Morse Theory and Cell Decomposition on Kähler Manifolds and Related Spaces

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In [6], by using a technique due to Frankel [4], we showed how the Kähler homogeneous structure of the spaces $G/C(T')$ may be exploited to give a non-degenerate Morse function which allows one to read $H_*(G/C(T'); \mathbb{Z})$ from the diagram of the compact semisimple Lie group G . Here we extend Frankel's technique to certain (not necessarily Kähler) submanifolds of Kähler manifolds. As a particular example, we shall compute the $\mathbb{Z}_2$ homology of the real flag manifolds.

We shall also discuss the question of cell decompositions of the spaces in question. In particular we shall show that two cell decompositions of the complex (and real) Grassmann manifolds due to Borel [1] and Ehresmann [3] are actually the same. This will be done via a cell decomposition coming from a Morse function.

We assume the reader has some knowledge of Lie theory, roots and the like.

§1. The technique of Frankel. Let M be a compact Kähler manifold with Kähler two-form Ω . Let $\Psi = \{\psi_t\}$ be a one-parameter group of isometries of M with fixed set F and Killing field (infinitesimal isometry) X . Then

Theorem. (Frankel [4].) *If either $F \neq \emptyset$ or the first real Betti number of M vanishes, then $\iota_X \Omega = -df$, where ι_X denotes interior product by X and f is a smooth function. The critical set of f is a finite union of disjoint nondegenerate critical manifolds in the sense of Bott [2] and coincides with F . If $x \in F$, the tangent space to M at x splits orthogonally as $TM_x = TF_x \oplus (\oplus e_r)$, where each e_r is two dimensional and invariant under both ψ_{t*} and the almost complex structure operator J . The index of f at x is twice the number of e_r that ψ_{t*} rotates negatively with respect to the orientation of J . Finally, the Morse-Bott inequalities are equalities:*

$$b_i(M; \mathbb{K}) = \sum b_{i-\lambda_\alpha}(F_\alpha; \mathbb{K})$$

where the sum runs over the components F_α of F , λ_α is the index of f on F_α and K is a field of coefficients.

As a simple example of this, consider $\mathbf{CP}(1)$, the complex projective line, viewed as the Riemann sphere. Ψ acts by rotations about the vertical axis counter-earthwise. f can be shown to be proportional to the ordinary height function of $S^2 \subseteq \mathbf{R}^3$ ($-\mathrm{d}f = \iota_X \Omega$ is equivalent to $\mathrm{grad} f = -JX$). It has a maximum at the north pole with index 2 and a minimum at the south pole of index 0.

$\mathbf{RP}(1)$ sits in $\mathbf{CP}(1)$ as a great circle through the poles. f restricts to $\mathbf{RP}(1)$ as a nondegenerate Morse function. We generalize this, noting that $\mathbf{RP}(1)$ is the fixed set of an involution of $\mathbf{CP}(1)$.

§2. The extension. Let Ψ be a one-parameter group of isometries of a compact Kähler manifold M satisfying the hypotheses of Frankel's Theorem. Let τ be an involution (isometry of order two) of M with N a component of the fixed set of τ .

Definition. Ψ acts τ -transversally to N if for each $p \in N$, JX_p is orthogonal to TN_p only if $X_p = 0$.

Thus, roughly, the flow generated by Ψ is transversal to N except at points of F .

Definition. Ψ acts τ -coherently with respect to N if it acts τ -transversally and if for each $p \in N \cap F$ there exists a decomposition of TM_p , invariant under τ_* , of the form

$$TM_p = TF_p \oplus V^+ \oplus V^-,$$

where the Hessian quadratic form of f is positive (resp. negative) definite on V^+ (resp. V^-).

Note that the index of f at p is $\dim(V^-)$.

Theorem. If ψ acts τ -coherently with respect to N , then $g = f|_N$ has only nondegenerate critical manifolds whose union is $F \cap N$. The index of g at $p \in F \cap N$ is $\dim(V^- \cap TN_p)$.

