Symmetric Singular Integral Operators

JOEL D. PINCUS

In this paper we study singular integral operators of the form

$$\mathfrak{L}x(\lambda) = A(\lambda)x(\lambda) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{k}(\lambda)k(\mu)}{\mu - \lambda} x(\mu) d\mu$$

when the coefficients $A(\lambda)$ and $k(\lambda)$ are permitted to be unbounded. We assume only that $A(\lambda)$ defines a self-adjoint multiplication operator, and that $\int_{-\infty}^{\infty} |k(\lambda)|^2 d\lambda < \infty$.

We give necessary and sufficient conditions for $\mathfrak{L}$, with a certain specified domain, to be maximally symmetric; and, when these conditions are satisfied, we compute the deficiency indices of $\mathfrak{L}$, give a complete spectral theory for the minimal self-adjoint dilation of $\mathfrak{L}$, and show how to decide when or if $\mathfrak{L}$ has a self-adjoint restriction to one of its invariant subspaces.

Let

$$A_n(\lambda) = \begin{cases} A(\lambda) & \text{if } |A(\lambda)| \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$k_n(\lambda) = \begin{cases} k(\lambda) & \text{if } |k(\lambda)| \leq n \\ 0 & \text{otherwise} \end{cases}$$

and set

$$L_n x(\lambda) = A_n x(\lambda) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{k}_n(\lambda)k_n(\mu)}{\mu - \lambda} x(\mu) d\mu.$$

L_n is a self-adjoint operator whose spectral theory has been completely elucidated in other papers. [1], [2].

It is known that the weak limit $w\text{-}\lim_{n \rightarrow \infty} (L_n - \ell)^{-1} = R(\ell)$ exists, as a bounded operator on $L_2(-\infty, \infty)$, whenever $\text{Im } \ell \neq 0$. [3].

Let $g_n(\nu, \lambda)$ be the characteristic function of the set of points (ν, λ) such that $A_n(\lambda) - |k_n(\lambda)|^2 < \nu < A_n(\lambda) + |k_n(\lambda)|^2$, $\lambda \in \text{supp. } k_n(\lambda)$. Let

$$E_n(\ell, z) = \exp \left(\frac{1}{2\pi i} \iint g_n(\nu, \mu) \frac{d\nu}{\nu - \ell} \frac{d\mu}{\mu - z} \right).$$

Theorem 0.

$$\left((I_n - \ell)^{-1} \left[\frac{k_n(t)}{t - w} \right], \left[\frac{k_n(t)}{t - \bar{z}} \right] \right) = \pi i [1 - E_n(\ell, z) \bar{E}_n(\bar{\ell}, \bar{z})] \frac{1}{z - w}. \quad [1], [2].$$

In general, $R(\ell) = \lim_{n \rightarrow \infty} (R_n - \ell)^{-1}$ exists in the weak operator topology for every non-real number ℓ , and $\lim_{\lambda \rightarrow \pm \infty} i\lambda R(i\lambda) = 1$ in the weak operator topology. [3]. Moreover,

$$\left(R(\ell) \left[\frac{k(t)}{t - w} \right], \left[\frac{k(t)}{t - \bar{z}} \right] \right) = \pi i [1 - E(\ell, w) \bar{E}(\bar{\ell}, \bar{z})] \frac{1}{z - w}$$

where

$$E(\ell, z) = \lim_{n \rightarrow \infty} E_n(\ell, z) = \exp \left(\frac{1}{2\pi i} \iint g(\nu, \mu) \frac{d\nu}{\nu - \ell} \frac{d\mu}{\mu - z} \right)$$

with $g(\nu, \lambda)$ being the characteristic function of the set of points (ν, λ) for which $A(\lambda) - |k(\lambda)|^2 < \nu < A(\lambda) + |k(\lambda)|^2$.

In Pincus-Rovnyak [3] necessary and sufficient conditions were given so that $R(\ell)$ was the resolvent of a self-adjoint transformation.

The present paper is concerned with the interesting case in which it is false that $R(\ell)$ is the resolvent of a self-adjoint transformation.

