A(e^{i\theta})y)$ for all $x, y \in \mathfrak{X}$.

2.3 Remark. In definition 2.2 we can show that

$$A(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{ik\theta},$$

where the A_k 's are bounded; in fact we can show that

$$\sum_{k=0}^{\infty} |A_k x|^2 \leq C |x|^2, \quad x \in \mathfrak{X},$$

for some finite constant C .

To prove these we proceed as follows. It is easy to see that $A(e^{i\theta}) = \sum_{k \geq 0} A_k e^{ik\theta}$, where the A_k 's are linear operators defined on $\mathfrak{X}$. We observe that $\sum_{k \geq 0} |A_k x|^2 = \int_c |A(e^{i\theta})x|^2 d\theta = \int_c (f(e^{i\theta})x)(x) d\theta < \infty$. To complete the proof it is sufficient to show that the operator $T : \mathfrak{X} \rightarrow L_2(\mathcal{K})$ defined by $Tx = A(e^{i\theta})x$ is bounded. By the standard method ([7] p. 85) one can show that T is closed, and being everywhere defined, it follows that T is bounded.

We now state some terminologies concerning Banach space valued stationary processes ([1], [2], [3], [12]). Let $\mathfrak{X}$ be a Banach space and $\mathcal{K}$ a Hilbert space. By a stationary process we mean a $B(\mathfrak{X}, \mathcal{K})$ valued sequence ξ_n , $-\infty < n < \infty$, of bounded linear operators on $\mathfrak{X}$ into $\mathcal{K}$ such that $\xi_n^* \xi_m$ depends only on $m - n$. The operator $R(m - n) = \xi_n^* \xi_m$ is called the covariance operator of the process. It is known [1] that we have the representation

$$(2.4) \quad R(m - n) = (1/2\pi) \int_c e^{-i(m-n)\theta} dF(e^{i\theta}),$$

where the spectral distribution F is a measure on the Borel sets of the unit circle

with values in $B^+(\mathfrak{X}, \mathfrak{X}^*)$, and the above integral is taken in the weak sense. When $F(\cdot)$ is absolutely continuous (a.c.) w.r.t. the Lebesgue measure, its density f is a bilinear functional on $\mathfrak{X} \times \mathfrak{X}$. f is called the spectral density of the process.

Let $f = f(e^{i\theta})$ be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. It is shown in [1] that there exists a stationary process ξ_n , $-\infty < n < \infty$, $\xi_n : \mathfrak{X} \rightarrow \mathfrak{K}$, for some Hilbert space $\mathfrak{K}$, whose spectral density is $f(e^{i\theta})$. Letting $U(\xi_n x) = \xi_{n+1}x$, for all $x \in \mathfrak{X}$ and $-\infty < n < \infty$, we obtain a unitary operator on

$$\mathfrak{S}(\xi_n \mathfrak{X}, -\infty < n < \infty),$$

where $\mathfrak{S}(\cdot)$ stands for the closed linear subspace generated by.

Let $u_n a = U^n a$ for each $a \in \mathfrak{A} = \mathfrak{S}(\xi_0 \mathfrak{X})$. It is clear that u_n , $-\infty < n < \infty$, is a stationary process on $\mathfrak{A}$ into $\mathfrak{K}$. Call F_u the spectral distribution of u_n . It is easy to see that $F_\xi = T^* F_u T$, where $T = \xi_0$ and F_ξ is the distribution of ξ_n .

2.5 Lemma. *With the above notation, the distribution F_u is a.c. w.r.t. the Lebesgue measure, and its density f_u is a bilinear functional on $\mathfrak{A} \times \mathfrak{A}$. Moreover we have*

$$(2.6) \quad (fx)(y) = f_u(Tx, Ty); \quad x, y \in \mathfrak{X}.$$

Proof. Let $a = Tx$ for some $x \in \mathfrak{X}$. Since $(F_\xi(\cdot)x)(x)$ is a.c. w.r.t. the Lebesgue measure and $(F_\xi(\cdot)x)(x) = (F_u(Tx))(Tx) = (F_u a)(a)$, it follows that $(F_u a)(a)$ is a.c. w.r.t. the Lebesgue measure for all $a \in T(\mathfrak{X})$. By §66 [5] we have that $(F_u a)(a)$ is a.c. on $\mathfrak{A}$. Now (2.6) easily follows.