The proof is straight forward—the transversality guarantees no extra critical points are introduced while the coherent decomposition guarantees nondegeneracy and computation of the indices.

§3. $G/C(T')$: a review of the Morse theory. In [6] we applied Frankel's technique to the Kähler homogeneous space $G/C(T')$ in the following manner, where G is a compact semisimple Lie group and T' is a subtorus.

Let T be a maximal torus of G that contains T' . Let Y be tangent to T' at the identity e so that $\exp(tY)$ is dense in T' . Choose a Weyl chamber $\mathfrak{F}$ for G that contains Y in its closure. Take the lexicographic ordering of the roots of

G induced by $\mathfrak{F}$. Denote the set of positive roots by Σ and the set of positive roots not vanishing identically on T'_* by Σ' . Thus $\alpha \in \Sigma'$ iff $\alpha(Y) > 0$. Let $K = C(T')$.

The Lie algebra $\mathfrak{g}$ of G splits canonically under the adjoint action of T as $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha} \mathfrak{a}_{\alpha}$. The tangent space to G/K at eK is naturally isomorphic to $\sum_{\alpha} \mathfrak{a}_{\alpha}$.

Choose $X \in \mathfrak{F}$ such that $\exp tX$ is dense in T . The maps $\psi_t : G/K \rightarrow G/K$ by $\psi_t(gK) = \exp(tX)gK$ give a one-parameter group of isometries. The fixed set is $\{gK \mid g \in N(T) = \text{normalizer of } T\}$ and is in one to one correspondence with the left cosets of W_K in W_G , where W_G (resp. W_K) is the Weyl group of G (resp. K).

If $g \in N(T)$, let φ_g be the corresponding element of W_G : $\varphi_g = \text{Ad}(g)|_T$. Then the index of Frankel's function f at gK is twice the number of roots $\alpha \in \Sigma'$ such that $\varphi_g \cdot \alpha < 0$ (i.e. $\alpha(\varphi_g^{-1}(X)) < 0$). (This statement may be converted to one about the diagram of G .)

§4. The complex flag manifold $\mathbf{CF}(n_1, n_2, \dots, n_k)$. If we set $\sum n_i = n$, the complex flag manifold $\mathbf{CF}(n_1, n_2, \dots, n_k)$ may be viewed in the form $G/C(T')$ where $G = SU(n)$ and T' is the k dimensional torus which is the center of $SU(n) \cap (U(n_1) \times \dots \times U(n_k)) = K$.

Now the critical points coming from §3 are given by gK with $g \in N(T)$. If $g \in N(T)$ there exists $g' \in N(T)$ such that $gK = g'K$ and g' is a matrix with exactly one 1 in each row and column and zeros elsewhere. (It may be necessary to introduce a negative sign on the entry of the last column so that g' has determinant $+1$ and is in $SU(n)$, not just in $U(n)$.)

We view a flag as an orthogonal decomposition of $\mathbf{C}^n$ into subspaces of dimension $n_1, n_2, \dots, n_k$. Under this, $g'K$ corresponds to the flag whose n_1 dimensional subspace is spanned by the first n_1 columns of g' , whose n_2 dimensional subspace is spanned by the next n_2 columns, etc. The fact that one may permute the first n_1 columns among themselves without changing their span corresponds to the fact that the critical points are in one to one correspondence with the elements of W_G/W_K , not W_G . (Note W_G is the symmetric group on n letters.)

Thus we may uniquely describe each critical point by a finite permutation σ where $\sigma(1) < \sigma(2) < \dots < \sigma(n_1)$, $\sigma(n_1 + 1) < \sigma(n_1 + 2) < \dots < \sigma(n_2)$, etc., and the $\sigma(i)$'s tell which standard basis vectors of $\mathbf{C}^n$ go into each subspace of the flag. This is equivalent to saying that up to sign in the last column, g' is the matrix $\delta_{\sigma(i), i}$. Denote g' by g_{σ} .