We will see that $R(\ell)$ is always a generalized resolvent corresponding to a symmetric operator, L , which we will identify with $\mathfrak{L}$ above on a dense domain.

It is easy to see that $E_n(z_1, z_2)$, and therefore, $E(z_1, z_2)$ satisfies the hypothesis of the following theorem:¹

Theorem 1. *Let $\mathfrak{E}(z_1, z_2)$ be an operator-valued analytic function defined on a Hilbert space h^2 for $z_1 \neq \bar{z}_1$, and $z_2 \neq \bar{z}_2$ and having the properties*

- 1) $\mathfrak{E}(z_1, z_2) \mathfrak{E}^*(\bar{z}_1, \bar{z}_2) = 1$.
- 2) *The function*

$$k(w_1, w_2; z_1, z_2) \equiv i \frac{[\mathfrak{E}(z_1, z_2) \mathfrak{E}^*(w_1, z_2) - \mathfrak{E}(z_1, \bar{w}_2) \mathfrak{E}^*(w_1, w_2)]}{[z_1 - \bar{w}_1][z_2 - \bar{w}_2]}$$

is a positive definite matrix function.

- 3) $\|1 - \mathfrak{E}(z_1, z_2)\|_h \leq C |\operatorname{Im} z_1|^{-1} |\operatorname{Im} z_2|^{-1}$
for some positive constant C . Then,

$$\mathfrak{E}(z_1, z_2) = 1 + \frac{1}{i} k(H_1 - z_1)^{-1} (H_2 - z_2)^{-1} k^*,$$

where H_1 and H_2 are self-adjoint transformations on a Hilbert space $\hat{\mathfrak{H}}$, having $k(w_1, w_2; z_1, z_2)$ as reproducing kernel, k is a bounded map of $\hat{\mathfrak{H}}$ into h , and

$$\begin{aligned} & (H_1 - z_1)^{-1} (H_2 - z_2)^{-1} - (H_2 - z_2)^{-1} (H_1 - z_1)^{-1} \\ &= (H_2 - z_2)^{-1} (H_1 - z_1)^{-1} \frac{k^* k}{i} (H_1 - z_1)^{-1} (H_2 - z_2)^{-1}. \end{aligned}$$

We will give a proof of this old result in Appendix A.

According to Theorem 1 there exists a Hilbert space $\hat{\mathcal{H}}$ and self-adjoint operators H_1 and H_2 on $\hat{\mathcal{H}}$ for which

$$E(z_1, z_2) = 1 + \frac{1}{\pi i} k(H_1 - z_1)^{-1}(H_2 - z_2)^{-1}k^*$$

where k^* is an operator of one dimensional range defined on a one-dimensional coefficient space. We can, without too much ambiguity, refer to k^* as a vector in $\hat{\mathcal{H}}$.

We will exhibit a subspace H of $\hat{\mathcal{H}}$ which we will be able to identify with $L_2[\text{supp } k(\lambda)]$. Then we will show that $R(\ell) = P(H_1 - \ell)^{-1}$ where P is the projection onto H . We then define $L = PH_1$.

After showing this, and developing some associated facts, we will begin considering a certain scattering problem: $H_2 \rightarrow H_2 - \mathfrak{D}$ where $\mathfrak{D} = (2/\pi)(H_1 - i)^{-1}k^*k(H_1 + i)^{-1}$.

Our fundamental observation is that the wave operator $W_- = s\text{-}\lim_{t \rightarrow -\infty} \exp[it(H_2 - \mathfrak{D})] \exp(-itH_2)P$ and the Cayley transform, W , of L are related by a single equation: $W = W_-f(PH_2P)$ where the function f is known explicitly.

This fact makes it possible to apply the known theory of Intertwining Isometries developed in Carey-Pincus [11] to show that $\hat{\mathcal{H}}$ is isometrically isomorphic in a natural way to the minimal unitary dilation space of W .

In fact we show that H_1 is the minimal self-adjoint power dilation of L .