§3. Main lemmas. In this section we will show the existence of a quasi square root for a $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. We then extend a lemma due to Helson ([6], p. 117) and the main lemma of Douglas [4] to the Banach space case.

Our first lemma is on the existence of a quasi square root.

3.1 Lemma. *Let f be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. Then there exists a Hilbert space $\mathfrak{K}$ and a measurable $B(\mathfrak{X}, \mathfrak{K})$ valued function $Q = Q(e^{i\theta})$ on the unit circle such that*

$$(3.2) \quad f(e^{i\theta}) = Q^*(e^{i\theta})Q(e^{i\theta}) \text{ a.e.,}$$

in the sense that

$$(f(e^{i\theta})x)(y) = (Q(e^{i\theta})x, Q(e^{i\theta})y), \quad x, y \in \mathfrak{X}.$$

Proof. By Lemma 2.5 there exists a Hilbert space $\mathfrak{A}$, an operator T in $B(\mathfrak{X}, \mathfrak{A})$ and a bilinear functional g on $\mathfrak{A} \times \mathfrak{A}$ with $\mathfrak{A} = \mathfrak{S}(T\mathfrak{X})$ such that

$$(1) \quad (f(e^{i\theta})x)(y) = g(e^{i\theta})(Tx, Ty) \quad \text{for } x, y \in \mathfrak{X}.$$

Let $\{x_i, 1 \leq i < \infty\}$ be a countable dense subset of $\mathfrak{X}$. Consider $\{Tx_i\}_{i \geq 1}$ and let $\{e_i\}_{i \geq 1}$ be the Gram-Schmidt orthogonalization of $\{Tx_i\}_{i \geq 1}$. Set

$$g_{ii}(e^{i\theta}) = g(e^{i\theta})(e_i, e_i).$$

It is clear that $g_{i,i}(e^{i\theta})$ defines a nonnegative matrix (not necessarily bounded). The result of p. 112 [6] can be applied to show the existence of a Hilbert space $\mathcal{K}$ and a sequence $\{F_i\}_{i \geq 1}$ of functions in $L_2(\mathcal{K})$ such that $g_{i,i}(e^{i\theta}) = (F_i(e^{i\theta}), F_i(e^{i\theta}))$. Following [6], p. 113, we obtain an operator A on the finite linear combinations of $\{e_i\}_{i \geq 1}$ by

$$(2) \quad A\left(\sum_{i=1}^N a_i e_i\right) = \sum_{i=1}^N a_i F_i.$$

It is clear that

$$(3) \quad \left|A\left(\sum_{i=1}^N a_i e_i\right)\right|^2 = \sum_{i,j=1}^N a_i \bar{a}_j g_{i,j}(e^{i\theta}).$$

Let $x \in \{x_i, 1 \leq i < \infty\}$. Then $Tx = \sum_{1 \leq i \leq N} a_i e_i$, for some $a_i, 1 \leq i \leq N < \infty$. Then by (3) we have

$$(4) \quad |ATx|^2 = \sum_{i,j=1}^N a_i \bar{a}_j g_{i,j}(e^{i\theta}).$$

By (1), (4) and bilinearity of g we have

$$(5) \quad |ATx|^2 = (f(e^{i\theta})x)(x), \quad x \in \{x_i, 1 \leq i < \infty\}.$$

Because f is weakly summable by (5) we have that the operator $AT : \mathfrak{X} \rightarrow L_2(\mathcal{K})$ is densely defined. For almost all θ 's we define $B(e^{i\theta})$ on $\{x_i, 1 \leq i < \infty\}$ by $B(e^{i\theta})x = (ATx)(e^{i\theta})$. Extend $B(e^{i\theta})$ to $\mathfrak{X}_0$, the finite linear combinations of x_i 's through linearity. From (5) we obtain

$$(6) \quad (B(e^{i\theta})x, B(e^{i\theta})y) = (f(e^{i\theta})x)(y), \quad x, y \in \mathfrak{X}_0.$$

