Differentiable Maps with 0-Dimensional Critical Set, II

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1. Introduction.

1.1. Theorem. *Let $f: M^n \rightarrow N^p$ be a C^n map with $n - p = 0, 1$ or 2 and $p \geq 2$, let $R_{p-1}(f)$ be its critical set, and let $\dim R_{p-1}(f) = \dim f(R_{p-1}(f)) \leq 0$. Then, at each $x \in M^n$, f is locally topologically equivalent to one of the following maps:*

- (a) *the projection map $\rho: \mathbf{R}^n \rightarrow \mathbf{R}^p$,*
- (b) *$\sigma: \mathbf{C} \rightarrow \mathbf{C}$ defined by $\sigma(z) = z^d$ ($d = 2, 3, \dots$), where $\mathbf{C}$ is the complex plane,*
- (c) *$\tau: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{R}$ defined by $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where $\bar{w}$ is the complex conjugate of w , or*
- (d) *a C^n map $\eta: \mathbf{R}^4 \rightarrow \mathbf{R}^2$ which is locally topologically equivalent to ρ except at one point.*

In particular, either f is locally topologically equivalent to ρ at every point, or $(n, p) = (2, 2)$, $(4, 3)$, or $(4, 2)$. Examples of maps η are the complex polynomial maps defined by $\eta(z, w) = z^i + w^k$, $j > k \geq 2$ [9, p. 3]. The case $n - p = 0$ or 1 is given in [4], and only $n - p = 2$ is considered here.

For background information see the introduction of [4]. In particular (1.1) answers a fortiori for $n - p \leq 2$ a question of Milnor [9, p. 100] (see [4, (1.7)]).

1.2. Conventions. A symbol such as M^n denotes a separable n -manifold, without boundary unless otherwise specified. A manifold with boundary may have an empty boundary. Homology is singular, cohomology is Alexander, coefficients are $\mathbf{Z}_2$, and $d^q(X) = \dim(H^q(X))$. The branch set $B_f \subset M^n$ is the set of points at which f fails to be locally topologically equivalent to projection ([4, (1.5)] or [5, (2.1)]). For detailed conventions and definitions see [4].

1.3. Outline. Let $f: M^{p+2} \rightarrow N^p$ ($p \geq 2$) be a map with $\dim B_f = \dim f(B_f) \leq 0$ and $f(B_f)$ locally tame. Under an additional assumption it was shown in [5] that B_f is discrete. In §2, starting with this result, or more precisely with

the last lemma of [4], it is proved that this assumption is not required (2.4), and that $B_f = \emptyset$ unless $p = 2$ ((2.2) and (2.4)). The main topological theorem of this paper is (2.6).

In §3 certain layer maps (3.1) $\alpha: \Gamma^4 \times \mathbb{R}^{p-2} \rightarrow \Lambda^2 \times \mathbb{R}^{p-2}$ are considered, and B_α is investigated with a study of the branch sets $B(\alpha_i)$ of the layers. Finally it is shown (3.4) that there is no layer map α with $\dim B_\alpha = \dim \alpha(B_\alpha) = 0$ (unless $p - 2 = 0$).

The differentiability hypothesis C^n is added in §4, and used to eliminate the tameness assumption on $f(B_f)$ in §2. Specifically, some technical lemmas on transverse regularity are first proved; if $f(B_f)$ is not locally tame (simple), there results from these lemmas (by restriction of f and change of coordinates) a layer map contradicting (3.4).

In this paper, as in its predecessors, as much as possible has been done within the topological (rather than smooth) category (see (2.6)).

2. Topological results.

2.1. Lemma [5, (3.9)]. *Let $p \geq 2$, let Δ be a closed p -simplex, let T_r be the set of all closed p -simplices in its r^{th} barycentric subdivision, and let M^{p+2} be a compact, connected $(p + 2)$ -manifold with boundary. Let $f: M^{p+2} \rightarrow \Delta$ be a map with $B_f \cap \partial M^{p+2} = \emptyset$, $\dim B_f = \dim f(B_f) \leq 0$, and $f(B_f) \cap \partial \tau = \emptyset$ for every $\tau \in T_r$ ($r = 0, 1, \dots$). Suppose that for each $x \in M^{p+2}$ the component $\Gamma(x)$ of $f^{-1}(f(x))$ containing x is not $\{x\}$.*

Then for each $\bar{x} \in B_f$ and $\epsilon > 0$, there are $r, \tau \in T_r$, a compact, connected $(p + 2)$ -manifold with boundary $K_1 \subset M^{p+2}$, and the restriction map $g: K_1 \rightarrow \tau$ such that (i) $B_f \cap K_1 = B_g = \{\bar{x}\}$, (ii) $g^{-1}(g(x))$ is the one-point union of q closed 2-cells with common point $\bar{x}$, and (iii) there is a compact, connected orientable 2-manifold $J^2 \neq D^2$ with $\partial J^2 \neq \emptyset$ and $g^{-1}(z) \approx J^2$ for each $z \in \tau - f(B_f)$.

2.2. Lemma. *If f satisfies the hypotheses of (2.1), then K_1 is contractible; if, in addition, $p \geq 3$, then $B_f = \emptyset$.*

Proof. Suppose $B_f \neq \emptyset$. By (2.1) we may suppose that $B_f = \{\bar{x}\}$, $\bar{y} = f(\bar{x})$, (1) $f^{-1}(\bar{y})$ is the one-point union of q ($q = 1, 2, \dots$) closed 2-cells with common point $\bar{x}$, and (2) the fiber J^2 is connected and orientable with nonempty boundary. Let $U \subset M^{p+2}$ be an open $(p + 2)$ -cell neighborhood of $\bar{x}$, let $\epsilon < d(\bar{x}, \text{bdy } U)$, and let r, τ , and K_1 be as given by (2.1).

For each $s = 0, 1, \dots, r$ let $\tau_s \in T_s$ be the unique p -simplex with $\tau \subset \tau_s$. There is a finite composition of elementary collapses [12, p. 141, (6.6.1)] mapping τ_s with triangulation $\mathfrak{T}_{s+1}$ onto τ_{s+1} . For each $\sigma \neq \tau_{s+1}$ in T_{s+1} there is a homeomorphism $\lambda: f^{-1}(\sigma) \approx J^2 \times \sigma$ with $\pi \circ \lambda$ (π the projection map) the restriction of f ([4, (1.9)] and [13, p. 53, Corollary 11.6]); it follows that there is a deformation retraction of $f^{-1}(\tau_s)$ onto $f^{-1}(\tau_{s+1})$, and thus (3) of $M^{p+2} = f^{-1}(\tau_0)$ onto $f^{-1}(\tau)$.