Our basic conclusions about the deficiency index of L then follows from known facts about the deficiency of intertwining isometries.

The present results are to be considered on extension of the determining function theory developed in [2], [5], [6].

For the orientation of the reader, we sketch the context in which the present investigation is set.

Let u and v be bounded self-adjoint operators on a separable Hilbert space $\mathcal{H}$ such that $vu - uv = (1/\pi i)C$ where C is positive and trace class. Assume that $\mathcal{H}$ is the smallest reducing subspace of both u and v which contains the range of the commutator C , i.e. assume $S \equiv u + iv$ is completely non-normal.

Introduce an auxiliary Hilbert space h so that $k^*k = C$. $k : \mathcal{H} \rightarrow h$ and $k^* : h \rightarrow \mathcal{H}$, and define the determining function of the pair $\{u, v\}$ by setting on h

$$\mathcal{E}(\ell, z) = 1_h + \frac{1}{\pi i} k(v - \ell)^{-1}(u - z)^{-1}k^*.$$

We list some of the known properties of this function:

$$\text{a) } \det \mathcal{E}(\ell, z) = \exp \left(\frac{1}{2\pi i} \iint G(\nu, \mu) \frac{d\nu}{\nu - \ell} \frac{d\mu}{\mu - z} \right) \quad [5]$$

for some function $G(\nu, \mu)$ called the principal function of S . Here $G(\nu, \mu) \geq 0$ a.e. and $\iint G(\nu, \mu) d\nu d\mu = 2 \text{ Tr } C$

b) $\text{Trace } [W(u, v), X(u, v)] = (i/2\pi) \iint \{W, X\} G(\nu, \mu) d\nu d\mu$ when W and X are polynomials in u and v , and $\{W, X\}$ is the Jacobian determinant,

$$\begin{vmatrix} \frac{\partial W}{\partial \mu} & \frac{\partial X}{\partial \mu} \\ \frac{\partial W}{\partial \nu} & \frac{\partial X}{\partial \nu} \end{vmatrix} \quad [7].$$

c) $G(\nu, \mu)$ is equal a.e. in a neighborhood of $z \notin \sigma_{\text{ess}}(S)$ to the index of $S-z$. [8].

d) If C has one dimensional range and u has finite spectral multiplicity then the essential boundary of the support of any almost everywhere determined version of $G(\nu, \mu)$ is $\sigma_{\text{ess}}(S)$. [8]

e) The essential support of any almost everywhere determined version of $G(\nu, \mu)$ is $\sigma(S)$. [9]

f) If C has one dimensional range, then $G(\nu, \mu)$ is a complete unitary invariant for S . The spectral multiplicity of both u and v is determined in an explicit way in this case from the principal function.

We remark that the determining function can be defined without the assumption that C is positive and that properties a , b , and c remain true. It follows from b above that the principal function G is invariant under trace class perturbation of S .

When C has one dimensional range and u has simple spectral multiplicity, v is always ([2], [10]) unitarily equivalent to a singular integral operator of the form

$$\alpha(\lambda)x(\lambda) + \frac{1}{\pi i} \int_{\sigma(u)} \frac{\overline{\beta(\lambda)}\beta(\mu)}{\mu - \lambda} x(\mu) d\mu$$

with real essentially bounded $\alpha(\lambda)$ and possibly complex $\beta(\lambda)$ in $L^\infty(\sigma(u))$.

In this representation u is multiplication by λ . Explicit calculation shows in this case that the principal function of the pair $\{u, v\}$ is the characteristic function of the set of points (ν, λ) for which $\alpha(\lambda) - |\beta(\lambda)|^2 < \nu < \alpha(\lambda) + |\beta(\lambda)|^2$, $\lambda \in \sigma(v)$.

Let $\mathfrak{A}$ be the C^* algebra generated by u and v . Result b) above shows in what sense $G(\nu, \mu) d\nu d\mu$ is an invariant for $\mathfrak{A}$, and it provides us with a change of variables formula for the principal function.

