Positive Newtonian Capacity and Stationary Viscous Incompressible Flow

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1. Introduction. Operating in Euclidean 3-space, we shall say $\mathbf{v} = (v_1, v_2, v_3)$ is a stationary viscous incompressible flow in an open set Ω if there is a pressure p such that

$$(1.1) \quad \mathbf{v} \text{ is in } C^2(\Omega), \text{ and } p \text{ is in } C^1(\Omega);$$

$$(1.2) \quad v_i, \partial v_i / \partial x_k, \text{ and } p \text{ are in } L^2(\Omega) \text{ for } j, k = 1, 2, 3;$$

$$(1.3) \quad \text{there is a non-zero constant } \nu \text{ such that in } \Omega \text{ the following equations are satisfied:}$$

$$\begin{cases} \nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p = 0 \\ \operatorname{div} \mathbf{v} = 0. \end{cases}$$

Let Z be a closed set contained in the interior of the unit ball, *i.e.*, $Z \subset B(0, 1)$ where $B(x, r)$ designates the open 3-ball with center x and radius r . We shall say Z is a removable set for stationary viscous incompressible flows if the following holds:

If $\mathbf{v}$ and p meet (1.1), (1.2), and (1.3) with $\Omega = B(0, 1) - Z$, then $\mathbf{v}$ and p can be defined at the points of Z so that $\mathbf{v}$ and p meet (1.1), (1.2), and (1.3) with $\Omega = B(0, 1)$.

In this paper, we intend to establish the *necessary* part of the following theorem:

Theorem. *A necessary and sufficient condition that a closed set $Z \subset B(0, 1)$ be removable for stationary viscous incompressible flows is that Z be of Newtonian capacity zero.*

The sufficiency part of the above theorem is an immediate corollary of [5], [6], and [2, p. 10].

The notion of Newtonian capacity zero for closed sets is a classical concept, but we give it here for completeness. The closed set $Z \subset B(0, 1)$ is of Newtonian capacity zero if it is not of positive Newtonian capacity. Z is a set of positive Newtonian capacity if there exists a non-negative Borel measure μ such that $\mu[B(0, 1) - Z] = 0$, $\mu(Z) = 1$, and

$$\int_Z |x - y|^{-1} d\mu(y) \text{ is a uniformly bounded function of } x \text{ in } B(0, 1) - Z.$$

The main result of this paper is obtained via a type of existence theorem which does not appear to be very prevalent in the general area of nonlinear partial differential equations. To obtain this theorem, we combine the functional analytic methods previously used in dealing with the Navier–Stokes equations (see [2, Chapter 5]) with various techniques used in the area of trigonometric series and its applications (see [3], [4], [5], and [6]).

2. Flows and multiple Fourier Series. To establish the necessary part of the above theorem, we shall need some lemmas connecting multiple Fourier series with the nonlinear stationary Navier–Stokes equations defined in (1.3).

We shall use the following notation:

$T_3 = \{x: -\pi \leq x_i < \pi, i = 1, 2, 3\}$; m will designate an integral lattice point; for a function g in $L^1(T_3)$, we shall set

$$(2.1) \quad \hat{g}(m) = (2\pi)^{-3} \int_{T_3} g(x) e^{-i(m, x)} dx.$$

Also, (x, y) will designate the usual inner product, and $(x, x)^{1/2}$ will be designated by $|x|$.

In order to establish our result, we shall have to introduce a number of functions.

We first introduce the function $q_i(x)$ defined in E_3 as follows:

$$(2.2) \quad q_i(x) = \lim_{t \rightarrow 0} \sum_{m \neq 0} i m_i e^{i(m, x) - |m|t} |m|^{-2}.$$

From [3, p. 72], we obtain that the following properties prevail for $j = 1, 2, 3$:

$$(2.3) \quad \begin{aligned} q_i(x) & \text{ is in } L^1(T_3); \\ q_i^{\wedge}(m) &= i m_i / |m|^2 \text{ for } m \neq 0; \\ q_i^{\wedge}(0) &= 0. \end{aligned}$$

In a similar manner, we introduce the function $H(x)$ defined in E_3 as follows:

$$(2.4) \quad H(x) = \lim_{t \rightarrow 0} \sum_{m \neq 0} e^{i(m, x) - |m|t} |m|^{-2}.$$

From [3, p. 72], we obtain that the following properties prevail:

$$(2.5) \quad \begin{aligned} H(x) & \text{ is in } L^1(T_N), \\ H^{\wedge}(m) &= |m|^{-2} \text{ for } m \neq 0 \text{ and} \\ H^{\wedge}(0) &= 0. \end{aligned}$$

From [3, p. 72], we obtain also that

$$(2.6) \quad H(x) \text{ and } q_i(x) \text{ are in } C^\infty[E_3 - \bigcup_m \{2\pi m\}],$$

and that

$$(2.7) \quad \Delta H(x) = 1 \text{ and } \Delta q_i(x) = 0 \text{ in } E_3 - \bigcup_m \{2\pi m\}.$$

Furthermore, we obtain from this same reference that there are positive constants α , α_i , β , and β_i such that

$$(2.8) \quad \begin{aligned} |H(x) - \alpha |x|^{-1}| &\leq \beta \\ |q_i(x) + \alpha_i x_i |x|^{-3}| &\leq \beta_i \quad \text{for } x \text{ in } T_3 - \{0\}. \end{aligned}$$