This shows that for almost all θ 's, $B(e^{i\theta})$ is well-defined and bounded on $\mathfrak{X}_0$. Hence *a.e.* $B(e^{i\theta})$ has a continuous extension to $\mathfrak{X}$. Call this extension $Q(e^{i\theta})$. It follows by (6) and the continuity of Q and f that

$$Q^*(e^{i\theta})Q(e^{i\theta}) = f(e^{i\theta}).$$

Furthermore, from the continuity of $Q(e^{i\theta})$ and (2), it follows that $Q(e^{i\theta})$ is weakly measurable. This completes the proof.

Lemma 3.1 allows us to make the following definition.

3.3 Definition. The function $Q = Q(\cdot)$ in lemma 3.1 is called a quasi square root of f .

In the absence of the square root, a quasi square root has almost all the desired properties. In 4.14 we provide an example of a nonnegative operator valued function on ℓ_1 into ℓ_∞ and construct a quasi square root for it. One of its applications is demonstrated in the following lemma where we extend to the Banach space a result due to Helson ([6], p. 117). First we introduce the following definition.

3.4 Definition. Let f be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. Let Q be a quasi square root of f with values in $B(\mathfrak{X}, \mathfrak{K})$. $\mathfrak{M}(Q)$ will denote the smallest subspace of $L_2(\mathfrak{K})$ invariant with respect to U (U is multiplication by $e^{i\theta}$) that contains the functions $\{Q(e^{i\theta})x : x \in \mathfrak{X}\}$. A function $p(e^{i\theta}) = \sum_{0 \leq n \leq N} e^{in\theta} x_n$ with $x_n \in \mathfrak{X}$ is said to be an analytic trigonometric polynomial and the set of all such polynomials will be denoted by $\mathfrak{U}$. It is clear that $\mathfrak{M}(Q)$ is the norm closure of $Q(\mathfrak{U})$ in $L_2(\mathfrak{K})$.

3.5 Lemma. Let f be a weakly summable $B(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. Then f is factorable iff for any Q , a quasi square root of f , $\mathfrak{M}(Q)$ contains no non-trivial reducing subspace of U .

Proof. Suppose $\mathfrak{M}(Q) \subseteq L_2(\mathfrak{K})$ contains no non-trivial reducing subspace. Then ([6] p. 61), it has the form $VH_2(\mathfrak{K})$, where V is a measurable isometric operator on $\mathfrak{K}$ into $\mathfrak{K}$. In particular $Qx = VG_x$, $G_x \in H_2(\mathfrak{K})$. Therefore

$$(1) \quad (fx)(x) = (Qx, Qx) = (VG_x, VG_x) = (G_x, G_x).$$

Define the operator $\Phi(e^{i\theta})$ on $\mathfrak{X}$ into $\mathfrak{K}$ by

$$(2) \quad \Phi(e^{i\theta})x = G_x(e^{i\theta}), \quad x \in \mathfrak{X} \text{ a.e.}$$

It is clear that $\Phi(e^{i\theta})$ is linear, and moreover by (1) and (2) we have

$$(3) \quad \begin{aligned} |\Phi(e^{i\theta})x|^2 &= |G_x(e^{i\theta})|^2 \\ &= |V(e^{i\theta})G_x(e^{i\theta})|^2 = (f(e^{i\theta})x)(x). \end{aligned}$$

Hence Φ is bounded. Then by (2) and (3) and the weak summability of f it follows that Φ is an analytic $B(\mathfrak{X}, \mathfrak{K})$ valued function. Hence

$$(4) \quad f(e^{i\theta}) = \Phi^*(e^{i\theta})\Phi(e^{i\theta}).$$

By (4) f is factorable.