If $P = \text{Cl } [f^{-1}(\tau) - K_1]$ and $g: P \rightarrow \tau$ is the restriction of f , then $B_g = \emptyset$, and as above there is (by (1))

$$\mu: (P, P \cap K_1) \approx ((qS^1 \times [0, 1]) \times \tau, (qS^1 \times \{0\}) \times \tau)$$

with $\pi \circ \mu = g$ and qS^1 the disjoint union of q copies of S^1 . As a result $(K_1, K_1 \cap f^{-1}(\bar{y}), K_1 - f^{-1}(\bar{y}))$ is a deformation retract of $(f^{-1}(\tau), f^{-1}(\bar{y}), f^{-1}(\tau) - f^{-1}(\bar{y}))$, and so (by (3)) of $(M^{p+2}, f^{-1}(\bar{y}), M^{p+2} - f^{-1}(\bar{y}))$. Since $K_1 \subset U \subset M^{p+2}$, M^{p+2} and K_1 are contractible and (4) inclusion induces $H_1(K_1 - f^{-1}(\bar{y})) \approx H_1(M^{p+2} - f^{-1}(\bar{y}))$.

From duality ([11, p. 342, (10)]) $H_2(U, U - f^{-1}(\bar{y})) \approx H_c^p(f^{-1}(\bar{y}) \cap U)$, which is 0 since $p \geq 3$ and $\dim f^{-1}(\bar{y}) \leq 2$; since U is a $(p+2)$ -cell the homology sequence of $(U, U - f^{-1}(\bar{y}))$ yields $H_1(U - f^{-1}(\bar{y})) = 0$. Since $K_1 \subset U \subset M^{p+2}$, $H_1(M^{p+2} - f^{-1}(\bar{y})) = 0$ by (4). Let $L = f^{-1}(\partial\Delta)$, and let $h: M^{p+2} - f^{-1}(\bar{y}) \rightarrow \Delta - \{\bar{y}\}$ be the restriction of f . Since $B_h = \emptyset$, $M^{p+2} - f^{-1}(\bar{y}) \approx L \times (0, 1]$ [4, (1.9)] and [13, p. 53, Theorem 11.4]), so that (5) $H_1(L) = 0$.

Let ξ be the bundle with projection map $f|L: L \rightarrow \partial\Delta$ (and fiber J^2), and let ζ be the subbundle with total space ∂L (and fiber ∂J^2). Consider the commutative diagram

$$(6) \quad \begin{array}{ccccc} H_{3-p}(\partial J^2) & \xrightarrow{\alpha} & H_1(\partial J^2) & \xrightarrow{\beta} & H_1(\partial L) \\ \downarrow \phi & & \downarrow \chi & & \downarrow \psi \\ H_{3-p}(J^2) & \xrightarrow{\gamma} & H_1(J^2) & \xrightarrow{\delta} & H_1(L) \end{array}$$

where β , δ , ϕ , χ , and ψ are induced by inclusion and the horizontal rows are parts of the Wang sequences [11, p. 482, Theorem 2] of ζ and ξ . Since J^2 is connected (by (2)), γ is an epimorphism (by (5)), and $p \geq 3$, either $H_1(J^2) \approx \mathbf{Z}_2$ and $p = 3$, or $H_1(J^2) = 0$. Since $\partial J^2 \neq \emptyset$ (by (2)), if $H_1(J^2) = 0$, then J^2 is a 2-cell [8, p. 43], contradicting (2.1) and (2). Thus $p = 3$ and $H_1(J^2) \approx \mathbf{Z}_2$, and since J^2 is orientable (2), $J^2 \approx S^1 \times [0, 1]$ [8, p. 43].

Let $Q = \text{Cl}[\partial(f^{-1}(\text{int } \Delta))]$, and let $e: Q \rightarrow \Delta$ be the restriction of f . Since $B_e = \emptyset$, there exists $\nu: Q \approx \Delta \times \partial J^2$ with $\pi \circ \nu = e$, so that $\partial L \approx \partial\Delta \times \partial J^2$. Thus the inclusion of ∂J^2 into ∂L is topologically equivalent [4, (1.3)] to the inclusion of $\{z\} \times \partial J^2$ in $S^{p-1} \times \partial J^2$ (where $z \in S^{p-1}$), β in (6) is an isomorphism (since $p = 3$), and $\alpha = 0$. Now ϕ is an epimorphism ($p = 3$ and J^2 is connected), and since γ is an epimorphism and $H_1(J^2) \approx \mathbf{Z}_2$, $\gamma \circ \phi \neq 0$. Since $\chi \circ \alpha = 0$, a contradiction of the commutativity results.

2.3. Definitions. A map $f: X \rightarrow Y$ is *quasi-monotone* if, for each connected open set $U \subset Y$ and component V of $f^{-1}(U)$, $f(V) = U$ [15, p. 151].

Define A_f by: $x \in A_f$ if and only if there are open neighborhoods U of $f(x)$ and V of $\Gamma(x)$ (2.1) such that $f|V: V \rightarrow U$ is topologically equivalent to the projection map $\pi: U \times \Gamma(x) \rightarrow U$. Thus $B_f \subset A_f$.

2.4. Both (2.1) and (2.2) are valid without the hypothesis " $\Gamma(x) \neq \{x\}$ for every $x \in M^{p+1}$ ".

Proof. Let $P_f = \{x \in M^{p+2}: \Gamma(x) = \{x\}\}$; we may suppose that $P_f \neq \emptyset$ ((2.1) and (2.2)). Note that $P_f \subset B_f$. There is a natural number k such that

$f^{-1}(y)$ has at most k components for every $y \in \Delta$ [14, p. 64, (2.5)]. Choose $\bar{y} \in f(P_f)$ such that $f^{-1}(\bar{y})$ has the maximum number h of components, and choose r ($r = 0, 1, \dots$) sufficiently large that, for the unique $\tau \in T_r$ (2.1) with $\bar{y} \in \tau$, $f^{-1}(\tau)$ has at least h components X_i ($i = 1, 2, \dots$), where $X_1 \cap P_f \neq \emptyset$. Since f is quasi-monotone ((2.3) and [14, p. 64, (2.5)]), each $f(X_i) = \tau$, so that $f^{-1}(\tau)$ has exactly h components and $f^{-1}(y) \cap X_i$ is a component of $f^{-1}(y)$ for each $y \in f(P_f)$.