Next, we set

$$(2.9) \quad G(x) = - \sum_{m \neq 0} e^{i(m, x)} |m|^{-4},$$

and observe that the series in (2.9) converges absolutely. Consequently, $G(x)$ is a continuous periodic function. We conclude from [3, Lemma 7], (2.4), (2.6), and well-known facts concerning subharmonic functions that

$$(2.10) \quad G(x) \text{ is in } C^\infty[E_3 - \bigcup_m \{2\pi m\}]$$

and furthermore that

$$(2.11) \quad \Delta G(x) = H(x) \text{ in } E_3 - \bigcup_m \{2\pi m\}.$$

From (2.8) and (2.11), we conclude with α the constant defined in (2.8) that

$$(2.12) \quad G(x) - 2^{-1}\alpha |x| \text{ is in } C^\infty[B(0, 2)].$$

Next, motivated by [2, p. 51], we set for $j, k = 1, 2, 3$,

$$(2.13) \quad \nu u_k^j(x) = \lim_{\epsilon \rightarrow 0} \sum_{m \neq 0} [-\delta_k^j + m_j m_k |m|^{-2}] |m|^{-2} e^{i(m, x) - |m|\epsilon},$$

where δ_k^j designates the Kronecker δ and, observe from (2.10) and [4, Lemma 2] that the above limit exists and is finite in $E_3 - \bigcup_m \{2\pi m\}$. (It clearly exists and is equal to $-\infty$ for $x = 2\pi m$ and $j = k$.)

From (2.9), (2.10), (2.11), and [4, Lemma 2], we see that

$$(2.14) \quad \nu u_k^j(x) = -\delta_k^j H(x) + \partial^2 G(x) / \partial x_j \partial x_k \text{ for } x \text{ in } E_3 - \bigcup_m \{2\pi m\},$$

and consequently from (2.10) and (2.12), we obtain that

$$(2.14') \quad \begin{aligned} \nu u_k^j &\text{ is in } L^1(T_3), \text{ and} \\ \nu u_k^j(x) &= [-\delta_k^j + m_j m_k |m|^{-2}] |m|^{-2} \quad \text{for } m \neq 0 \\ \nu u_k^j(0) &= 0. \end{aligned}$$

Next, we say that g is in $W_2^1(T_3)$ if g is in $L^2(T_3)$ and if $\sum |g^\wedge(m)|^2 |m|^2 < \infty$.

We shall refer to the function in $L^2(T_3)$ whose Fourier series is $\sum im_i g^\wedge(m) e^{i(m, x)}$ by g_{x_i} . It is clear that $\int_{T_3} g(x) \psi_{x_i}(x) dx = - \int_{T_3} g_{x_i}(x) \psi(x) dx$ if ψ is a periodic function (of period 2π in each variable) which is in $C^\infty(E_3)$.

The following lemma then prevails:

Lemma 1. *Let g be in $W_2^1(T_3)$ and suppose $g^\wedge(0) = 0$. Then there is an absolute constant K , independent of g , such that*

$$\int_{T_3} g^4 dx \leq K \left\{ \int_{T_3} g^2 dx \right\}^{1/2} \left\{ \int_{T_3} |\nabla g|^2 dx \right\}^{3/2}.$$

Proof. To establish the lemma, we define for $t > 0$

$$(2.15) \quad g(x, t) = \sum_m g^\wedge(m) e^{i(m, x) - |m|t},$$

and observe that $g(x, t)$ is a periodic function in $C^\infty(E_3)$.

Next, we extend g by periodicity of period 2π in each variable to all of E_3 and observe from [3, pp. 55–6] that

$$(2.16) \quad \lim_{t \rightarrow 0} g(x, t) = g(x) \quad \text{almost everywhere in } E_3.$$

Also we have that for $t > 0$,

$$(2.17) \quad \begin{aligned} \int_{T_3} |g(x, t)|^2 dx &\leq \int_{T_3} |g(x)|^2 dx \\ \int_{T_3} |\nabla g(x, t)|^2 dx &\leq \int_{T_3} |\nabla g(x)|^2 dx. \end{aligned}$$

Furthermore since $g^\wedge(0) = 0$,

$$(2.18) \quad \begin{aligned} \int_{T_3} |g(x, t)|^2 dx &\leq \int_{T_3} |\nabla g(x, t)|^2 dx \\ \int_{T_3} |g(x)|^2 dx &\leq \int_{T_3} |\nabla g(x)|^2 dx. \end{aligned}$$

Next, we choose a function $\lambda(x)$ with the following properties:

$$(2.19) \quad \begin{aligned} \lambda &= 1 \quad \text{in } B(0, 1) \\ \lambda &= 0 \quad \text{in } E_3 - B(0, 2) \\ \lambda &\text{ is in } C^\infty(E_3) \text{ and } 0 \leq \lambda(x) \leq 1 \text{ for all } x. \end{aligned}$$

For x^0 , a given point in E_3 , we set $\psi(x) = \lambda(x - x^0)$. Then from [2; Lemma 2, p. 9], we have

$$(2.20) \quad \begin{aligned} \int_{B(x^0, 1)} |g(x, t)|^4 dx &\leq \int_{B(x^0, 2)} |\psi(x)g(x, t)|^4 dx \\ &\leq 4 \left\{ \int_{B(x^0, 2)} |\psi(x)g(x, t)|^2 dx \right\}^{1/2} \left\{ \int_{B(x^0, 2)} |\nabla \psi(x)g(x, t)|^2 dx \right\}^{3/2}. \end{aligned}$$

Observing that $|\nabla \lambda(x)|$ is uniformly bounded in E_3 , we conclude from (2.17), (2.18), (2.19), and (2.20) that there is a constant K' , independent of x^0 and g , such that

$$(2.21) \quad \int_{B(x^0, 1)} |g(x, t)|^4 dx \leq K' \left\{ \int_{T_3} g^2(x) dx \right\}^{1/2} \left\{ \int_{T_3} |\nabla g(x)|^2 dx \right\}^{3/2}.$$

The conclusion to the lemma then follows immediately from Fatou's lemma, (2.16), (2.21), and the fact that a finite number of unit three-balls cover T_3 .