Recall that the positive roots of $SU(n)$ are $\alpha_{ij} = x_i - x_j$, where $i < j$. (There is no problem with the choice of Y and $\mathfrak{F}$ to make this true.) $\alpha_{ij} \in \Sigma'$ iff for some r , $i \leq n_r < j$. Thus the index of f at $g_{\sigma}K$ is $2 \# \{\alpha_{ij} \in \Sigma' \mid \sigma(i) > \sigma(j)\} \equiv 2\lambda_{\sigma}$.

As an example consider $\mathbf{CF}(1, 1, 2)$. $\Sigma' = \{x_1 - x_2, x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4\}$. Table 1 gives the various σ that determine the critical points and their indices. Hence the Betti numbers (any coefficients) of $\mathbf{CF}(1, 1, 2)$ are 1, 0, 2, 0, 3, 0, 3, 0, 2, 0, 1.

Table 1

σ	$2\lambda_\sigma$
(1, 2, 3, 4)	0
(1, 3, 2, 4)	2
(1, 4, 2, 3)	4
(2, 1, 3, 4)	2
(2, 3, 1, 4)	4
(2, 4, 1, 3)	6
(3, 1, 2, 4)	4
(3, 2, 1, 4)	6
(3, 4, 1, 2)	8
(4, 1, 2, 3)	6
(4, 2, 1, 3)	8
(4, 3, 1, 2)	10

§5. The real flag manifold $\mathbf{RF}(n_1, \dots, n_k)$. The group automorphism of $SU(n)$ that sends each matrix to its complex conjugate induces an involution τ on $\mathbf{CF}(n_1, \dots, n_k)$. Denote the action of τ on gK by $g^\tau K$.

Claim 1. *The component, F_1 , of the fixed set of τ that contains eK is diffeomorphic to $\mathbf{RF}(n_1, \dots, n_k)$.*

Proof. Let $h \in SO(n) \subset SU(n)$. gK is fixed by τ iff $gK = g^\tau K$ iff $g^{-1}g^\tau \in K$. But $g^{-1}g^\tau = g^{-1}h^{-1}hg^\tau = (hg)^{-1}(hg)^\tau$ since $h^\tau = h$. Thus if gK is fixed so is hgK , and each component of the fixed set is invariant under the left action of $SO(n)$. The claim will follow if we can show that $M \equiv \{hK \mid h \in SO(n)\} = F_1$, since M is clearly diffeomorphic to

$$SO(n)/SO(n) \cap (O(n_1) \times \dots \times O(n_k)) = \mathbf{RF}(n_1, \dots, n_k).$$

The tangent space to G/K at eK is essentially $\sum_{\alpha} a_{\alpha}$. If $\alpha = x_i - x_j$, then a_{α} is all complex matrices of the form

$$i \begin{bmatrix} & i & & j & \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & z & \cdots \\ & \vdots & & \vdots & \\ j & \cdots & -\bar{z} & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \\ & \vdots & & \vdots & \end{bmatrix} = W_z,$$

where $z \in \mathbf{C}$. $\tau_* W_z = W_z$ and this is W_z iff $z \in \mathbf{R}$. Hence W_z is fixed by τ_* iff W_z is real skew symmetric, i.e. tangent to $SO(n)$. Since the fixed set is totally geodesic, this implies that $F_1 \subset M$. But clearly $M \subset F$, and thus $F_1 = M$.

Claim 2. *The group Ψ of §§3–4 acts τ -transversally to M .*

Proof. First note that if $gK \in M$ is not fixed by ψ_t , then for small positive t , $\psi_t(gK) \notin M$. Hence X is not tangent to M at gK unless $X = 0$ (i.e. gK fixed). The result will follow by establishing that $T_{gK}(SU(n)/K) \simeq T_{gK}(M) \oplus JT_{gK}(M)$, where $g \in SO(n)$. Since J is left invariant, we need only show this result for $g = e$. J acts on a_α , $\alpha \in \Sigma'$, by $JW_x = W_{ix}$. Since $T_{eK}(M)$ is spanned by matrices of the form W_x , $x \in \mathbf{R}$, the result is immediate.