The present paper utilizes this correspondence between a change of variables for the principal function and a change of generators in $\mathfrak{A}$, but because we deal with unbounded operators the underlying non-commutative Banach algebra corresponding to the principal function operates in a dilation space rather than in the original Hilbert space.

We now proceed with our study of the singular integral operators.

Consider the subspace H of $\hat{\mathcal{H}}$ spanned by vectors of the form $\sum_i a_i (H_2 - z_i)^{-1} k^*$.

Lemma 1. H_2 restricted to H has absolutely continuous spectral measure.

Proof:

$$\frac{1}{2} \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} [E(\zeta + i\eta, z) - E(\zeta - i\eta, z)] d\zeta = k(H_2 - z)^{-1}k^*.$$

But we can also verify that

$$\frac{1}{2} \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} [E(\zeta + i\eta, z) - E(\zeta - i\eta, z)] d\zeta = \frac{1}{2} \int_{-\infty}^{\infty} \frac{|k(\mu)|^2}{\mu - z} d\mu$$

where $2|k(\mu)|^2 = \int_{-\infty}^{\infty} g(\nu, \mu) d\nu$, and $k(H_2 - z)^{-1}k^* = \int_{-\infty}^{\infty} d||E_{\lambda}k^*||^2/(\lambda - z)$. Thus, $(d/d\lambda) ||E_{\lambda}k^*||^2 = (d/d\lambda) \int_{-\infty}^{\lambda} |k(\mu)|^2 d\mu$.

We can conclude that in the spectral representation of H_2 restricted to H , E_{λ} is represented by multiplication by $\chi(-\infty, \lambda)$, the characteristic function of $(-\infty, \lambda)$ and $(H_2 - z)^{-1}k^*$ is represented by $(\lambda - z)^{-1}\bar{k}(\lambda)$.

Let, henceforth, $N(k^*)$ denote the set of points λ where $k(\lambda) = 0$. The lemma which we have just proved implies that H is isometrically isomorphic to $L^2(N^c(k^*))$. We will denote the isometry by $\mathcal{J}$. So $\mathcal{J} : H \rightarrow L^2(N^c(k^*))$ and $\mathcal{J}$ is the diagonalizing isometry for H_2 on H .

Let us also henceforth denote the projection of $\mathfrak{H}$ onto H by P .

Lemma 2. *H is the absolutely continuous subspace of H_2 .*

Proof: The following identity follows at once from a direct computation, using the definition of the determining function, and the commutator identity relating $(H_1 - \ell)^{-1}$ and $(H_2 - z)^{-1}$:

$$E(\ell, z)\bar{E}(\ell, z) = 1 + \frac{1}{\pi} \left[\frac{\ell - \bar{\ell}}{i} \right] k(H_1 - \ell)^{-1}(H_2 - z)^{-1}(H_1 - \bar{\ell})^{-1}k^*.$$

Thus,

$$\begin{aligned} \frac{\operatorname{Im} E(\ell, z)\bar{E}(\ell, \bar{z})}{\operatorname{Im} \ell} &= \frac{E(\ell, z)\bar{E}(\ell, \bar{z}) - \bar{E}(\ell, z)E(\ell, \bar{z})}{[\ell - \bar{\ell}]} \\ &= \frac{1}{\pi i} \{k(H_1 - \ell)^{-1}[(H_2 - z)^{-1} - (H_2 - \bar{z})^{-1}](H_1 - \bar{\ell})^{-1}k^*\} \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu_t(t)}{(t - x)^2 + y^2} \end{aligned}$$

for a unique positive measure $d\mu_t(t)$ which is given explicitly by the relation $(1/2) d\mu_t(t) = d||E_t(H_1 - \bar{\ell})^{-1}k^*||^2$, E_t being the H_2 resolution of the identity.

Assume for the moment, the truth of the following proposition:

Proposition. *The absolutely continuous part of $d||E_t(H_1 - \bar{\ell})^{-1}k^*||^2$ is $d||E_t P(H_1 - \bar{\ell})^{-1}k^*||^2$.*

We will show that any measure of the form $d||E_t(1 - P)x||^2$ with x arbitrary is purely singular.