Now assume f is factorable, say $f = \Phi^*\Phi$, where Φ is an analytic $B(\mathfrak{X}, \mathfrak{K})$ valued function. Let Q be a $B(\mathfrak{X}, \mathfrak{K})$ valued function which is a quasi square root of f . We can compare Φ and Q as follows. Define

$$(5) \quad V(\Phi p) = Qp, \quad p \in \mathfrak{U}.$$

We have $(V(\Phi p), V(\Phi p)) = (Qp, Qp) = (2\pi)^{-1} \int_0^{2\pi} (f(e^{i\theta})p(e^{i\theta})(p(e^{i\theta}))) d\theta = (\Phi p, \Phi p)$. Hence we can extend V to an isometry on $\mathfrak{M}(\Phi)$ onto $\mathfrak{M}(Q)$, where $\mathfrak{M}(\Phi) = \mathfrak{S}(\Phi p, p \in \mathfrak{U})$. This mapping commutes with multiplication with $e^{i\theta}$. Now $\mathfrak{M}(\Phi)$ contains no non-trivial reducing subspace of the shift U , because it is a part of $H_2(\mathfrak{K})$. Hence its image, $\mathfrak{M}(Q)$, under V cannot contain a non-trivial reducing subspace.

Now we can extend the main lemma of Douglas as follows.

3.6 Lemma. Let f be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle. Then f is factorable iff for each non-zero function $g \in \mathfrak{M}(Q)$, the measure of $Z_g = \{e^{i\theta}, g(e^{i\theta}) = 0\}$ is positive.

Proof. This follows from lemma 3.5 and the fact that an invariant subspace of the shift U contains a non-trivial reducing subspace of U iff it contains a non-zero function g for which the measure of Z_g is positive.

§4. Main theorems. In this section we extend most of the results of Douglas [4] to the Banach space case. Lemma 3.6 is repeatedly used in the course of the proof of our theorems.

4.1 Theorem. *Let f_1 and f_2 be weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle and Q_1 and Q_2 be $B(\mathfrak{X}, \mathfrak{K})$ valued quasi square roots of f_1 and f_2 respectively such that*

$$(a) \quad f_2(e^{i\theta}) \geq f_1(e^{i\theta}) \text{ a.e.},$$

$$(b) \quad \mathfrak{N}(Q_2^*) \supseteq \mathfrak{N}(Q_1^*) \text{ a.e.},$$

$$(c) \quad \varphi(e^{i\theta}) |Q_1(e^{i\theta})x| \geq |Q_1^*(e^{i\theta})Q_2(e^{i\theta})x|, \quad \text{a.e.},$$

where φ is some nonnegative scalar valued function. If f_1 is factorable, then f_2 is factorable.

Proof. An argument similar to the one used in [4], Theorem 1, can be given to show the existence of a function $h \in \mathfrak{M}(Q_1)$ such that $Z_g \setminus Z_h$ and $Z_h \setminus Z_g$ have both zero measure. Now applying lemma 3.5 we conclude that f_2 is factorable.

4.2 Remark. In case $\mathfrak{X}$ is a Hilbert space with $Q_1 = f_1^{1/2}$, $Q_2 = f_2^{1/2}$, (a) and (b) imply that $\mathfrak{N}(f_1) = \mathfrak{N}(f_2)$. Also, in this case, condition (c) is the same as $\varphi(e^{i\theta})f_1(e^{i\theta}) \geq Q_2(e^{i\theta})f_1(e^{i\theta})Q_2(e^{i\theta})$ a.e.. Hence our result 4.1 extends the main theorem of Douglas [4].

The following theorem does not seem to follow from Theorem 4.1. However we provide a direct proof of it based on lemma 3.5.

4.3 Theorem. *Let f_1 and f_2 be weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle such that*

$$f_2(e^{i\theta}) \geq f_1(e^{i\theta}) \geq \varphi(e^{i\theta})f_2(e^{i\theta}) \text{ a.e.},$$

where $\varphi(e^{i\theta})$ is a positive scalar valued function. If f_1 is factorable, then f_2 is factorable.