Claim 3. *Ψ acts τ -coherently on M .*

Proof. Let $gK \in M$ be fixed by Ψ . Then we have that $g^r = g$. Hence, $\tau_* \cdot L_{g^*} = L_{g^*} \cdot \tau_*$. But $T_{gK}(G/K) = L_{g^*} \sum_{\Sigma'} a_\alpha$ and each a_α is invariant under τ_* and J . The claim follows.

Hence, by §2, our function on $\mathbf{C}F(n_1, \dots, n_k)$ restricts to $\mathbf{R}F(n_1, \dots, n_k)$ with the indices being the λ_σ of §4.

Let c_i be the number of critical points of index i . The Morse inequalities state that $b_i \leq c_i$ for all i , where the b_i are the Betti numbers (for any field of coefficients).

Claim 4. *If the coefficients are $\mathbf{Z}_2$, the Morse inequalities are equalities.*

Proof. (by induction on n). The case $n = 1$ is trivial.

The involution of $\mathbf{R}^n$ given by $\varphi(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_k)$ induces an involution on $\mathbf{R}F(n_1, \dots, n_k)$. The fixed set is $\bigcup_{i=1}^k M_i$, where M_i is diffeomorphic to $\mathbf{R}F(n_1, \dots, n_i - 1, \dots, n_k)$. By a well known theorem of transformation groups due to Floyd,

$$\sum_i b_i(\bigcup_i M_i; \mathbf{Z}_2) \leq \sum_i b_i(\mathbf{R}F(n_1, \dots, n_k); \mathbf{Z}_2).$$

The Morse inequalities imply that

$$\sum b_i(\mathbf{R}F(n_1, \dots, n_k); \mathbf{Z}_2) \leq \sum c_i = \frac{n!}{n_1! \dots n_k!}.$$

By the induction hypothesis $\sum b_i(M_i; \mathbf{Z}_2) = (n-1)!/n_1! \dots (n_i-1)! \dots n_k!$. Summing this expression over j and combining with the above formulas give

$$\frac{n!}{n_1! \dots n_k!} \leq \sum b_i(\mathbf{R}F(n_1, \dots, n_k); \mathbf{Z}_2) \leq \frac{n!}{n_1! \dots n_k!}.$$

Hence, we must have Morse equalities when $\sum n_i = n$ also. Thus we have

Theorem. $b_i(\mathbf{R}F(n_1, \dots, n_k); \mathbf{Z}_2) = b_{2i}(\mathbf{C}F(n_1, \dots, n_k); \mathbf{Z}_2) = b_{2i}(\mathbf{C}F(n_1, \dots, n_k); \mathbf{Z})$ (recall $H_*(\mathbf{C}F(n_1, \dots, n_k))$ is torsion free). In terms of Poincaré polynomials, $P_{\mathbf{Z}}(\mathbf{C}F(n_1, \dots, n_k); t) = P_{\mathbf{Z}_2}(\mathbf{R}F(n_1, \dots, n_k); t^2)$.

§6. The case of critical manifolds. Consider the one-parameter group of isometries of $\mathbf{C}^n$ given by $\varphi_t(z_1, \dots, z_n) = (e^{it}z_1, z_2, \dots, z_n)$. This induces a

one-parameter group of isometries of $\mathbf{CF}(n_1, \dots, n_k)$ whose critical set is a disjoint union of submanifolds diffeomorphic to $\mathbf{CF}(n_1, \dots, n_i - 1, \dots, n_k)$, $i = 1, 2, \dots, k$. The index on $\mathbf{CF}(n_1, \dots, n_i - 1, \dots, n_k)$ can be shown to be $2\mu_i$, where $\mu_i = n_1 + n_2 + \dots + n_{i-1}$ ($\mu_1 = 0$). The action of φ_i is τ -coherent so we have similar results with indices μ_i for the real case.

Following Bott [2] we have

Theorem. $\mathbf{CF}(n_1, \dots, n_k)$ has the same homotopy type as $\mathbf{CF}(n_1 - 1, n_2, \dots, n_k)$ with $2\mu_i$ —dimensional vector bundles over $\mathbf{CF}(n_1, \dots, n_i - 1, \dots, n_k)$ attached, $i = 2, 3, \dots, k$. A similar result holds for the real case with the bundles having dimension μ_i .