Let $X = X_1$ and let $\alpha: X \rightarrow \tau$ be the restriction of f ; then (1) X is a compact, connected $(p + 2)$ -manifold with boundary and $B_\alpha \cap \partial X = \emptyset$ (by (2.1)), (2) $\emptyset \neq P_\alpha \subset B_\alpha$, and (3) $\alpha^{-1}(y)$ is a single point for each $y \in \alpha(P_\alpha) \neq \emptyset$. Suppose that P_α is not closed. Then there exists $x_n \rightarrow x$ with $x_n \in P_\alpha$ and $x \notin P_\alpha$, so that $\dim \Gamma_\alpha(x) > 0$ (2.1). Since $\dim B_\alpha \leq 0$, there exists $u \in \Gamma_\alpha(x) - B_\alpha$ and an open neighborhood U of u disjoint from B_α . Since $\alpha(U)$ is an open neighborhood of $\alpha(x)$, for n sufficiently large $\alpha(x_n) \in \alpha(U)$; thus $\alpha^{-1}(\alpha(x_n)) \cap U$ is not a single point, contradicting (3). Hence P_α is closed.

Suppose $p \geq 3$. For each $x \in B_\alpha - P_\alpha$ choose s and $\sigma \in T_s$ such that $\alpha(x) \in \sigma \subset \tau$ and the component L of $\alpha^{-1}(\sigma)$ containing x is disjoint from P_α . If $\zeta: L \rightarrow \sigma$ is the restriction of α , then $B_\zeta = \emptyset$ by (4.1). As a result $P_\alpha = B_\alpha$, so that ((2.3) and [4, (4.6)]) (4) $A_\alpha = B_\alpha$.

Given $y \in \alpha(A_\alpha)$ and W an open neighborhood of y in $\text{int } \tau$, choose s and $\rho \in T_s$ such that $y \in \rho \subset W$. Let ρ_m be the union of the $(p - 1)$ -skeleton of ρ in $\mathfrak{T}_{s+m}$ with $\{\xi \in T_{s+m} : \xi \subset \rho \text{ and } \alpha(A_\alpha) \cap \xi = \emptyset\}$ ($m = 0, 1, \dots$). By [11, p. 402, Lemma 15], (ρ, ρ_m) is 2-connected, and from the homotopy sequence $\pi_1(\rho_m) = 0$. Since $\bigcup_m \rho_m = \rho - \alpha(A_\alpha)$ and $\text{int } \rho - \alpha(A_\alpha)$ has the same homotopy type, $\pi_1(\text{int } \rho - \alpha(A_\alpha)) = 0$. It follows from [6, p. 75, (2.3)] with $U = \text{int } \rho$ that $A_\alpha = \emptyset$. According to (2) and (4), $\emptyset \neq P_\alpha \subset B_\alpha = A_\alpha$, and a contradiction results.

Thus we may suppose that $p = 2$. Let $\alpha = \bar{\gamma} \circ \bar{\beta}$ be the monotone-light factorization of α , where $\bar{\beta}: X \rightarrow Y$ and $\bar{\gamma}: Y \rightarrow \tau$ [15, p. 141, (4.1)], let $\beta: X - P_\alpha \rightarrow Y - \bar{\beta}(P_\alpha)$ and $\gamma: Y - \bar{\beta}(P_\alpha) \rightarrow \tau - \alpha(P_\alpha)$ be the restrictions of $\bar{\beta}$ and $\bar{\gamma}$, respectively, and let m be the minimal natural number such that $\alpha^{-1}(y)$ has at most m components for each $y \in \tau$ [14, p. 64, (2.5)]. Then [6, (4.5)(d)] β is monotone and γ is an m -to-1 covering map.

In the Leray spectral sequence of the map β [4, p. 140, (6.1)], $E_2^{s,t} = 0$ unless $s = 0$ or 1, and $t = 0, 1$, or 2. Thus $d_r = 0$ for each $r \geq 2$, so that $E_2^{s,t} \approx E_\infty^{s,t}$ for all s, t , and (using [2, p. 257, (12)]) we obtain the short exact sequence

$$(5) \quad 0 \rightarrow E_2^{1,0} \rightarrow H^1(X - P_\alpha; \mathbb{Z}_2) \rightarrow E_2^{0,1} \rightarrow 0.$$

Since β is monotone, the Leray sheaf $\mathcal{H}^0$ [4, p. 137, (4.1)] is the constant sheaf $\mathbb{Z}_2$, so that (6) $E_2^{1,0} \approx H^1(Y - \beta(P_\alpha); \mathbb{Z}_2)$.

We use $\mathbb{Z}_2$ coefficients below. Since $P_\alpha \subset B_\alpha$ by (1), $\dim P_\alpha = 0$ and $H^2(P_\alpha) = 0$, and since $H^2(P_\alpha \cup \partial X) \approx H^2(P_\alpha) \oplus H^2(\partial X) \approx H^2(\partial X)$ by (1) and ∂X is a compact manifold, $H^2(P_\alpha \cup \partial X)$ is finite [11, p. 342, Theorem 11; p. 243,

Theorem 3; p. 340, Corollary 5]. Now $H^3(X)$ is also finite, and it follows from the cohomology sequence that $H^3(X, P_\alpha \cup \partial X)$ is finite. From the Universal Coefficient Theorem [11, p. 243, Theorem 3] $H^1(X - P_\alpha) \approx H_1(X - P_\alpha)$ and from (1) $H_1(X - P_\alpha) \approx H_1(\text{int } X - P_\alpha) = H_1(X - (P_\alpha \cup \partial X)) \approx H^3(X, P_\alpha \cup \partial X)$ by Lefschetz Duality [11, p. 297, Theorem 19], so that $H^1(X - P_\alpha)$ is finite. By (5) and (6) $H^1(Y - \beta(P_\alpha))$ is finite, so that $H_1(Y - \beta(P_\alpha))$ is also finite. Since $H_1(Y - \beta(P_\alpha)) \approx H_1(Y - (\beta(P_\alpha) \cup \partial(Y - \beta(P_\alpha)))) \approx H^1(Y, \beta(P_\alpha) \cup \partial(Y - \beta(P_\alpha)))$ [11, p. 297, Theorem 19] and $\tilde{H}^0(Y) = 0$, it follows from the cohomology sequence that $\tilde{H}^0(\beta(P_\alpha) \cup \partial(Y - \beta(P_\alpha)))$ is finite. Thus $\beta(P_\alpha)$ is finite, so that P_α is finite by (3). As a result, by choosing τ sufficiently small, we may suppose that P_α is a single point $\bar{x}$.