Proof. By lemma 3.5 it is sufficient to prove that for a non-zero g in $\mathfrak{M}(Q_2)$, the measure of Z_g is zero. Let $Q_2 p_n \rightarrow g$ in $L_2(\mathfrak{K}_2)$; then $f_1 < f_2$ implies that there exists $h \in L_2(\mathfrak{K}_1)$ such that $Q_1 p_n \rightarrow h$ in $L_2(\mathfrak{K}_1)$. Choose a subsequence of p_n , denoting it again by p_n , such that

$$(1) \quad \begin{cases} Q_2(e^{i\theta})p_n(e^{i\theta}) & \text{converges a.e. in } \mathfrak{K}_2 \text{ to } g(e^{i\theta}) \\ Q_1(e^{i\theta})p_n(e^{i\theta}) & \text{converges a.e. in } \mathfrak{K}_1 \text{ to } h(e^{i\theta}). \end{cases}$$

By (1) we have, a.e., $|h(e^{i\theta})|_{\mathfrak{K}_1} \leq |g(e^{i\theta})|_{\mathfrak{K}_2}$. Hence the measure of $Z_g \setminus Z_h = 0$. Similarly, by (1) and assumption, for almost all θ 's we have

$$\begin{aligned}
 |g(e^{i\theta})| &= \lim_{n \rightarrow \infty} |Q_2(e^{i\theta})p_n(e^{i\theta})| \\
 &\leq \frac{1}{\varphi(e^{i\theta})} \lim_{n \rightarrow \infty} |Q_1(e^{i\theta})p_n(e^{i\theta})| \\
 &= \frac{1}{\varphi(e^{i\theta})} |h(e^{i\theta})|.
 \end{aligned}$$

Hence the measure of $Z_h \setminus Z_g$ is zero. So we have shown that $Z_h = Z_g$ a.e., Now by lemma 3.5 and the factorability of f_1 it follows that f_2 is factorable.

The following theorem is a slight extension of Theorem 4.1.

4.4 Theorem. *Let f_1 and f_2 be weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle such that*

- (a) $f_2(e^{i\theta}) \geq m(e^{i\theta})f_1(e^{i\theta})$ a.e. where $m(e^{i\theta})$ is nonnegative scalar valued and $\int_{\epsilon} \log m(e^{i\theta}) d\theta > -\infty$,
- (b) $\Re(Q_2^*) \supseteq \Re(Q_1^*)$, a.e.,
- (c) $\varphi(e^{i\theta}) |Q_1(e^{i\theta})x| \geq |Q_2^*(e^{i\theta})Q_1(e^{i\theta})x|$ a.e.,

where φ is a nonnegative scalar valued function.

If f_1 is factorable, then f_2 is factorable.

Proof. The proof is a combination of a standard method and theorem 4.1. Let $t(e^{i\theta}) = 1 \wedge m(e^{i\theta})$. Then $0 \leq t(e^{i\theta}) \leq 1$ a.e. and $\int \log t(e^{i\theta}) d\theta > -\infty$. By Szegő's theorem there exists $k = k(e^{i\theta}) \in H^2$ such that $t(e^{i\theta}) = |k(e^{i\theta})|^2$. Assuming $f_1(e^{i\theta}) = \Phi^*(e^{i\theta})\Phi(e^{i\theta})$ we have

$$t(e^{i\theta})f_1(e^{i\theta}) = (k(e^{i\theta})\Phi(e^{i\theta}))^*(k(e^{i\theta})\Phi(e^{i\theta})).$$

Applying theorem 4.1 to $f_1(e^{i\theta})$ and $t(e^{i\theta})f_1(e^{i\theta})$ we conclude the factorability of f_2 .

We now state the following extension of Theorem 4.1, whose proof is omitted.

4.5 Theorem. *Let f_1 and f_2 be weakly summable $R^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle such that*

$$f_2(e^{i\theta}) \geq m(e^{i\theta})f_1(e^{i\theta}) \geq \varphi(e^{i\theta})f_2(e^{i\theta}) \text{ a.e.,}$$

where m and φ are positive scalar valued functions, with $\int_{\epsilon} \log m(e^{i\theta}) d\theta > -\infty$. If f_1 is factorable, then f_2 is factorable.

Now we shall give some Devinatz type theorems. First we introduce the following definition.

