Saturation Theorems Related to the Mean Ergodic Theorem

D. LEVIATAN

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1. Introduction. Let X be a Banach space and T denote a bounded operator from X into X . The mean ergodic theorem on the behavior of the sequence of arithmetical means of the iterates T^n of a power bounded T (i.e. $\|T^n\| \leq M$, $n = 0, 1, 2, \dots$ for some M) has been extended and generalized in many directions. Among these is a theorem due to Yosida [4] who replaced the power-bounded operator by an operator T with $\|T^n\| = o(n)$ as $n \rightarrow \infty$. Observing that the arithmetical means are the Cesàro $(C, 1)$ transform of the iterates T^n the following has been shown recently by Ramanujan and the author [3].

Theorem A. *Let X be a Banach space and T a bounded operator from X into X . Let (a_{nk}) be a regular summability matrix and $\{\gamma_n\}$ an increasing sequence of non-negative reals. Assume that*

- (1) $T_n x = \sum_{k=0}^{\infty} a_{nk} T^k x$ exists for each $x \in X$ and each n , and that $\{T_n x\}$ ($n \geq 0$) is weakly relatively compact;
- (2) $\sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| \gamma_{k+1} \rightarrow 0$, as $n \rightarrow \infty$;
- (3) $\|T^n\| \leq \gamma_n$.

Then for each $x \in X$, $T_n x \rightarrow Px$ as $n \rightarrow \infty$, where P is the projection of X onto N , the null space of $I - T$, parallel to $\bar{R}$, the closure of the range of $I - T$. Moreover $X = N \oplus \bar{R}$.

If T is power bounded and X is reflexive, then (1) is automatically satisfied. If the matrix (a_{nk}) is the Cesàro matrix (C, r) ($r \geq 1$) or the Abel "matrix", then (see [3] § 4) $\{\gamma_n\}$ may be taken as any sequence satisfying $\gamma_n = o(n)$ as $n \rightarrow \infty$; and this reduces to Yosida's result [4] for $(C, 1)$.

Recently Butzer and Westphal [1] considered the question of how rapidly the $(C, 1)$ —means of a power-bounded operator T on a reflexive space X converge to P in the strong sense; and also how well the Abel means of a power

bounded operator T on X , not necessarily reflexive, approximate P in the strong sense where P is suitably defined on $N \oplus \bar{R}$, which is not necessarily all of X . In view of Theorem A they posed the question whether it is possible to replace the power bounded T by an operator T satisfying $\|T^n\| = o(n)$ as $n \rightarrow \infty$. It is our purpose here to partially answer this question. While in section 2 we shall give a complete answer in the Abel case and in fact show that if X is reflexive the situation is exactly the same as in the power-boundedness case, the Cesàro means cause some trouble. In section 3 we shall give a complete answer for the $(C, 2)$ means rather than the $(C, 1)$ means.

In the sequel X will always denote a Banach space and T a bounded operator from X into X .

2. Saturation and Abel means. The Abel means of the iterates of T are defined by

$$(4) \quad T_r x = (1 - r) \sum_{n=0}^{\infty} r^n T^n x, \quad x \in X, \quad 0 < r < 1,$$

provided the series on the right converges.

Our result for the Abel means reads as follows.

Theorem 1. *Let T satisfy $\|T^n\| = o(n)$ ($n \rightarrow \infty$) and assume that $\{T_r x\}$ ($0 < r < 1$) is weakly relatively compact. Then*

- (i) *if $\|T_r x - Px\| = o(1 - r)$, as $r \rightarrow 1-$, then $x \in N$;*
- (ii) *the set $\{(1 - r)^{-1}(T_r x - Px)\}$ ($0 < r < 1$) is weakly relatively compact if and only if $x \in N \oplus R$.*

An immediate consequence is the following (compare with [1] Corollary 3).

Corollary 1. *Let X be reflexive and let T satisfy the assumptions of Theorem 1. Then*

$$\|T_r x - Px\| = O(1 - r), \text{ as } r \rightarrow 1-, \text{ if and only if } x \in N \oplus R.$$

To put Corollary 1 in the context of approximation theory it will read as follows.

Corollary 1'. *Let X be reflexive and let T satisfy the assumptions of Theorem 1. Then the approximation process $\{T_r x, 0 < r < 1\}$ is saturated with order $O(1 - r)$ as $r \rightarrow 1-$ and the saturation class is equal to $N \oplus R$.*

For the proof of Theorem 1 we need the following result, the proof of which is analogous to that of [1, Prop. 1].