Since $\bar{x} = \alpha^{-1}(\alpha(\bar{x}))$ by (3) and α is quasi-monotone ((2.3) and [14, p. 64, (2.5)]), the unique simplex $\sigma_s \in T_s$ with $\alpha(\bar{x}) \in \sigma_s$ has $\alpha^{-1}(\sigma_s)$ connected ($s = 0, 1, \dots$). Let $U \subset X$ be an open 4-cell neighborhood of $\bar{x}$, and let s be sufficiently large that (7) $\alpha^{-1}(\sigma_s) \subset U$. By the naturality of (5) and (6)

$$(9) \quad \begin{array}{ccc} 0 \rightarrow & H^1(Y - \{\bar{\beta}(\bar{x})\}) & \rightarrow H^1(X - \{\bar{x}\}) \\ & \downarrow \phi & \downarrow \psi \\ 0 \rightarrow & H^1(\gamma^{-1}(\sigma_s) - \{\bar{\beta}(\bar{x})\}) & \rightarrow H^1(\alpha^{-1}(\sigma_s) - \{\bar{x}\}) \end{array}$$

is commutative, where the rows are exact and ϕ and ψ are induced by inclusion. From (7) and (8) ψ is the 0-homomorphism. Since γ is an m -to-1 covering map over $\tau - \{\alpha(\bar{x})\}$ and $\tau - \{\alpha(\bar{x})\} \approx S^1 \times (0, 1] \approx \sigma_s - \{\alpha(\bar{x})\}$, $Y - \{\bar{\beta}(\bar{x})\} \approx S^1 \times (0, 1] \approx \gamma^{-1}(\sigma_s) - \{\bar{\beta}(\bar{x})\}$ and ϕ is induced by a homotopy equivalence. A contradiction of (8) and the commutativity of (9) results.

2.5. Definition. A 0-dimensional subset $A \subset N^p$ is *locally simple* if, for every $y \in N^p$, there is a closed p -cell neighborhood Δ of y with triangulation, such that $A \cap \partial\tau = \emptyset$ for each $\tau \in T_\tau$ (2.1). (See [5, (2.4)].)

2.6. Theorem. Let $f: M^n \rightarrow N^p$ be a map with $0 \leq n - p \leq 2$, $p \geq 2$, $\dim(B_f) = \dim(f(B_f)) = 0$, and $f(B_f)$ locally simple or locally tame in N^p [5, (2.4)]. Then f satisfies the conclusion of (1.1) and (for $n - p = 2$) of (2.1).

Proof. For $n - p = 0$ or 1 (2.6) is included in [4, (4.9)], so we may suppose that $n - p = 2$. If $\bar{x} \in B_f$, then let $K^{p+2} \subset M^{p+2}$ be the manifold-with-boundary and $g = f|K^{p+2}: K^{p+2} \rightarrow U$ be the proper map given by [4, (2.6)] with $\bar{x} \in \text{int } K^{p+2}$, $B_o \cap \partial K^{p+2} = \emptyset$, and U an open p -cell neighborhood of $f(\bar{x})$. Note that locally tame implies locally simple by [5, (2.4)]. By (2.5) there is a closed p -cell $\Delta \subset U$ such that $f(\bar{x}) \in \text{int } \Delta$ and $f(B_f) \cap \partial\tau = \emptyset$ for each $\tau \in T_\tau$. If L^{p+2} is the component of $g^{-1}(\Delta)$ containing $\bar{x}$ and $h: L^{p+2} \rightarrow \Delta$ is the restriction of g , then the conclusion follows by applying (2.4) to h .

3. Technical topological lemmas for the differential case.

3.1. Definition. An $\alpha: \Gamma^{n-k} \times \mathbf{R}^k \rightarrow \Lambda^{p-k} \times \mathbf{R}^k$ is a *layer map* if for each $t \in \mathbf{R}^k$, $\alpha(\Gamma^{n-k} \times \{t\}) \subset \Lambda^{p-k} \times \{t\}$. The restriction map

$$\alpha | (\Gamma^{n-k} \times \{t\}): \Gamma^{n-k} \times \{t\} \rightarrow \Lambda^{p-k} \times \{t\}$$

is denoted by α_t , and is usually viewed as a map of Γ^{n-k} into Λ^{p-k} (so that $\alpha(x, t) = (\alpha_t(x), t)$).

3.2. Lemma. Let $\alpha: \Gamma^4 \times \mathbf{R}^{p-2} \rightarrow \Lambda^2 \times \mathbf{R}^{p-2}$ be a layer map with $\dim(B_\alpha \cap \alpha^{-1}(y, t)) \leq 0$ and $\dim(\alpha(B_\alpha) \cap (\Lambda^2 \times \{t\})) \leq 0$ for each $y \in \Gamma^2$ and $t \in \mathbf{R}^{p-2}$. Then

(a) each $B(\alpha_t)$ is discrete.

(b) Let $\bar{u} \in B(\alpha_0)$ and $\delta > 0$. Then there are a connected $(p+2)$ -manifold-with-boundary $K \subset \Gamma^4 \times \mathbf{R}^{p-2}$ with $\bar{u} \in \text{int } K$ and $\text{diam } K < \delta$, and an open p -cell $D \subset \Lambda^2 \times \mathbf{R}^{p+2}$ such that $\beta = \alpha | K: K \rightarrow D$ is a proper map with $B_\alpha \cap \partial K = \emptyset$.

(c) Moreover, if $\Delta \subset \Lambda^2$ is a closed (topological) 2-simplex and $\epsilon > 0$ with $\Delta \times \bar{S}(0, \epsilon) \subset D$ and $(\partial \Delta \times \bar{S}(0, \epsilon)) \cap \beta(B_\beta) = \emptyset$, then for some η , $0 < \eta \leq \epsilon$, $d^q(\beta^{-1}(\Delta \times \{t\}))$ is independent of $t \in \bar{S}(0, \eta)$ ($q = 0, 1, \dots$).

Proof. For each $x \in \Gamma^4$, there is a closed 4-cell $E \subset \Gamma^4$ with $x \in \text{int } E$. Since each $\alpha_t(E \cap B(\alpha_t))$ is a closed, at most 0-dimensional set in Λ^2 , $\alpha_t(B(\alpha_t) | \text{int } E)$ trivially satisfies local simpleness (2.5), and from (2.6) $E \cap B(\alpha_t)$ is discrete. Since x is arbitrary, conclusion (a) results.

According to [4, (2.6)] conclusion (b) is satisfied. Moreover, there is a closed collar F of ∂K in K with $F \cap B_\alpha = \emptyset$, and, in fact, $B(\alpha | F) = \emptyset$ and $B(\alpha | \text{Cl}_K[K - F]) \subset K - F$. (To see this, note that [4, (2.6)] results easily from [4, (2.5)]. In the proof of [4, (2.5)] Q has a bicollar P disjoint from B_f , and [4, (2.4)] is applied to that P .)