4.6 Definition. Let f be a $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle; we say that

- (a) f has a "conjugate analytic null function" if
 - (i) $\Re(f(e^{i\theta}))$ is complementary a.e. and
 - (ii) $|P(e^{i\theta})x|^2 = (\Phi(e^{i\theta})x, \Phi(e^{i\theta})x); x \in \mathfrak{X}$,

where P is the projection into the complement of $\Re(f(e^{i\theta}))$ along $\Re(f(e^{i\theta}))$, and Φ is an analytic $B(\mathfrak{X}, \mathfrak{X})$ valued function.

(b) f has a “quasi conjugate analytic null function” if condition (a) (i) holds and (a) (ii) is replaced by

$$\begin{aligned} (\Phi(e^{i\theta})x, \Phi(e^{i\theta})x) &\leq |P(e^{i\theta})x|^2 \\ &\leq \varphi(e^{i\theta})(\Phi(e^{i\theta})x, \Phi(e^{i\theta})x), \quad x \in \mathfrak{X}, \end{aligned}$$

where φ is a positive scalar valued function.

We remark that in the Hilbert space case the terminologies “conjugate analytic null”, “quasi conjugate analytic null” and “conjugate analytic range” are all equivalent.

The following result is a generalization of Theorem 2 [4] to the Banach space case.

4.7 Theorem. *Let f be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle such that*

(a) *f has a quasi conjugate analytic null function.*

(b) *$(f(e^{i\theta})x)(x) \geq n(e^{i\theta})|P(e^{i\theta})x|^2$,*

where $n(e^{i\theta})$ is a positive valued function with $\int_{\mathbb{T}} \log n(e^{i\theta}) d\theta > -\infty$. Then f is factorable.

Proof. Let $t(e^{i\theta}) = 1 \wedge n(e^{i\theta})$. Then as in the proof of theorem 4.4, $t(e^{i\theta}) = |p(e^{i\theta})|^2$, with $p(e^{i\theta}) \in H^2$. Since $f = Q^*Q$, $\Re(f) = \Re(Q)$ and $(f(x))(y) = 0$, if either x or y is in $\Re(Q)$, we have

$$(1) \quad n(e^{i\theta}) |P(e^{i\theta})x|^2 \leq (f(e^{i\theta})x)(x) \leq |f(e^{i\theta})|^2 |P(e^{i\theta})x|^2.$$

By (1) and our assumptions we have

$$(\Phi(e^{i\theta})x, \Phi(e^{i\theta})x) \leq (f(e^{i\theta})x)(x) \leq |f(e^{i\theta})| \varphi(e^{i\theta})(\Phi(e^{i\theta})x, \Phi(e^{i\theta})x),$$

where φ is as in the definition 4.6 (b). Hence we have

$$\begin{aligned} (2) \quad (p(e^{i\theta})\Phi(e^{i\theta}))^*(p(e^{i\theta})\Phi(e^{i\theta})) &\leq f(e^{i\theta}) \\ &\leq \psi(e^{i\theta})(p(e^{i\theta})\Phi(e^{i\theta}))^*(p(e^{i\theta})\Phi(e^{i\theta})), \end{aligned}$$

where $\psi = |f|\varphi/t$. Hence by theorem 4.3, f is factorable.

Using this result we can prove the following theorem which generalizes Devinatz' theorem ([6], p. 119) to the Banach space case.

4.8 Theorem. *Let f be a weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle such that $f^{-1}(e^{i\theta})$ exists a.e. and is bounded. If $\int_{\mathbb{T}} \log [|f^{-1}(e^{i\theta})|^{-1}] d\theta > -\infty$, then f is factorable.*

Proof. Let us denote $|f^{-1}(e^{i\theta})|^{-1}$ by $n(e^{i\theta})$, so we have

$$(1) \quad n(e^{i\theta}) |x|^2 \leq (f(e^{i\theta})x)(x) \quad \text{and} \quad \int_{\mathbb{T}} \log n(e^{i\theta}) d\theta > -\infty.$$