Proposition 1. *Let T satisfy $\|T^n\| = o(n)$ ($n \rightarrow \infty$) and assume that $\{T_r x\}$ ($0 < r < 1$) is weakly relatively compact. Then*

- (a) *if $\|\sum_{n=0}^{\infty} r^n T^n x\| = o(1)$, as $r \rightarrow 1-$, then $x = 0$;*
- (b) *the set $\{\sum_{n=0}^{\infty} r^n T^n x\}$ ($0 < r < 1$) is weakly relatively compact if and only if $x \in R$.*

Proof. First it should be noted that since $\|T^n\| = o(n)$, then $\sum_{n=0}^{\infty} r^n T^n x$ exists for $0 < r < 1$ and $x \in X$. The following identity which plays a crucial role in the proof, is straightforward (see [1] (3.1)).

$$(5) \quad \sum_{n=0}^{\infty} r^n T^n (I - T)x = (1/r) \left(x - (1 - r) \sum_{n=0}^{\infty} r^n T^n x \right), \quad 0 < r < 1.$$

Now if $\|\sum_{n=0}^{\infty} r^n T^n x\| = o(1)$, then it follows by the continuity of $I - T$ that

$$\lim_{r \rightarrow 1-} \sum_{n=0}^{\infty} r^n T^n (I - T)x = (I - T)O = 0$$

and

$$\lim_{r \rightarrow 1-} (1 - r) \sum_{n=0}^{\infty} r^n T^n x = 0.$$

Hence by virtue of (5) $x = 0$. This proves (a). In order to prove (b) assume first that $x \in R$. Then $x = (I - T)y$ for some $y \in X$. By (5)

$$(6) \quad \begin{aligned} \sum_{n=0}^{\infty} r^n T^n x &= \sum_{n=0}^{\infty} r^n T^n (I - T)y \\ &= (1/r) \left(y - (1 - r) \sum_{n=0}^{\infty} r^n T^n y \right) \end{aligned}$$

Now it follows by Theorem A and the discussion following it, that the right-hand side of (6) tends to a limit $(y - Py)$ as $r \rightarrow 1-$ and so evidently $\{\sum_{n=0}^{\infty} r^n T^n x\}$ ($0 < r < 1$) is weakly relatively compact. Conversely, suppose that $\{\sum_{n=0}^{\infty} r^n T^n x\}$ ($0 < r < 1$) is weakly relatively compact. Then there exist a sequence $r_i \rightarrow 1-$ and a $y \in X$ such that for all $f \in X^*$

$$(7) \quad \lim_{i \rightarrow \infty} \left\langle \sum_{n=0}^{\infty} r_i^n T^n x, f \right\rangle = \langle y, f \rangle.$$

We proceed as Butzer and Westphal [1] observing that (7) is valid with f replaced by $(I - T^*)f$ and this yields

$$(8) \quad \lim_{i \rightarrow \infty} \left\langle \sum_{n=0}^{\infty} r_i^n T^n (I - T)x, f \right\rangle = \langle (I - T)y, f \rangle$$

for all $f \in X^*$. On the other hand the weak relative compactness of $\{\sum_{n=0}^{\infty} r^n T^n x\}$ ($0 < r < 1$) implies that $\|\sum_{n=0}^{\infty} r^n T^n x\| = O(1)$ as $r \rightarrow 1-$ and so by (5)

$$\lim_{r \rightarrow 1-} \sum_{n=0}^{\infty} r^n T^n (I - T)x = x.$$

Combining this with (8) we get

$$\langle x, f \rangle = \langle (I - T)y, f \rangle, \quad f \in X^*;$$

thus by the Hahn-Banach theorem $x = (I - T)y$ or $x \in R$. This completes our proof.

Remark. As was pointed out by the referee the proof of Proposition 1 actually shows that conditions (b) hold if and only if the set $\{\sum_{n=0}^{\infty} r^n T^n x, 0 < r < 1\}$ is relatively compact in the norm as well as in the weak topology.

Proof of Theorem 1. By virtue of Theorem A, $X = N \oplus \bar{R}$ and therefore each $x \in X$ can be represented uniquely as $x = x_1 + x_2$, $x_1 \in N$, $x_2 \in \bar{R}$. For $x_1 \in N$ we have $Tx_1 = x_1$ and so $T_r x_1 = x_1$ and $Px_1 = x_1$. Consequently

$$T_r x - Px = T_r x_2.$$

Now if $\|T_r x - Px\| = o(1 - r)$ as $r \rightarrow 1$, it follows that $\|\sum_{n=0}^{\infty} r^n T^n x_2\| = o(1)$ which by Theorem 1 implies $x_2 = 0$ or $x = x_1 \in N$. The set $\{(1 - r)^{-1} T_r x - Px\} (0 < r < 1)$ is the set $\{\sum_{n=0}^{\infty} r^n T^n x_2\} (0 < r < 1)$ and so it is weakly relatively compact if and only if $x_2 \in R$, that is, if and only if $x \in N \oplus R$. This completes the proof.