Let Δ and ϵ be as given in (c). Since β is proper, $\beta(B_\beta)$ is closed, and there is a 2-simplex $\Delta' \subset \text{int } \Delta$ with

$$(\text{Cl } [\Delta - \Delta'] \times \bar{S}(0, \epsilon)) \cap \beta(B_\beta) = \emptyset.$$

Let $\Upsilon = \beta^{-1}(\Delta \times S(0, \epsilon))$, let $\gamma: \Upsilon \rightarrow \Delta \times S(0, \epsilon)$ be restriction of β , let $\Upsilon_t = \Upsilon \cap (\Gamma^4 \times \{t\})$, and let $\gamma_t: \Upsilon_t \rightarrow \Delta$ be defined by restriction of γ .

Since $B(\alpha | F) = \emptyset$, $\gamma | \Upsilon \cap F$ is a bundle map over $\Delta \times S(0, \epsilon)$ [4, (1.9)], and thus is a product bundle [13, p. 53, Corollary (11.6)]. Its fiber is a collar of the boundary of a 2-manifold, and hence $\text{Cl } [\Upsilon_t - F]$ is a deformation retract of Υ_t for each $t \in S(0, \epsilon)$. Let

$$\Phi = \text{Cl } [\gamma - F] \cap \gamma^{-1}(\Delta' \times S(0, \epsilon)), \text{ let}$$

$$\Xi = \text{Cl } [\gamma - F] \cap \gamma^{-1}(\text{Cl } [\Delta - \Delta'] \times S(0, \epsilon)),$$

and let $\zeta: \Xi \rightarrow \text{Cl } [\Delta - \Delta'] \times S(0, \epsilon)$ be restriction of γ . Then $B_\zeta = \emptyset$, so that ζ is a bundle map over $\text{Cl } [\Delta - \Delta'] \times S(0, \epsilon) \approx [0, 1] \times S^1 \times S(0, \epsilon)$. Thus each Φ_t is a deformation retract of $\text{Cl } [\Upsilon_t - F]$, and so (1) each Φ_t is a deformation retract of Υ_t .

Let $U \subset \mathcal{T}_0 - \Phi_0$ be a closed collar of $\partial\mathcal{T}_0$, let $V \subset \Phi_0 - B_\gamma$ be a closed bicollar of $\partial\Phi_0$, let $\Psi = \text{Cl}[\mathcal{T}_0 - U]$, and let $\Omega = \text{Cl}[\Phi_0 - V]$; there is an η , $0 < \eta \leq \epsilon$, sufficiently small that

$$(2) \quad \Omega \times S(0, \eta) \subset \Phi \subset \Psi \times S(0, \eta) \subset \mathcal{T}.$$

Since $\Omega \times \{0\} \subset \Phi_0 \subset \Psi \times \{0\} \subset \mathcal{T}_0$, each a deformation retract of the next one, $H^q(\Omega \times \{t\}) \approx H^q(\Psi \times \{t\})$, so that $d^q(\Phi_t) \geq d^q(\Psi)$ from (2). From (1), on the other hand, $d^q(\Psi) \geq d^q(\Phi_t) = d^q(\mathcal{T}_t)$, and as a result $d^q(\mathcal{T}_t) = d^q(\Psi)$ for all t , i.e. (conclusion (c)) $d^q(\mathcal{T}_t)$ is independent of $t \in S(0, \eta)$.

3.3. Lemma. *Under the hypotheses of (3.2), if $B_\alpha \neq \emptyset$, then $\alpha(B_\alpha)$ contains a $(p - 2)$ -cell.*

(In reading the following proof, note that we do not know whether or not B_α is $\cup\{B(\alpha_t)\}$ —but see (2) below.)

Proof. By (3.2)(a) each $B(\alpha_t)$ is discrete, and by (2.6) $\Gamma(x) \neq \{x\}$ for each α_t , so that (1) $\Gamma_\alpha(x) \neq \{x\}$ for every $x \in \Gamma^4 \times \mathbb{R}^{p-2}$.

According to [4, (4.10)]

$$(2) \quad B_\alpha = \text{Cl}[\cup\{B(\alpha_t) : t \in \mathbb{R}^{p-2}\}],$$

so that for some t there is $\bar{u} \in B(\alpha_t)$, say $t = 0$. Since $B(\alpha_0)$ is discrete, there is a K_1 given by (2.6) and (2.1) for α_0 ; choose $\delta > 0$ such that $S(\bar{u}, \delta) \cap (\Gamma^4 \times \{0\}) \subset K_1$, and let K , D , and β be as given by (3.2)(b). Since $\text{diam } K < \delta$, $\beta^{-1}(\beta(\bar{u}))$ is contained in the one-point union of a finite number of closed 2-cells by (2.6), so that (3) $d^2(\beta^{-1}(\beta(\bar{u}))) = 0$ and $B(\beta_0) = \{\bar{u}\}$.

Since β is proper, $\beta(B_\beta) \cap (\Lambda^2 \times \{0\})$ is a closed 0-dimensional set in a 2-manifold, so there is a (topological) 2-simplex $\Delta \subset \Lambda^2$ with $\beta(\bar{u}) \in \Delta \times \{0\}$, $\Delta \times \{0\} \subset D$, and $(\partial\Delta \times \{0\}) \cap \beta(B_\beta) = \emptyset$. For some $\epsilon > 0$ $(\partial\Delta \times \bar{S}(0, \epsilon)) \cap \beta(B_\beta) = \emptyset$. By (3.2)(c) for some $\eta > 0$, (4) $d^q(\beta^{-1}(\Delta \times \{t\}))$ is independent of $t \in S(0, \eta)$.

Let $\mathcal{T}$ be the component of $\beta^{-1}(\Delta \times S(0, \eta))$ containing $\bar{u}$, let $\mathcal{T}_t = \mathcal{T} \cap (\Gamma^4 \times \{t\})$, and let $\gamma : \mathcal{T} \rightarrow \Delta \times S(0, \eta)$ and $\gamma_t : \mathcal{T}_t \rightarrow \Delta \times \{t\}$ be restrictions of β . According to (4), (5) $d^q(\mathcal{T}_t)$ is independent of t . By (1) and [4, (4.5)], γ is monotone, so that (6) each γ_t is monotone and each $\mathcal{T}_t$ is connected [15, p. 138, (2.2)].

Since $B(\gamma_0) = \{\bar{u}\}$ by (3), $\gamma_0|(\Delta - \gamma_0(\bar{u}))$ is a bundle map [4, (1.9)] over $\Delta - \gamma_0(\bar{u}) \approx S^1 \times [0, 1)$. As a result [13, p. 53, Corollary (11.6)] for any 2-simplex τ with $\gamma_0(\bar{u}) \in \text{int } \tau$ and $\tau \subset \text{int } \Delta$, $\gamma_0^{-1}(\tau)$ is a deformation retract of $\mathcal{T}_0 = \gamma_0^{-1}(\Delta)$. By the Continuity Theorem [11, p. 318, Theorem 6] $d^2(\gamma_0) = d^2(\gamma^{-1}(\gamma(\bar{u}))) = 0$ by (3). From (5), (7) $d^2(\mathcal{T}_t) = 0$ for each t .