Denoting the positive quantity $\int_c n(e^{i\theta}) d\theta$ by N^2 , from (1) we obtain

$$(2) \quad N^2 |x|^2 \leq \int_c (f(e^{i\theta})x)(x) d\theta.$$

By [1] there exists a Hilbert space $\mathcal{K}$ and an operator T in $B(\mathfrak{X}, \mathcal{K})$ such that

$$(3) \quad (T^*Tx)(x) = \int_c (f(e^{i\theta})x)(x) d\theta.$$

By (2), (3) and boundedness of T we have

$$(4) \quad N |x| \leq |Tx| \leq M |x| \quad \text{for all } x \in \mathfrak{X},$$

where $0 < N \leq M < \infty$. We note that $\mathfrak{N}(f(e^{i\theta})) = \{0\}$ so that the projection operator occurring in the last theorem is I . Then (1) and (4) guarantee the validity of the hypotheses of theorem 4.7. Hence f is factorable.

4.9 Remark. As we have seen above, the condition $\int_c \log [|f^{-1}(e^{i\theta})|^{-1}] d\theta > -\infty$ implies the existence of a Hilbert space $\mathcal{K}$ and a bounded linear operator T on $\mathfrak{X}$ onto $\mathcal{K}$ which is one to one. This means that the topology of $\mathfrak{X}$ can be obtained through an inner product. Hence one could also obtain our theorem 4.8 by appealing directly to the Hilbert space case.

Since no necessary and sufficient conditions have been found for the factorability of a positive operator valued function in the Banach space case (some result related to this question for the Hilbert space case is known [11]), it is useful to know under what condition the finite sum, limit and series of factorable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions is factorable. Based on lemma 3.6 proofs similar to the ones used in [4] can be given to establish the following theorems.

4.10 Theorem. Let f_1 and f_2 be weakly summable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle. If f_1 and f_2 are factorable, then $f = f_1 + f_2$ is factorable.

4.11 Theorem. Let $\{f_i\}_{i \geq 1}$ be an increasing sequence of factorable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function on the unit circle and φ be a nonnegative scalar valued function such that

$$(a) \quad \lim_{i \rightarrow \infty} f_i(e^{i\theta}) = f(e^{i\theta}) \quad a.e.,$$

$$(b) \quad \lim_{n \rightarrow \infty} \int_c (f_i(e^{i\theta})x)(x) d\theta < \infty, \quad \text{for all } x \in \mathfrak{X},$$

$$(c) \quad |f(e^{i\theta})| \leq \varphi(e^{i\theta}) \quad a.e..$$

Then f is factorable.

4.12 Theorem. Let $\{f_i\}_{i \geq 1}$ be a sequence of factorable $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued functions on the unit circle and φ be a nonnegative scalar valued function such that

$$(a) \quad \sum_{i=1}^{\infty} f_i(e^{i\theta}) = f(e^{i\theta}),$$

$$(b) \quad \sum_{j=1}^{\infty} \int_c (f_j(e^{i\theta})x)(x) d\theta < \infty, \quad x \in \mathfrak{X},$$

$$(c) \quad |f(e^{i\theta})| \leq \varphi(e^{i\theta}) \quad a.e..$$

Then f is factorable.

4.13. Remark. Let Ψ be a $B(\mathfrak{X}, \mathfrak{K})$ valued function such that for each $x \in \mathfrak{X}$, $\Psi(e^{i\theta})x \in L_2(\mathfrak{K})$ and

$$\Psi = \sum_{k \geq 0} A_k e^{-in\theta}.$$

We say that Ψ is a conjugate analytic function. If there exists a conjugate analytic function Ψ such that $f = \Psi^* \Psi$, then f is said to be conjugately-factorable. We remark that all our results have dual statements concerning conjugate-factoring.

4.14 Example. Here we consider an example of a weakly summable $B(\mathfrak{X}, \mathfrak{X}^*)$ valued function f on the unit circle. Using the technique of lemma 3.1 we construct a quasi square root of f . We adhere to the notations of lemma 3.1.