Next, we observe that the following lemma holds:

Lemma 2. *Let g be in $W_2^{-1}(T_3)$ and suppose $\hat{g}(0) = 0$. Then there is an absolute constant K , independent of g , such that*

$$(2.22) \quad \int_{T_3} g^6 dx \leq K \left\{ \int_{T_3} |\nabla g|^2 dx \right\}^3.$$

The proof of Lemma 2 follows from [2, Lemma 3, p. 10] in a manner similar to the proof of Lemma 1. We leave the details to the reader.

Next, we let ${}_0W_2^{-1}(T_3)$ designate the Hilbert space introduced in Lemmas 1 and 2, i.e.,

$$(2.23) \quad {}_0W_2^{-1}(T_3) = \{g: g \text{ in } W_2^{-1}(T_3) \text{ and } \hat{g}(0) = 0\}$$

where the inner product $\langle g, h \rangle$ is defined by

$$(2.24) \quad \langle g, h \rangle = (2\pi)^{-3} \int_{T_3} (g_{x_1} h_{x_1} + g_{x_2} h_{x_2} + g_{x_3} h_{x_3}) dx \text{ for } g \text{ and } h \text{ in } {}_0W_2^{-1}(T_3).$$

The following lemma (see [2, p. 13]) then prevails:

Lemma 3. *Let g and g_k be in ${}_0W_2^{-1}(T_3)$ for $k = 1, 2, \dots$. Suppose g_k is weakly convergent to g in ${}_0W_2^{-1}(T_3)$. Then*

$$(2.25) \quad \lim_{k \rightarrow \infty} \int_{T_3} |g - g_k|^4 dx = 0.$$

Proof. With no loss in generality, we can suppose that $g = 0$. Then it follows from the Banach–Steinhaus theorem that there is a constant M such that

$$(2.26) \quad \sum_m |g_k^\wedge(m)|^2 |m|^2 \leq M \quad \text{for } k = 1, 2, \dots.$$

Also we have from weak convergence to zero that

$$(2.27) \quad \lim_{k \rightarrow \infty} g_k^\wedge(m) = 0 \quad \text{for every } m.$$

Next, we observe that

$$(2.28) \quad \sum_m |g_k^\wedge(m)|^2 \leq \sum_{|m| \leq N} |g_k^\wedge(m)|^2 + N^{-2} \sum_{N \leq |m|} |g_k^\wedge(m)|^2 |m|^2.$$

We consequently obtain from (2.26), (2.27), and (2.28) that

$$\limsup_{k \rightarrow \infty} \sum_m |g_k^{\wedge}(m)|^2 \leq MN^{-2}.$$

Since N is arbitrary, we conclude that

$$(2.29) \quad \lim_{k \rightarrow \infty} \int_{T_s} |g_k|^2 dx = 0.$$

On the other hand from Lemma 1 and (2.26) we obtain that there is a constant K such that

$$\int_{T_s} g_k^4 dx \leq K \left\{ \int_{T_s} g_k^2 dx \right\}^{1/2} M^{3/2} \quad \text{for } k = 1, 2, \dots.$$

The conclusion to the lemma then follows immediately from this last fact and (2.29).

Next, we define $\mathfrak{D}(T_3)$ as follows:

$$(2.30) \quad \mathfrak{D}(T_3) = \{\psi : \psi \text{ is in } C^\infty(E_3), \text{ and } \psi \text{ is periodic of period } 2\pi \\ \text{in each variable}\}.$$

The following lemma then holds:

Lemma 4. *Let f_j be in ${}_0W_2^1(T_3) \cap \mathfrak{D}(T_3)$ for $j = 1, 2, 3$. Set*

$$(2.31) \quad v_i(x) = (2\pi)^{-3} \sum_{k=1}^3 \int_{T_s} u_i^k(x-y) f_k(y) dy \quad \text{for } i = 1, 2, 3$$

and

$$(2.32) \quad p(x) = (2\pi)^{-3} \sum_{k=1}^3 \int_{T_s} q_k(x-y) f_k(y) dy.$$

Then $p(x)$ and $v_i(x)$ are in ${}_0W_2^1(T_3) \cap \mathfrak{D}(T_3)$ for $j = 1, 2, 3$ and

$$(2.33) \quad v \Delta v_i - \partial p / \partial x_i = f_i, \quad j = 1, 2, 3, \quad \sum_{i=1}^3 \partial v_i / \partial x_i = 0.$$

Proof. From (2.3) and (2.14'), we see that the integrals on the right hand side of (2.31) are well-defined and that

$$\int_{T_s} u_i^k(x-y) f_k(y) dy = \int_{T_s} u_i^k(y) f_k(x-y) dy$$

and

$$\int_{T_s} q_k(x-y) f_k(y) dy = \int_{T_s} q_k(y) f_k(x-y) dy.$$

As a consequence, $p(x)$ and $v_i(x)$ are in $\mathfrak{D}(T_3)$. Also

$$(2.34) \quad \begin{aligned} v_i^\wedge(m) &= \sum_{k=1}^3 u_i^k(m) f_k^\wedge(m), \quad j = 1, 2, 3, \\ p^\wedge(m) &= \sum_{k=1}^3 q_k(m) f_k^\wedge(m). \end{aligned}$$

From (2.34), we see that $v_i^\wedge(0) = p^\wedge(0) = 0$; consequently, v_i and p are also in ${}_0W_2^{-1}(T_3)$.