From (3) for $z \in \Delta - \gamma(B_\gamma)$ sufficiently near $\gamma(\bar{u})$, the connected (by (6)) 2-manifold $\gamma^{-1}(z)$ has non-empty boundary, i.e. $\partial J^2 \neq \emptyset$. Let T_r (as in (2.1)) be the set of all closed 2-simplices in the r^{th} barycentric subdivision of Δ . For each $t \in S(0, \eta)$ there is a homeomorphism h_t of Δ onto itself such that $(\partial\sigma \times$

$\{t\} \cap \gamma(B_\gamma) = \emptyset$ for each $\sigma \in h_i(T_r)$ ($r = 0, 1, \dots$). According to [5, (3.7)a] inclusion induces an epimorphism

$$H^2(\Upsilon_t) = H^2(\gamma_i^{-1}(\Delta)) \rightarrow \sum \{H^2(\gamma_i^{-1}(\sigma): \sigma \in h_i(T_r)\}$$

and, from (7), $d^2(\gamma_i^{-1}(\sigma)) = 0$ for $\sigma \in h_i(T_r)$ ($r = 0, 1, \dots$). From the Continuity Theorem [11, p. 318, Theorem 6] $d^2(\gamma_i^{-1}(y)) = 0$ for every y and t , so that each γ_i satisfies the standing hypotheses in [5, (3.1)].

Let $m = d^1(J^2)$, and $e_i(\sigma) = m - d^1(\gamma_i^{-1}(\sigma))$. Then (8) $e_i(\sigma) \geq 0$, $e_i(\sigma) = 0$ if and only if $\gamma_i(B(\gamma_i)) \cap \sigma = \emptyset$, and

$$e_i(\sigma) = \sum \{e_i(\sigma): \tau \in h_i(T_{r+s}) \text{ and } \tau \subset \sigma\}$$

($\sigma \in h_i(T_r)$, $r = 0, 1, \dots$) according to [5, (3.8) and (3.7)(b)]. By (5) $e_i(\Delta)$ is independent of t , call this number b , and thus from the Continuity Theorem [11, p. 318, Theorem 6] $\gamma_i(B(\gamma_i))$ has at most b points.

(9) Choose t with the maximal number of points (say $t = 0$), and choose r sufficiently large that these points $y_i \in \Delta$ ($i = 1, 2, \dots, k$) are in distinct simplices $\sigma_i \in h_0(T_r)$. For $\xi > 0$ sufficiently large, $(\partial\sigma_i \times \bar{S}(0, \xi)) \cap \gamma(B_\gamma) = \emptyset$. Since $y_i \in \sigma_i$, $e_0(\sigma_i) \neq 0$ by (8). From (3.2)(c) applied to β with $\Delta = \sigma_i$ and $\epsilon = \xi$, $d^1(\gamma_0^{-1}(\sigma_i)) = d^1(\gamma_i^{-1}(\sigma_i))$, i.e. $e_0(\sigma_i) = e_i(\sigma_i)$, so that $e_i(\sigma_i) \neq 0$ also; by (8) $\gamma_i(B(\gamma_i)) \cap \sigma_i \neq \emptyset$. From the maximality, $\gamma_i(B(\gamma_i)) \cap \sigma_i$ has exactly one point $y_{i,t}$.

We wish to prove that (10) $Y_i = \{(y_{i,t}, t): t \in S(0, \xi)\}$ is closed in $\sigma_i \times S(0, \xi)$ ($i = 1, 2, \dots, k$). Suppose the contrary, and $(v, t) \in \bar{Y}_i - Y_i$. Then $v \notin \gamma_i(B(\gamma_i))$, so there is a simplex $\sigma \in h_i(T_r)$ with $v \in \sigma$ and $\sigma \cap \gamma_i(B(\gamma_i)) = \emptyset$. Thus $e_i(\sigma) = 0$ by (8). Since $(\partial\sigma \times \{t\}) \cap \gamma(B_\gamma) = \emptyset$, $(\partial\sigma \times \{s\}) \cap \gamma(B_\gamma) = \emptyset$ for some $\zeta > 0$ and every $s \in \bar{S}(t, \zeta)$. Again from (3.2)(c) applied to β with $\Delta = \sigma$ and $\epsilon = \zeta$ (sufficiently small) $e_s(\sigma) = 0$ for every $s \in S(t, \zeta)$, so that $(\sigma \times S(t, \zeta)) \cap Y_i = \emptyset$ by (8), contradicting the choice of v . Thus Y_i is closed.

Define $\xi_i: S(0, \eta) \rightarrow Y_i$ by $\xi_i(t) = (y_{i,t}, t)$. By (10) ξ_i is continuous, and since it is one-to-one and onto, it is a homeomorphism, i.e. Y_i is a $(p - 2)$ -cell. From (2) (applied to γ)

$$\gamma(B_\gamma) \cap U_i(\sigma_i \times S(0, \eta)) = \cup_i Y_i,$$

so that $\gamma(B_\gamma)$ contains a $(p - 2)$ -cell. (In fact, let $W \subset S(0, \eta)$ be open and in (9) choose t to be maximal only in W . The same argument works, so in a dense open set of $\Delta \times S(0, \eta)$, $\gamma(B_\gamma)$ is locally a $(p - 2)$ -cell.)

3.4. Corollary. *If $\alpha: \Gamma^4 \times \mathbb{R}^{p-1} \rightarrow \Lambda^2 \times \mathbb{R}^{p-2}$ is a layer map with $\dim B_\alpha = \dim(\alpha(B_\alpha)) = 0$, then $p = 2$.*

4. Differential results.

4.1. Lemma. *Let $f: M^n \rightarrow N^p$ be C^m , let K^q be a C^m q -manifold ($m = 1, 2, \dots$; or $m = \infty$, or $m = \omega$; $q = 0, 1, \dots, p - 1$), let ρ be a C^m diffeomorphism of a region in N^p onto $K^q \times \mathbb{R}^{p-q}$, and let Ω be a compact subset of $f^{-1}(\rho^{-1}(K^q \times \{0\}))$.*

If $f| \Omega$ is transverse regular on $\rho^{-1}(K^q \times \{0\})$, then there are $\epsilon > 0$, a C^∞ $(n - p + q)$ -manifold (or empty set) L , and a C^m diffeomorphism σ of $L \times S(0, \epsilon)$ onto a neighborhood of Ω in M^n such that $\rho \circ f \circ \sigma$ is a layer map.

This is a generalization of [6, p. 80, (3.5)] and [3, p. 376, (2.7)]. “ $f| \Omega$ is transverse regular” means that f is transverse regular at x for each $x \in \Omega$.