Let $\mathfrak{X} = \ell^1$, $\mathfrak{K} = \ell^2$. Hence $\mathfrak{X}^* = \ell^\infty$. Take the partition $E_n = [\pi(2^n - 2)/2^{n-1}, \pi(2^n - 1)/2^{n-1}]$, $1 \leq n < \infty$, of $[0, 2\pi)$. For each $x = (x_1, x_2, \dots) \in \ell^1$ we define

$$f(e^{i\theta})x = (x_1, x_2, \dots, x_n, 0, 0, \dots) \quad \text{if } \theta \in E_n.$$

Since $(f(e^{i\theta})x)(x) = \sum_{1 \leq i \leq n} |x_i|^2$ for $\theta \in E_n$, we see that f is a nonnegative operator valued function. It is clear that $f = f(e^{i\theta})$ is measurable and $|f(e^{i\theta})|_B \leq 1$ for each θ . Now we give the construction of a quasi square root of f as outlined in lemma 3.1. We first obtain the operator T . Since $(Tx, Tx) = \sum_{i \geq 1} (1/2^{i-1}) |x_i|^2$, one can see that $T : \ell^1 \rightarrow \ell^2$ can be taken as

$$Tx = (x_1, x_2/2^{1/2}, x_3/(2^2)^{1/2}, \dots).$$

Hence the function g in the proof of lemma 3.1 is given by

$$g(e^{i\theta})(a, b) = \sum_{j=1}^n 2^{j-1} a_j \bar{b}_j, \quad \theta \in E_n;$$

$$a = (a_1, a_2, a_3, \dots) \quad \text{and} \quad b = (b_1, b_2, b_3, \dots) \quad \text{in } \ell^2.$$

Consider the countable set $\{x_i\}_{i \geq 1}$ in the proof of lemma 3.1 to be given by $x_i = (\delta_{ji})_{j=1}^\infty$. Then $Tx_i = (\delta_{ji}/(2^{j-1})^{1/2})_{j \geq 1}$. Now the Gram-Schmidt orthogonalization of $\{Tx_i\}_{i \geq 1}$ becomes $\{e_i\}_{i \geq 1}$, where $e_i = (\delta_{ji})_{j \geq 1}$. Hence the matrix valued function $[g_{ij}]_{i,j \geq 1}$ is given by

$$g_{i,j}(e^{i\theta}) = g(e^{i\theta})(e_i, e_j) = \begin{cases} 2^{i-1} & \text{if } i = j. \\ 0 & \text{if } i \neq j \end{cases}$$

Now it is easy to see that the functions $F_i \in L_2(\mathfrak{K})$ of the proof of lemma 3.1, can be taken to be the constant functions $F_i(e^{i\theta}) = ((2^{i-1})^{1/2} \delta_{ji})_{j \geq 1}$. The operator A is given by $A(e^{i\theta})(a_1, a_2, a_3, \dots) = (a_1, a_2 2^{1/2}, a_3 (2^2)^{1/2}, \dots, a_n (2^{n-1})^{1/2}, 0, 0,$

$\dots) \theta \in E_n$. Now for each $\theta \in E_n$, we have

$$\begin{aligned} B(e^{i\theta})x &= (ATx)(e^{i\theta}) = A(x_1, x_2/2^{1/2}, x_3/(2^2)^{1/2}, \dots) \\ &= (x_1, x_2, \dots, x_n, 0, 0, \dots). \end{aligned}$$

Hence we have

$$Q(e^{i\theta})x = (x_1, x_2, \dots, x_n, 0, 0, \dots), \theta \in E_n.$$

Obviously $Q(e^{i\theta})$ is measurable and

$$(f(e^{i\theta})x)(y) = (Q(e^{i\theta})x, Q(e^{i\theta})y), x, y \in \mathfrak{X}.$$

Clearly $Q(e^{i\theta})$ is bounded. So we have obtained an explicit form for this quasi square root.

4.15 Note. As we mentioned in the introduction, the factorization of f , a $B^+(\mathfrak{X}, \mathfrak{X}^*)$ valued function, arises naturally in the study of stationary processes taking values in a Banach space. In the Hilbert space case the analytic factor plays a crucial role in the prediction theory of stationary processes. In this case a necessary and sufficient condition for the factorability of f is known. These problems for the Banach space case are related to our present work; however to avoid a digression we will defer their discussion to another paper.

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