Next, from (2.3), (2.13), and (2.34), we observe that for $m \neq 0$,

$$-v |m|^2 v_i^\wedge(m) - im_i p^\wedge(m) = f_i^\wedge(m), \quad j = 1, 2, 3,$$

and

$$\sum_{i=1}^3 im_i v_i^\wedge(m) = 0.$$

(2.33) follows from these facts and the fact that all derivatives of p and v_i can be obtained by differentiating their Fourier series under the summation sign. The proof of the lemma is therefore complete.

Lemma 5. *Let f_j be in ${}_0W_2^{-1}(T_3)$ for $j = 1, 2, 3$. Let v_i and p be defined respectively by (2.31) and (2.32). Then v_i and p are in ${}_0W_2^{-1}(T_3)$ and*

$$(2.35) \quad \begin{aligned} \int_{T_3} [v_i \Delta \psi + p \psi_{x_i}] dx &= \int_{T_3} f_i \psi dx \\ \int_{T_3} [v_1 \psi_{x_1} + v_2 \psi_{x_2} + v_3 \psi_{x_3}] dx &= 0 \text{ for all } \psi \text{ in } \mathfrak{D}(T_3). \end{aligned}$$

Proof. (2.34) holds as before, and it is immediate from this that v_i and p are in ${}_0W_2^{-1}(T_3)$.

To establish (2.35), we introduce $v_i(x, t)$, $f_i(x, t)$, and $p(x, t)$ for $t > 0$ in a manner analogous to (2.15). But then the analogue of (2.34) holds for these functions, and consequently the analogue of (2.33) is also true. We obtain therefore that (2.35) holds when $v_i(x)$, $p(x)$, and $f_i(x)$ are replaced respectively by $v_i(x, t)$, $p(x, t)$, and $f_i(x, t)$. But, as is well-known [3; Remark 2, p. 76],

$$\begin{aligned} \int_{T_3} |v_i(x, t) - v_i(x)| dx &\rightarrow 0 \quad \text{as } t \rightarrow 0, \\ \int_{T_3} |p(x, t) - p(x)| dx &\rightarrow 0 \quad \text{as } t \rightarrow 0, \\ \int_{T_3} |f_i(x, t) - f_i(x)| dx &\rightarrow 0 \quad \text{as } t \rightarrow 0, \end{aligned}$$

and the conclusion of the lemma follows.

Next, we introduce the Hilbert space $\mathcal{H}(T_3)$. We say $\mathbf{v} = (v_1, v_2, v_3)$ is in $\mathcal{H}(T_3)$ if v_j is in ${}_0W_2^1(T_3)$ for $j = 1, 2, 3$ and if furthermore

$$v_{1x_1} + v_{2x_2} + v_{3x_3} = 0.$$

The inner product $[\mathbf{v}, \mathbf{w}]$ in $\mathcal{H}(T_3)$ is defined as follows:

$$(2.36) \quad [\mathbf{v}, \mathbf{w}] = \sum_{i=1}^3 \langle v_i, w_i \rangle$$

where $\langle v_i, w_i \rangle$ is defined by (2.24).

Next, we observe from Lemma 2 that for v in $\mathcal{H}(T_3)$, v_j is in $L^0(T_3)$ for $j = 1, 2, 3$. Consequently as in [2, p. 116], we obtain that for fixed $\mathbf{v}$ in $\mathcal{H}(T_3)$ the expression

$$(2.37) \quad (2\pi)^{-3} \sum_{i=1}^3 \sum_{k=1}^3 \int_{T_3} v_i v_j \phi_{ix_k} dx$$

defined for all ϕ in $\mathcal{H}(T_3)$ gives rise to a bounded linear functional on $\mathcal{H}(T_3)$.

Consequently, there exists an element $A\mathbf{v}$ in $\mathcal{H}(T_3)$ such that $[A\mathbf{v}, \phi]$ is equal to the expression in (2.37) for every ϕ in $\mathcal{H}(T_3)$.

Next, we observe that the following lemma holds:

Lemma 6. *Let A be the operator defined on $\mathcal{H}(T_3)$ by (2.37), i.e., $[A\mathbf{v}, \phi]$ equals the expression in (2.37) for every ϕ in $\mathcal{H}(T_3)$. Then A is a completely continuous operator.*

The proof of Lemma 6 follows from Lemmas 1, 2, and 3 as in [2, p. 117]. We leave the details to the reader.

Next, we establish the following lemma:

Lemma 7. *Let A be the operator defined in Lemma 6. Then $[A\mathbf{v}, \mathbf{v}] = 0$ for $\mathbf{v}$ in $\mathcal{H}(T_3)$.*

To establish Lemma 7, we first note the following Remark.

Remark 1. *Let g be in $L^\gamma(T_3)$, $1 \leq \gamma < \infty$ and let $g(x, t)$ be defined as in (2.15). Then*

$$\int_{T_3} |g(x, t) - g(x)|^\gamma dx \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

The proof of Remark 1 follows easily from the proof of [3; Remark 2, p. 76].

Next, we say g is in $W_{3/2}^1(T_3)$ if g is in $L^{3/2}(T_3)$ and if there are functions f_k , $k = 1, 2, 3$, in $L^{3/2}(T_3)$ such that

$$(2.38) \quad \int_{T_3} g(x) \psi_{x_k}(x) dx = - \int_{T_3} f_k(x) \psi(x) dx \quad \text{for all } \psi \text{ in } \mathfrak{D}(T_3).$$

We shall refer to $f_k(x)$ as $g_{x_k}(x)$.