Proof. By restriction of f to an open neighborhood of Ω in M^n , we may suppose that f is transverse regular on $\rho^{-1}(K^q \times \{0\})$. Also we may suppose [3, p. 376, (2.7)] that M^n is a C^∞ n -manifold, $\Lambda = f^{-1}(\rho^{-1}(K^q \times \{0\}))$ is a C^∞ submanifold, and ([10, p. 20] and [7]) M^n is a complete Riemannian manifold.

Let L and J be open neighborhoods of Ω in Λ such that $\bar{L} \subset J$ and $\bar{J}$ is compact, let ν be the normal bundle of J in M^n , and let $E(\delta)$ be the set of vectors of length less than δ in the total space of ν . For $\delta > 0$ sufficiently small, there is a C^∞ diffeomorphism w (the exponential map) of $E(\delta)$ onto an open (tubular) neighborhood of J in M^n . For $x \in J$ let ν_x be the fiber of ν over x . Then for $\epsilon > 0$ sufficiently small and $r \in S(0, \epsilon)$ each $f^{-1}(\rho^{-1}(K^q \times \{r\}))$ and $w(\nu_x \cap E(\delta))$ meet (transversely) in one point, call it $\sigma(x, r)$. The map $\sigma: J \times S(0, \epsilon) \rightarrow M^n$ is C^m , (since Λ is a C^∞ submanifold), $\sigma| (J \times \{0\})$ is one-to-one, and (from the transverse regularity) σ has maximal rank at each point of $J \times \{0\}$. Thus we may suppose that σ has maximal rank, and from the Inverse Function Theorem and [10, p. 49, (5.7); the word “homeomorphically” should be inserted after “maps”] $\sigma| (\bar{L} \times S(0, \epsilon))$ is a C^m embedding for ϵ sufficiently small; $\sigma| L \times S(0, \epsilon)$ is the desired map.

4.2. Remark. To read the proof of (4.3) the reader will need [6] at hand. Both [6, p. 78, (3.4) and p. 80, (3.7)(a)] are true if we omit the hypothesis “proper”, let $\Omega \subset M^n$ be compact, and replace the statement “ f is transverse regular” by “ $f| \Omega$ is transverse regular”, in both statements and the proofs of these lemmas. Since $f(\Omega)$ is compact, we may suppose that Γ in [6, p. 78, (3.4)] is compact, so that only the first three paragraphs of the proof are needed.

4.3. Lemma. Let $f: M^n \rightarrow N^p$ be C^m , let $\Omega \subset M^n$ be compact, let Y be closed in N^p , and let (K, L) be a finite polyhedral pair in (a C^m triangulation of) N^p such that $L \cap Y = \emptyset$ and $f| \Omega$ is transverse regular on each open simplex of L . Let U be an open neighborhood of $K - L$ with $\bar{U}$ compact, and let $\epsilon > 0$. Then either

(a) there is a C^m diffeomorphism A of N^p such that A agrees with the identity I off U , $\rho(A, I) < \epsilon$ (the coarse C^m topology is metric on this space of diffeomorphisms [10, p. 25, (3.5)]), $f| \Omega$ is transverse regular on each open simplex of $A(K)$, and $A(K^{p-1}) \cap Y = \emptyset$; or

(b) there is an embedding $\lambda: S^m \times \mathbf{R}^{p-m} \rightarrow N^p$ for some $m = 0, 1, \dots, p-1$ such that $f| \Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \emptyset$ for each $t \in \mathbf{R}^{p-m}$.

(K^{p-1} is the $(p-1)$ -skeleton of K .)

Proof. By [6, (3.4), p. 80] and (4.2) there is a C^n diffeomorphism A such that (1) A agrees with I off U and $d(A, I) < \epsilon$, and (2) $f|_\Omega$ is transverse regular on each open simplex of $A(K)$. The rest of the proof is paragraphs three through six of [6, (3.7), p. 80] with Ψ replaced by K and X replaced by Y .

4.4. Lemma. *Let $f: M^n \rightarrow N^p$ be C^n , let $\Omega \subset M^n$ be compact, and let $Y \subset N^p$ be closed. Then either*

- (a) *Y is a locally simple 0-dimensional subspace, or*
- (b) *there is an embedding $\lambda: S^m \times \mathbf{R}^{p-m} \rightarrow N^p$ for some $m = 0, 1, \dots, p-1$ such that $f|_\Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \emptyset$ for each $t \in \mathbf{R}^{p-m}$.*

Proof. Suppose that there is no such embedding λ , and let $\bar{y} \in Y$. Let T be a C^n triangulation of N^p such that $\bar{y}$ is in the interior of the closed p -simplex K_0 , let K_r be the r^{th} barycentric subdivision of K_0 , and let $L_r \subset K_r$ be the polyhedron formed by the $(p-1)$ -skeleton of K_{r-1} (i.e. $L_r = K_r \cap K_{r-1}^{p-1}$). Choose $U_0 \subset N^p$ open with $K_0 \subset U_0$ and $\bar{U}_0$ compact, and δ with $0 < \delta < d(\bar{y}, \partial K_0)$. Let A_0 be as given by (4.3) for $K = K_0$, $L = L_0 = \emptyset$, $U = U_0$, and $\epsilon = \delta/2$. In general, let A_r be as given in (4.3) for $K = A_{r-1} \circ \dots \circ A_0(K_{r-1})$, $L = A_{r-1} \circ \dots \circ A_1(L_{r-1})$, $U = U_0 - L$, and $\epsilon = 2^{-r-1}\delta$.

The space of all C^n maps of N^p into itself which agree with the identity off U , topologized with the (coarse = fine) C^n topology, has a metric ρ with $\rho(g, h) \geq d(g(x), h(x))$ for all $x \in N^p$ [3, p. 381, (3.4) and its proof], and we may suppose this is the metric of (4.3). Thus $\lim_r A_r \circ \dots \circ A_0$ exists and is a C^n map, and similarly A^{-1} is C^n , so that A is a C^n diffeomorphism. Since $A_s \circ \dots \circ A_0$ agrees with $A_r \circ \dots \circ A_0$ on L_r for $s \geq r$, $A(L_r) \cap Y = \emptyset$ for every r and $\bar{y} \in A(\text{int } K_0) = A_0(\text{int } K_0)$, and thus Y is simple at $\bar{y}$.

4.5. Lemma. *Let $f: M^{p+2} \rightarrow N^p$ be C^{p+2} with $p \geq 2$, and let $\dim B_f = \dim f(B_f) \leq 0$. Then the conclusion of (1.1) is satisfied.*

Proof. Suppose the contrary, and that f is a counterexample with minimal $p \geq 2$. Let $\bar{x} \in B_f$. By (2.6) it suffices to prove that B_f is discrete at $\bar{x}$.