In terms of homology, in the language of Poincaré polynomials, this result is

Corollary.

$$P_{\mathbf{Z}}(\mathbf{CF}(n_1, \dots, n_k); t) = \sum_{i=1}^k t^{2\mu_i} P_{\mathbf{Z}}(\mathbf{CF}(n_1, \dots, n_i - 1, \dots, n_k); t).$$

$$P_{\mathbf{Z}_2}(\mathbf{RF}(n_1, \dots, n_k); t) = \sum_{i=1}^k t^{\mu_i} P_{\mathbf{Z}_2}(\mathbf{CF}(n_1, \dots, n_i - 1, \dots, n_k); t).$$

It is known that the Poincaré polynomials for the quaternionic flag manifolds are found by replacing t by t^2 in the corresponding complex polynomial. Hence the corollary is true for the quaternionic case if we replace $\mathbf{C}$ by $\mathbf{H}$ and $t^{2\mu_i}$ by $t^{4\mu_i}$. We do not know if the stronger homotopy result holds for the quaternionic case. In [7] we give a proof that it does if $k = 2$ (Grassmann manifold).

§7. Cell decompositions. Let f be a smooth real valued function on a Riemannian manifold M . One may then discuss the gradient vector field and the integral curves of this field. In particular, one may consider the set of all points lying on integral curves that have a particular nondegenerate critical point x as upper limit point. Morse [5] showed that this set $MC(x)$ is diffeomorphic to an open cell of dimension equal to the index of f at x . The tangent space to the cell of x is the space spanned by the eigenvectors of the Hessian that have negative eigenvalues. If f has only nondegenerate critical points and M is compact, the Morse cells give a cell decomposition of M . Note that a change in the Riemannian metric may change the cells.

For the case of the complex flag manifolds it can be shown that $JX = -\text{grad } f$ is tangent to the real flag manifolds as they are embedded. Hence the integral curves of $g = f|_{\mathbf{RF}}$ are integral curves of f also. Thus the Morse cells of g are the intersection of the Morse cells of f with the real flag manifolds.

In [6] we showed that the Morse cells coming from our function of §3 coincide with certain cells found by Borel [1] by a group action. In particular this implied that the Morse cells are complex analytic submanifolds. The proof tacitly used the following lemma, which we shall use below to prove that the Borel cell decomposition agrees with the classical Schubert decomposition of the complex Grassmann manifolds.

Lemma. Let f be a nondegenerate function on a Riemannian manifold M with p a critical point. Let B be the Morse cell of f through p . Suppose N is a connected submanifold of M which is invariant under the flow generated by $\text{grad } f$ (and hence a union of gradient lines of maximal extension) and which contains p . If $TN_p = TB_p$, then $B \subset N$.

Proof. Consider $g = f|_N$ and $h = f|_B$. h has p as a nondegenerate absolute maximum by the construction of B . Note that $\text{grad } g = \text{grad } f|_N$, so that p is the only critical point of g .

If $H(f)$ denotes the Hessian quadratic form of f at p , then

$$H(g) = H(f)|_{TN_p \times TN_p} = H(f)|_{TB_p \times TB_p} = H(h),$$

which is negative definite. Hence p is a nondegenerate relative maximum of g and the Morse cell of g at p , which is made up entirely of integral curves of $\text{grad } g = \text{grad } f$ that terminate at p , must be all of B . Hence $B \subset N$.