Proof of Lemma. Now let $\mathbf{v}$ be given in $\mathcal{H}(T_3)$. Then we observe from Lemma 2, Remark 1, and (2.38) that, for $j = 1, 2, 3$,

$$(2.39) \quad w_i = v_i^2 \text{ is in } L^3(T_3) \cap W_{3/2}^1(T_3),$$

and

$$(2.40) \quad w_{ixk} = 2v_i v_{ixk}.$$

Next, we observe that for $t > 0$,

$$(2.41) \quad \int_{T_3} v_k(x, t) w_{ixk}(x, t) dx = - \int_{T_3} v_{kxk}(x, t) w_i(x, t) dx.$$

Since v_k is in $L^3(T_3) \cap W_{3/2}^1(T_3)$, we have from (2.39), (2.41), and Remark 1 that

$$(2.42) \quad \int_{T_3} v_k(x) w_{ixk}(x) dx = - \int_{T_3} v_{kxk}(x) w_i(x) dx.$$

But then from this fact and (2.40), we conclude that

$$2 \int_{T_3} v_k(x) v_i(x) v_{ixk}(x) dx = - \int_{T_3} v_{kxk}(x) v_i^2(x) dx.$$

Now from (2.37) and this last fact, we have that

$$\begin{aligned} (2\pi)^3 [A\mathbf{v}, \mathbf{v}] &= \sum_{i=1}^3 \sum_{k=1}^3 \int_{T_3} v_k v_i v_{ixk} dx \\ &= -2^{-1} \sum_{i=1}^3 \sum_{k=1}^3 \int_{T_3} v_i^2 v_{kxk} dx = -2^{-1} \sum_{i=1}^3 \int_{T_3} v_i^2 \left(\sum_{k=1}^3 v_{kxk} \right) dx = 0, \end{aligned}$$

and the proof of the lemma is complete.

Next, using (2.14), we set

$$(2.43) \quad U_{iki}(x) = \partial u_k^i(x) / \partial x_i \text{ for } x \text{ in } E_3 - \bigcup_m \{2\pi m\}$$

and observe from (2.6) and (2.10) that

$$(2.44) \quad U_{iki}(x) \text{ is in } C^\infty[E_3 - \bigcup \{2\pi m\}].$$

As a consequence of (2.11), (2.12), and (2.14), we conclude that there is a constant α' such that for $i, j, k = 1, 2, 3$,

$$(2.45) \quad |U_{iki}(x)| \leq \alpha' |x|^{-2} \text{ for } x \text{ in } B(0, 2) - \{0\}.$$

Observing that U_{iki} is periodic in $E_3 - \bigcup_m \{2\pi m\}$, we conclude from (2.44), (2.45), and [2; Lemma 5, p. 13] that the following holds:

Remark 2. Let $g(x)$ be in $L^{3/2}(T_3)$, and set

$$f(x) = (2\pi)^{-3} \int_{T_3} g(y) U_{iki}(x - y) dy.$$

Then f is defined almost everywhere in T_3 , and f is in $L^3(T_3)$.

We leave the details of the proof of Remark 2 to the reader.

Next, we establish the following lemma:

Lemma 8. Let $\mathbf{f} = (f_1, f_2, f_3)$ where f_i is in $L^{3/2}(T_3)$ and $f_i^\wedge(0) = 0$ for $j = 1, 2, 3$. Set $\mathbf{v} = (v_1, v_2, v_3)$ where v_i is defined almost everywhere in T_3 by (2.31). Then

$$(2.46) \quad \mathbf{v} \text{ is in } \mathfrak{H}(T_3)$$

and

$$(2.47) \quad -\nu[\mathbf{v}, \Phi] = \sum_{i=1}^3 (2\pi)^{-3} \int_{T_3} f_i \Phi_i dx \quad \text{for } \Phi \text{ in } \mathfrak{H}(T_3).$$

Proof. From (2.31), we observe

$$(2.48) \quad v_i^\wedge(m) = \sum_{k=1}^3 u_i^{k\wedge}(m) f_k^\wedge(m).$$

From the hypothesis of the lemma, (2.43), and Remark 2, and the fact that $|U_{ik}^\wedge(m)| = |m_i u_k^{i\wedge}(m)|$, we have in particular that,

$$\sum_{m \neq 0} |m_i|^2 |u_i^{k\wedge}(m) f_k^\wedge(m)|^2 < \infty, \quad i, j, k = 1, 2, 3.$$

We consequently conclude from (2.48) that

$$(2.49) \quad v_i \text{ is in } {}_0W_2^{-1}(T_3), \quad j = 1, 2, 3.$$

Next, for $t > 0$, we set

$$(2.50) \quad \begin{aligned} v_i(x, t) &= \sum_m v_i^\wedge(m) e^{i(m, x) - |m|t}, \\ f_i(x, t) &= \sum_m f_i^\wedge(m) e^{i(m, x) - |m|t}, \end{aligned}$$

and define $p(x, t)$ by (2.32) with $f_k(y)$ replaced by $f_k(y, t)$. We also note from (2.48) that $v_i(x, t)$ is equal to the expression on the right hand side of the equality sign in (2.31) when $f_k(y)$ is replaced by $f_k(y, t)$. As a consequence we obtain from Lemma 5 that for $t > 0$

$$(2.51) \quad \sum_{k=1}^3 \int_{T_3} v_k(x, t) \psi_{x_k}(x) dx = 0, \quad \text{for } \psi \text{ in } \mathfrak{D}(T_3).$$

Letting $t \rightarrow 0$ in (2.51) and using Remark 1, we conclude that

$$\sum_{k=1}^3 \int_{T_3} v_k(x) \psi_{x_k}(x) dx = 0$$

for ψ in $\mathfrak{D}(T_3)$. But this implies that $\sum_{k=1}^3 v_{kx_k} = 0$ and this fact in conjunction with (2.49) establishes (2.46).