By [4, (2.6)] there are a connected (topological) manifold $K^{p+2} \subset M^{p+2}$ with boundary such that $\bar{x} \in \text{int } K^n$ and an open p -cell $D \subset N^p$ such that $h = f|_{K^{p+2}}: K^{p+2} \rightarrow D$ is proper with $B_h \cap \partial K^{p+2} = \emptyset$. Let $\Delta \subset D$ be a closed p -cell neighborhood of $f(x)$ and let $\Omega \subset \text{int } K^{p+2}$ be the compact set $h^{-1}(\Delta) \cap B_h$. If $h(B_h) \cap \text{int } \Delta$ is simple (2.5) in $\text{int } \Delta$, the conclusion results from (2.6); thus we may suppose that it is not simple, so that $p \geq 3$.

By (4.4) applied to $h|_{\text{int } K^{p+2}}$, $Y = \Delta \cap h(B_h) = h(\Omega)$, Ω as above, there is a C^{p+2} embedding $\lambda: S^m \times \mathbf{R}^{p-m} \rightarrow D$, $m \leq p-1$, such that $h|_\Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap h(\Omega) = \lambda(S^m \times \{t\}) \cap \Delta \cap h(B_h) \neq \emptyset$ for each $t \in \mathbf{R}^{p-m}$. Let $\Omega' = \Omega \cap h^{-1}(\lambda(S^m \times \{0\}))$. Apply (4.1) to obtain by restriction of h to a neighborhood of Ω' and change of coordinates a C^{p+2} layer map

$$g: L^{m+2} \times \mathbf{R}^{p-m} \rightarrow S^m \times \mathbf{R}^{p-m} \quad \text{with}$$

$\dim B_g = \dim g(B_g) \leq 0$ and $g(B_g) \cap (S^m \times \{t\}) \neq \emptyset$ for each $t \in \mathbf{R}^{p-m}$. Since $B_g \cap ((L^{n-p+m}) \times \{t\}) \supset B(g_t)$, $\dim B(g_t) = \dim g_t(B(g_t)) = 0$ for each t .

If $m \geq 2$, it follows from the minimality assumption that each $B(g_t)$ is discrete, and from its nonemptiness and (2.6) that $m = 2$. In this case ($m \geq 2$) let $\Gamma^4 = L^4$, $\Lambda^2 = S^2$, and $g = \alpha$. If $m = 0$ or 1 , let $\Gamma^4 = L^{2+m} \times \mathbf{R}^{2-m}$, let $\Lambda^2 = S^m \times \mathbf{R}^{2-m}$, and view g as a layer map α of $\Gamma^4 \times \mathbf{R}^{p-2}$ into $\Lambda^2 \times \mathbf{R}^{p-2}$. Since $p \geq 3$, a contradiction results from (3.4).

4.6. Proof of (1.1). For $n - p = 0$ or 1 , (1.1) is [4, (1.1)]. For $n - p = 2$, $B_f \subset R_{p-1}(f)$ according to the Rank Theorem [4, (1.6)], and (1.1) in this case results from (4.5).

4.7. Remark. Is the hypothesis $\dim f(B_f) \leq 0$ needed in (1.1) for $n - p = 2$? If so, then there is a counterexample with $n = 4$ and $p = 2$.

Proof. If there is an example, pick one with minimal $p \geq 2$; by (1.1) $\dim f(B_f) > 0$. We argue as in the proof of (4.5), except that we use [6, p. 80, (3.7)(a)] and (4.2) in place of (4.4) to obtain the layer map g . From the minimality assumption, $m = 0$ or 1 . If m were 0 , from the transverse regularity $\text{imag } \lambda$ would be disjoint from the critical set, and by the Rank Theorem [6, (1.6)] from $h(B_h)$, contradicting the choice of λ . Thus $m = 1$. Using a natural inclusion $\mathbf{R} \subset \mathbf{R}^{p-m}$, and restricting g to $L^3 \times \mathbf{R}$, we obtain the desired example.

4.8. Remark. If $f: \mathbf{C}^m \rightarrow \mathbf{C}$ ($m \geq 1$, $\mathbf{C}$ the complex plane) is a polynomial map with isolated critical point at 0 , then $\{0\} = B_f$.

(Such maps are discussed in [9]. By [9, p. 13, Remark] $f^{-1}(f(0))$ is not a smooth manifold; if it were not a topological manifold, the conclusion would result. The analogous real polynomial statement is false, e.g. $f(x) = x^3$.)

Proof. For $m = 1$ the remark is well-known [1, p. 179] so suppose that $m \geq 2$. Let $D_\epsilon = \bar{S}(0, \epsilon)$, $V = f^{-1}(f(0))$, and $E = \partial D_\epsilon - V \subset \mathbf{C}^m$, where E is the total space of the fiber bundle (over S^1) of [9, p. 43, (4.8)]. According to [9, p. 57, (6.5)] its fiber has the homotopy type of a bouquet $S^{m-1} \vee S^{m-1} \vee \dots \vee S^{m-1}$ of $(m-1)$ -spheres, at least one in number [9, p. 60, (7.3)] since $\{0\}$ is a critical point, and from the homotopy sequence of a fibering (1) $\pi_{m-1}(E) \neq 0$ if $m \geq 3$, and $\pi_1(E) \approx \mathbf{Z} \oplus \mathbf{Z}$ or is non-abelian if $m = 2$. For $\epsilon > 0$ sufficiently small and any δ with $0 < \delta < \epsilon$, $(D_\epsilon - V, D_\delta - V)$ is homeomorphic to $E \times ([0, 1], [1/2, 1])$ [9, p. 18, (2.10)], so that (2) the homomorphism induced by inclusion of $\pi_{m-1}(D_\delta - V)$ into $\pi_{m-1}(D_\epsilon - V)$ is an isomorphism and $\pi_{m-1}(D_\delta - V) \approx \pi_{m-1}(E)$.

From the Rank Theorem [4, (1.6)] $B_f \subset \{0\}$. Suppose $0 \notin B_f$. Then there are an open $2m$ -cell neighborhood U of 0 with $U \subset D_\epsilon$ and $(U, U \cap V)$ homeomorphic with $(\mathbf{R}^{2m}, \mathbf{R}^{2(m-1)})$, and $\delta > 0$ with $D_\delta \subset U$. By (2) the homomorphism

$$i_\# : \pi_{m-1}(D_\delta - V) \rightarrow \pi_{m-1}(U - V)$$

induced by inclusion has $\text{imag } i_\# \approx \pi_{m-1}(E)$, while $\pi_{m-1}(U - V) = 0$ for $m \geq 3$ and $\mathbf{Z}$ for $m = 2$; (1) is thus contradicted.

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