The Borel cell decomposition of G/K is found in the following manner. As before, the Lie algebra of G is $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha} \mathfrak{a}_{\alpha} \cdot \mathbb{C} \otimes \mathfrak{a}_{\alpha}$ is spanned by two complex vectors E_{α} and $E_{-\alpha}$ where $E_{\alpha} + E_{-\alpha}$ and $i(E_{\alpha} - E_{-\alpha})$ span $\mathfrak{a}_{\alpha}$ in the standard way. Let $\mathfrak{l}$ be the subalgebra spanned by $\mathbb{C} \otimes \mathfrak{t}$ and $\{E_{\alpha} \mid \alpha \in \Sigma\}$ while $\mathfrak{l}_Y$ is spanned by $\mathfrak{l}$ and $\{E_{-\alpha} \mid \alpha(Y) = 0\}$ (Y is as in §3). Let $G^{\mathbb{C}}$ be a (complex) Lie group whose Lie algebra is $\mathbb{C} \otimes \mathfrak{g}$ and which contains G as a subgroup (corresponding to the subalgebra $1 \otimes \mathfrak{g}$). Let L and L_Y be subgroups of $G^{\mathbb{C}}$ whose Lie algebras are $\mathfrak{l}$ and $\mathfrak{l}_Y$.

$G/K \simeq G^{\mathbb{C}}/L_Y$. L acts on $G^{\mathbb{C}}/L_Y$ complex analytically on the left. The orbits of this action are the Borel cells.

To show that the Borel cells and the Morse cells arising from the function on §3 coincide, one proves that each Borel cell contains exactly one critical point, that each Borel cell is invariant under the flow generated by JX and that the tangent space to the Morse cell at a critical point is contained in the tangent space to the Borel cell through that point. Hence $\dim MC(p) \leq \dim BC(p)$ for all critical points p . Since both the Morse cells and the Borel cells give cell decompositions that generate the homology, the two cells through the same critical point must have the same dimension. Thus $TMC(p)_p = TBC(p)_p$ and by the lemma $MC(p) \subset BC(p)$. Since both sets of cells partition we have $MC(p) = BC(p)$ for all p .

Translating back to G/K , this says that the Morse cell through the critical point gK has as tangent space $L_{\phi_*}(\sum \mathfrak{a}_{\alpha})$, where the sum runs over all $\alpha \in \Sigma$ with $\phi_* \cdot \alpha < 0$.

§8. The Grassman manifolds. A point in the complex Grassmannian $CG(p, q) \equiv CF(p, q)$ may be identified with a p -dimensional subspace of $\mathbb{C}^{p+q}$ (and its orthogonal complement). The critical points are those planes spanned by p of standard basis vectors of $\mathbb{C}^{p+q}$.

Consider the critical point spanned by the $\sigma(1), \sigma(2), \dots, \sigma(p)$ standard

basis vectors. The Schubert cell through the point is the collection of all planes spanned by the columns of matrices in column echelon form. (See figure 1)

FIGURE 1

$$\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \cdot \\ \cdot \\ \cdot \\ \sigma_p \end{array} \left[\begin{array}{cccccccc} * & * & * & \cdot & \cdot & \cdot & \cdot & * \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & * & * & \cdot & \cdot & \cdot & \cdot & * \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & & \cdot & \cdot & \cdot & \cdot & * \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{array} \right]$$

It is a straight forward computation to verify that the Schubert cells are invariant under the left action of the maximal torus and hence under the isometries ψ_i .

If a given critical plane is spanned by the $\sigma(1), \dots, \sigma(p)$ standard basis vectors, let $\sigma(p+1), \dots, \sigma(p+q)$ be the rest of the integers less than $p+q$, arranged in increasing order (as are $\sigma(1), \dots, \sigma(p)$). Denote the critical plane by $g_\sigma K$ as before. If $\alpha \in \Sigma'$, $\varphi_{\sigma\sigma} \cdot \alpha < 0$ and $X \in a_\alpha$, it is straight forward, but computationally and notationally messy, to show that $g_\sigma \exp(tX)K$ is in the Schubert cell $SC(\sigma)$, at least for small t . Hence $L_{\sigma\sigma} a_\alpha$ is tangent to $SC(\sigma)$ at $g_\sigma K$. Thus $\dim SC(\sigma) \geq \dim BC(\sigma)$ and as before we must have equality. Hence the lemma implies that $SC(\sigma) = BC(\sigma)$ and the Borel and Schubert decompositions coincide.

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