Next, we let ϕ be in $\mathcal{H}(T_3)$ and define $\phi_i(x, t)$ in a manner analogous to (2.50). Then $\phi_i(x, t)$ is in $\mathcal{D}(T_3) \cap \mathcal{H}(T_3)$, and we conclude from (2.35) and (2.24) that

$$\begin{aligned} -\nu \langle v_i(x, t), \phi_i(x, t) \rangle + (2\pi)^{-3} \int_{T_3} p(x, t) \phi_{ix_i}(x, t) dx \\ = (2\pi)^{-3} \int_{T_3} f_i(x, t) \phi_i(x, t) dx. \end{aligned}$$

Summing over j and using the fact that $\phi(x, t)$ is in $\mathcal{H}(T_3)$, we conclude that

$$(2.52) \quad -\nu \sum_{i=1}^3 \langle v_i(x, t), \phi_i(x, t) \rangle = (2\pi)^{-3} \sum_{i=1}^3 \int_{T_3} f_i(x, t) \phi_i(x, t) dx.$$

Observing from Lemma 2 that ϕ_i is in $L^6(T_3)$ and using Remark 1, we let $t \rightarrow 0$ in (2.52) and conclude that

$$-\nu \sum_{i=1}^3 \langle v_i(x), \phi_i(x) \rangle = (2\pi)^{-3} \sum_{i=1}^3 \int_{T_3} f_i(x) \phi_i(x) dx.$$

This last fact in conjunction with (2.36) establishes (2.47), and the proof of the lemma is complete.

3. Proof of the necessary condition of the theorem. Let $Z \subset B(0, 1)$ be a given closed set of positive Newtonian capacity. Also with no loss in generality, we can suppose that Z is of 3-dimensional Lebesgue measure zero.

From [1, pp. 88–9] we obtain that there exists a non-negative Borel measure μ with the following properties:

$$(3.1) \quad \mu(T_3 - Z) = 0;$$

$$(3.2) \quad \mu(Z) = 1;$$

$$(3.3) \quad f_i'(x) = \int_Z (x_i - y_i) |x - y|^{-3} d\mu(y), \text{ for } x \text{ in } E_3 - Z, \\ \text{is in } L^2 \text{ on compact subsets of } E_3 \text{ for } j = 1, 2, 3.$$

We view μ as defined on T_3 and set

$$(3.4) \quad \hat{\mu}(m) = (2\pi)^{-3} \int_{T_3} e^{-i(m, x)} d\mu(x).$$

Next, with $q_i(x)$ defined by (2.3), we set

$$(3.5) \quad f_i(x) = (2\pi)^{-3} \int_Z q_i(x - y) d\mu(y) \quad \text{for } x \text{ in } T_3 - Z,$$

and observe from (2.8) and (3.3) that

$$(3.6) \quad f_i \text{ is in } L^2(T_3) \quad \text{for } j = 1, 2, 3.$$

Also, from (3.5), we have that

$$(3.7) \quad \hat{f}_i(m) = \hat{q}_i(m) \hat{\mu}(m).$$

Consequently from (2.3), (3.6), and (3.7), we have that

$$(3.8) \quad \sum_{m \neq 0} |\mu^\wedge(m)|^2 |m|^{-2} < +\infty.$$

Next, with $\mathfrak{D}(T_3)$ defined by (2.30), we select a function λ with the following properties:

$$(3.9) \quad \begin{aligned} & \text{(i) } \lambda \text{ is in } \mathfrak{D}(T_3); \\ & \text{(ii) } \lambda \geq 0; \\ & \text{(iii) } \int_{T_3} \lambda(x) dx = 1; \\ & \text{(iv) } \lambda(x) = 0 \text{ for } x \text{ in } B(0, 3/2); \\ & \text{(v) } \lambda(x) = 0 \text{ for } x \text{ in } T_3 - B(0, 2). \end{aligned}$$

Next, we define the Borel measure η on T_3 by

$$(3.10) \quad d\eta(x) = d\mu(x) - \lambda(x) dx.$$

Consequently,

$$(3.11) \quad \eta^\wedge(m) = \mu^\wedge(m) - \lambda^\wedge(m),$$

and we have from (3.2), (3.8), (3.9), and (3.10) that

$$(3.12) \quad \eta^\wedge(0) = 0$$

and

$$(3.13) \quad K_\eta^2 = \sum_{m \neq 0} |\eta^\wedge(m)|^2 |m|^{-2} < +\infty.$$

Next, we observe from (2.14') that

$$(3.14) \quad |u_i^{1^\wedge}(m)| \leq 2 |m|^{-2} \text{ for } m \neq 0.$$

Consequently, we have from (3.13) and (3.14) that

$$(3.15) \quad \sum_{m \neq 0} |u_i^{1^\wedge}(m) \eta^\wedge(m)|^2 |m|^2 \leq 4K_\eta^2 < +\infty.$$

But then from (2.14') and (3.15), we obtain that

$$(3.16) \quad \text{there exists } F_i \text{ in } {}_0W_2^1(T_3) \text{ such that } F_i^\wedge(m) = u_i^{1^\wedge}(m) \eta^\wedge(m).$$

Next, we note from (2.14') and an easy calculation that

$$\sum_{i=1}^3 m_i u_i^{1^\wedge}(m) = 0 \quad \text{for all } m.$$

Consequently, we have from (3.16) and this last fact that

$$(3.17) \quad \mathbf{F} = (F_1, F_2, F_3) \text{ is in } \mathfrak{H}(T_3).$$

Next, we let A designate the completely continuous operator defined on $\mathfrak{H}(T_3)$ in Lemma 6, and we set

$$(3.18) \quad A'\mathbf{v} = \nu^{-1}[A\mathbf{v} + \mathbf{F}] \quad \text{for } \mathbf{v} \text{ in } \mathfrak{H}(T_3).$$

Since $\mathbf{F}$ is in $\mathcal{H}(T_3)$, we conclude from (3.18) that

$$(3.19) \quad A' \text{ is a completely continuous operator on } \mathcal{H}(T_3).$$