Remark. As was pointed out by the referee, Theorem 1 and Proposition 1 remain valid if the assumption $\|T^n\| = o(n)$ is replaced by the assumption $\|T^n x\| = o(n)$ for each $x \in X$.

3. Saturation and (C, α) means. Let $\alpha \geq 1$. The Cesàro means of order α , the (C, α) means, of the iterates of T are defined by (see [2] p. 96–98)

$$T_n^{(\alpha)}(x) = \binom{n + \alpha}{\alpha}^{-1} \sum_{k=0}^{\infty} \binom{n - k + \alpha - 1}{\alpha - 1} T^k x.$$

The following two identities play a crucial role in proving the results here and are easily proven.

$$(9) \quad \binom{n + \alpha}{\alpha}^{-1} \sum_{k=0}^{\infty} \binom{n - k + \alpha}{\alpha} T^k (I - T)x = \frac{\binom{n + 1 + \alpha}{\alpha}}{\binom{n + \alpha}{\alpha}} (x - T_{n+1}^{(\alpha)} x),$$

and

$$(10) \quad \binom{n + \alpha + 2}{\alpha + 1} T_{n+1}^{(\alpha+1)}(x) - \binom{n + \alpha + 1}{\alpha + 1} T_n^{(\alpha+1)} x = \binom{n + 1 + \alpha}{\alpha} T_{n+1}^{(\alpha)} x.$$

We first have

Theorem 2. Let T satisfy $\|T^n\| = o(n)$ ($n \rightarrow \infty$) and assume that $\{T_n^{(\alpha)} x\}$ is weakly relatively compact. Then

- (i) if $\|T_n^{(\alpha+1)} x - Px\| = o(1/n)$ as $n \rightarrow \infty$, then $x \in N$;
- (ii) the set $\{n(T_n^{(\alpha+1)} x - Px)\} (n \geq 0)$ is weakly relatively compact if and only if $x \in N \oplus R$.

Comparing Theorems 1 and 2 we see that while Theorem 1 involves dealing with a single summability method Theorem 2 involves two methods namely

the (C, α) and $(C, \alpha + 1)$ methods. The assumption on $\{T_n^{(\alpha)}x\}$ is stronger than an analogous assumption on $\{T_n^{(\alpha+1)}x\}$ but we cannot do without it in the proof. This can be seen easily by comparing (5) and (9). Also since (9) involves both the (C, α) and $(C, \alpha + 1)$ methods we need (10) in the proof. Otherwise the proof is along the lines of the proof of Theorem 1 and is therefore omitted.

An immediate consequence of Theorem 2 is

Corollary 2. *Let X be reflexive and let T satisfy the assumptions of Theorem 2. Then $\|T_n^{(\alpha+1)}x - Px\| = O(1/n)$, as $n \rightarrow \infty$, if and only if $x \in N \oplus R$.*

Since the strength of the (C, α) methods increases with α and since the Abel method is stronger than all the Cesàro methods (see [2] Theorems 43 and 55), we may conclude with

Theorem 3. *Let X be reflexive and let T satisfy $\|T^n\| = o(n)$ ($n \rightarrow \infty$). Assume that for some $\alpha \geq 1$ the set $\{T_n^{(\alpha)}x\}$ ($n \geq 0$) is weakly relatively compact. Then for each $\beta \geq \alpha + 1$ the approximation process $\{T_n^{(\beta)}x, n \geq 0\}$ is saturated with order $O(1/n)$ and the Abel approximation process $\{T_r x, 0 < r < 1\}$ is saturated with order $O(1 - r)$. The saturation class in all cases is $N \oplus R$.*

REFERENCES

1. P. L. BUTZER & U. WESTPHAL, *The mean ergodic theorem and saturation*, Indiana Univ. Math. J. **20** (1971), 1163–1174.
2. G. H. HARDY, *Divergent series*, Oxford, 1949.
3. D. LEVIATAN & M. S. RAMANUJAN, *A generalization of the mean ergodic theorem*, Studia Math. **39** (1971), 113–117.
4. K. YOSIDA, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Tokyo **14** (1938), 292–294.

Tel-Aviv University

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