Next, let ξ be a real number with $0 \leq \xi \leq 1$ and suppose there is a $\mathbf{v}^\xi$ in $\mathcal{H}(T_3)$ such that

$$(3.20) \quad \mathbf{v}^\xi = \xi A' \mathbf{v}^\xi.$$

Then from (3.18), (3.20), and Lemma 7, we have

$$(3.21) \quad \begin{aligned} [\mathbf{v}^\xi, \mathbf{v}^\xi] &= \xi \nu^{-1} \{ [A \mathbf{v}^\xi, \mathbf{v}^\xi] + [\mathbf{F}, \mathbf{v}^\xi] \} \\ &= \xi \nu^{-1} [\mathbf{F}, \mathbf{v}^\xi]. \end{aligned}$$

But then from (3.15) and (3.16), we have that $||[\mathbf{F}, \mathbf{v}^\xi]|| \leq 2(3)^{1/2} K_\eta [\mathbf{v}^\xi, \mathbf{v}^\xi]^{1/2}$. Since $0 \leq \xi \leq 1$, we conclude from (3.21) that

$$(3.22) \quad [\mathbf{v}^\xi, \mathbf{v}^\xi]^{1/2} \leq \nu^{-1} 2(3)^{1/2} K_\eta.$$

From (3.19), (3.20), (3.22), and the Leray–Schauder principle [2, p. 32], we obtain that there exists a $\mathbf{v}$ in $\mathcal{H}(T_3)$ such that $A' \mathbf{v} = \mathbf{v}$. From (3.18), we consequently conclude

$$(3.23) \quad \text{there exists a } \mathbf{v} \text{ in } \mathcal{H}(T_3) \text{ such that } \nu \mathbf{v} = A \mathbf{v} + \mathbf{F}.$$

Next, we set

$$(3.24) \quad g_j = \sum_{k=1}^3 v_k v_{j x_k} \quad \text{for } j = 1, 2, 3$$

and observe from (2.39) and (2.40) that

$$(3.25) \quad g_j \text{ is in } L^{3/2}(T_3), \quad \text{and} \quad g_j \wedge(0) = 0.$$

Next, we set

$$(3.26) \quad V_j(x) = (2\pi)^{-3} \sum_{k=1}^3 \int_{T_3} u_i^k(x-y) g_k(y) dy,$$

and observe from Lemma 8 that

$$(3.27) \quad \mathbf{V} = (V_1, V_2, V_3) \text{ is in } \mathcal{H}(T_3)$$

and furthermore that

$$(3.28) \quad \nu[\mathbf{V}, \boldsymbol{\phi}] = - \sum_{i=1}^3 (2\pi)^{-3} \int_{T_3} g_i \phi_i dx \quad \text{for } \boldsymbol{\phi} \text{ in } \mathcal{H}(T_3).$$

In (2.41) and (2.42), we showed

$$(3.29) \quad \text{if } w \text{ and } h \text{ are in } L^3(T_3) \cap W_{3/2}^1(T_3), \text{ then } \int_{T_3} w h_{x_k} dx = - \int_{T_3} w_{x_k} h dx.$$

Setting $h = v_i$ and $w = v_k \phi_i$, we conclude in particular from Lemma 2 and (3.29) that

$$-\int_{T_s} v_k \phi_i v_{ixk} dx = \int_{T_s} v_i [v_k \phi_{ixk} + \phi_i v_{kxk}] dx.$$

But then we conclude from this last fact and (3.24) that

$$(3.30) \quad -\int_{T_s} g_i \phi_i dx = \sum_{k=1}^3 \int_{T_s} v_i v_k \phi_{ixk}.$$

Next, we obtain from (2.37), Lemma 6, (3.28) and (3.30) that

$$(3.31) \quad \nu[\mathbf{V}, \phi] = [A\nu, \phi] \quad \text{for } \phi \text{ in } \mathcal{H}(T_3).$$

But then from (3.23) and (3.31), we have that

$$(3.32) \quad \nu[\mathbf{V}, \phi] = [\nu\mathbf{v} - \mathbf{F}, \phi] \quad \text{for } \phi \text{ in } \mathcal{H}(T_3).$$

We conclude from (3.32) that

$$(3.33) \quad \mathbf{v} = \mathbf{V} + \nu^{-1}\mathbf{F}.$$

Next, we define the functions $P(x)$ and $P'(x)$ almost everywhere on T_3 as follows:

$$(3.34) \quad P(x) = (2\pi)^{-3} \sum_{k=1}^3 \int_{T_s} q_k(x-y) g_k(y) dy,$$

$$(3.35) \quad P'(x) = (2\pi)^{-3} \int_{T_s} q_1(x-y) d\eta(y).$$

From (2.8), we have that $q_k(x) = \mathbf{0}(|x|^{-2})$ as $x \rightarrow 0$. We consequently conclude (as we did in Remark 2) from (3.25), (3.34), and [2; Lemma 5, p. 13] that

$$(3.36) \quad P(x) \text{ is in } L^3(T_3).$$

From (3.35), we have that $P' \wedge(m) = q_1 \wedge(m) \eta \wedge(m)$. But then we obtain from (2.3) and (3.13) that $\sum_{m \neq 0} |P' \wedge(m)|^2 < +\infty$, and therefore that

$$(3.37) \quad P'(x) \text{ is in } L^2(T_3).$$

Next, we set

$$\eta(x, t) = \sum_{m \neq 0} \eta \wedge(m) e^{i(m, x) - |m|t},$$

and define $P(x, t)$ and $P'(x, t)$ in a similar manner.

Then it follows from Lemma 4, (3.12), (3.16), and (3.26) that for $t > 0$,

$$(3.38) \quad \nu \Delta V_i(x, t) - P_{xi}(x, t) = g_i(x, t), \quad j = 1, 2, 3,$$

and

$$(3.39) \quad \begin{aligned} \nu \Delta F_1(x, t) - P_{x_1}'(x, t) &= \eta(x, t) \\ \nu \Delta F_2(x, t) - P_{x_2}'(x, t) &= 0 \\ \nu \Delta F_3(x, t) - P_{x_3}'(x, t) &= 0. \end{aligned}$$

Next, we set

$$(3.40) \quad p(x) = P(x) + \nu^{-1}P'(x).$$

Using the well-known fact that

$$\lim_{t \rightarrow 0} \int_{T_3} \psi(x) \eta(x, t) dx = \int_{T_3} \psi(x) d\eta(x) \quad \text{for } \psi \text{ in } \mathfrak{D}(T_3),$$

we conclude from (3.33), (3.38), (3.39), and (3.40) that

$$(3.41) \quad \begin{aligned} \int_{T_3} [\nu_1 \Delta \psi + p \psi_{x_1}] dx &= \int_{T_3} \psi [\nu^{-1} d\eta(x) + g_1 dx] \\ \int_{T_3} [\nu_j \Delta \psi + p \psi_{x_j}] dx &= \int_{T_3} \psi g_j dx, \quad j = 2, 3, \quad \text{for } \psi \text{ in } \mathfrak{D}(T_3). \end{aligned}$$

Next, we set

$$(3.42) \quad w_{jk} = v_j v_k.$$

Then it follows from (3.24) that $g_i = \sum_{k=1}^3 w_{ik} x_k$ and that w_{jk} is in $L^3(T_3) \cap W_{3/2}^1(T_3)$. We consequently conclude from (3.29) that

$$(3.43) \quad \int_{T_3} \psi g_i dx = - \sum_{k=1}^3 \int_{T_3} w_{ik} \psi_{x_k} dx \quad \text{for } \psi \text{ in } \mathfrak{D}(T_3).$$

Now, let $B(y, r)$, $r > 0$, be an open 3-ball such that

$$(3.44) \quad B(y, r) \subset B(0, 3/2),$$

and let

$$(3.45) \quad \zeta \text{ be in } C_0^\infty[B(y, r)].$$

It is clear that there exists a ψ in $\mathfrak{D}(T_3)$ such that

$$(3.46) \quad \begin{aligned} \psi &= \zeta \quad \text{in } B(y, r) \\ \psi &= 0 \quad \text{in } T_3 - B(y, r). \end{aligned}$$

We consequently conclude from (3.10), (3.11), and (3.41) through (3.46) that

$$(3.47) \quad \begin{aligned} \int_{B(y, r)} \left[\nu_1 \Delta \zeta + p \zeta_{x_1} + \sum_{k=1}^3 w_{1k} \zeta_{x_k} \right] dx &= \int_{B(y, r)} \nu^{-1} \zeta d\mu(x), \\ \int_{B(y, r)} \left[\nu_j \Delta \zeta + p \zeta_{x_j} + \sum_{k=1}^3 w_{jk} \zeta_{x_k} \right] dx &= 0, \quad j = 2, 3, \\ &\quad \text{for } \zeta \text{ in } C_0^\infty[B(y, r)]. \end{aligned}$$

Recalling that $Z \subset B(0, 1)$ and that μ has its support on Z , we conclude from (3.36), (3.37), (3.40), (3.42), (3.44), (3.47), and [6, Theorem 3] that

$$(3.48) \quad \begin{aligned} &\text{there exists } \mathbf{v}' \text{ in } C^2[B(0, 3/2) - Z] \text{ and } p' \text{ in } C^1[B(0, 3/2) - Z] \text{ such} \\ &\text{that the following holds: (i) } (\mathbf{v}', p') \text{ satisfies the equations in (1.3) in} \\ &B(0, 3/2) - Z; \text{ (ii) } \mathbf{v}' = \mathbf{v} \text{ and } p' = p \text{ almost everywhere in} \\ &B(0, 3/2) - Z. \end{aligned}$$

From (3.36), (3.37), and (3.40), we have that p is in $L^2(T_3)$, and we know that $\mathbf{v}$ is in $\mathcal{H}(T_3)$. We consequently conclude from (3.48) that

$$(3.49) \quad (\mathbf{v}', p') \text{ meets conditions (1.1), (1.2), and (1.3) in } B(0, 1) - Z.$$

Suppose Z were a removable set for stationary viscous incompressible flows. Then we would obtain from (3.49) that

$$(3.50) \quad v' \text{ and } p' \text{ can be defined on } Z \text{ so that } (\mathbf{v}', p') \text{ meets conditions (1.1), (1.2), and (1.3) in } B(0, 1).$$

But then it is easy to see that (3.50) implies, in particular, the following fact:

$$(3.51) \quad \int_{B(0,1)} \left[v_1' \Delta \zeta + p' \zeta_{x_1} + \sum_{k=1}^3 v_1' v_k' \zeta_{x_k} \right] dx = 0$$

for ζ in $C_0^\infty[B(0, 1)]$.

Now recalling that by assumption, the 3-dimensional Lebesgue measure of Z is zero, we see from (3.48) that $\mathbf{v} = \mathbf{v}'$ and $p = p'$ almost everywhere in $B(0, 1)$. Consequently, using (3.42) and (3.44) and comparing (3.51) with the first equation in (3.47), we see that

$$(3.52) \quad \int_{B(0,1)} \zeta d\mu(x) = 0 \quad \text{for } \zeta \text{ in } C_0^\infty[B(0, 1)].$$

But (3.52) contradicts (3.2). Z , therefore, is not a removable set for stationary viscous incompressible flows, and the necessary condition of the theorem is complete.

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