Because (by the fundamental identity for determining functions $(H_1 - \ell)^{-1} \cdot (H_2 - z)^{-1}k^* = (H_2 - z)^{-1}(H_1 - \ell)^{-1}k^*E(\ell, z)$) the smallest invariant subspace for both H_1 and H_2 which contains the range of the commutator k^*k is generated by finite sums of the form $\sum_{i,j} a_{ij}(H_2 - \ell_j)^{-1}(H_1 - \bar{q}_i)^{-1}k^*$, we can restrict our attention to vectors x having the form $x = (H_1 - \bar{\ell})^{-1}(H_2 - q)^{-1}k^*$.

Thus,

$$\begin{aligned} & \|E_\Delta(1 - P)(H_1 - l_i)^{-1}(H_2 - q_i)^{-1}k^*\| \\ &= \|E_\Delta(1 - P)(H_2 - q_i)^{-1}(H_1 - l_i)^{-1}k^*\| \|E(l_i, q_i)\| \\ &\leq \frac{|E(l_i, q_i)|}{|\operatorname{Im} q_i|} \|E_\Delta(1 - P)(H_1 - l_i)^{-1}k^*\|. \end{aligned}$$

where Δ is any Borel set. Let $\sum_i$ be the support set of the measure $\|E_\Delta(1 - P)(H_1 - l_i)^{-1}k^*\|$. By the proposition $\sum_i$ is a Lebesgue null set. Thus if the intersection of Δ and $\sum_i$ is empty the inequality above shows that $\|E_\Delta(1 - P)(H_1 - l_i)^{-1}(H_2 - q_i)^{-1}k^*\| = 0$. Thus the support of $\|E_\Delta(1 - P) \sum_{i,j} a_{ij}(H_1 - l_i)^{-1}(H_2 - q_j)^{-1}k^*\|$ is contained in $\cup \sum_i$, which is a null set.

We turn to the proof of the proposition. Let us simply make a direct calculation of the Radon-Nikodym derivative, $(d/dt)\mu_t(t)$, of the measure $d\mu_t(t)$.

$$\lim_{y \downarrow 0} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu_t(t)}{(t - x)^2 + y^2} = \frac{d}{dt} \mu_t(t).$$

But it is easy to show that [3]

$$\begin{aligned} & \frac{E(\ell, t + i0)\bar{E}(\ell, t - i0) - E(\ell, t - i0)\bar{E}(\ell, t + i0)}{\ell - \bar{\ell}} \\ &= \begin{cases} \frac{1}{2} |E(\ell, t + i0) - E(\ell, t - i0)|^2 |k(t)|^{-2} & t \notin N(k^*) \\ 0 & t \in N(k). \end{cases} \end{aligned}$$

Thus, if $H(\ell, t) \equiv E(\ell, t + i0) - E(\ell, t - i0)$,

$$\frac{d}{dt} \|E_t(H_1 - \bar{\ell})^{-1}k^*\|^2 = \begin{cases} \frac{1}{4} |H(\ell, t)|^2 |k(t)|^{-2} & t \notin N(k^*) \\ 0 & t \in N(k^*) \end{cases}$$

But $gR(\bar{\ell})k^* = gP(H_1 - \bar{\ell})^{-1}k^* = (1/2)(\bar{H}(\ell, t)/k(t))$, by Theorem 2 below. Thus,

$$\frac{d}{dt} \|E_tP(H_1 - \bar{\ell})^{-1}k^*\|^2 = \frac{d}{dt} \int_{-\infty}^t \frac{1}{4} \frac{|H(\ell, \lambda)|^2}{|k(\lambda)|^2} d\lambda.$$

Hence,

$$\frac{d}{dt} \|E_t(H_1 - \bar{\ell})^{-1}k^*\|^2 = \frac{d}{dt} \|E_tP(H_1 - \bar{\ell})^{-1}k^*\|^2$$

and $\|E_t(1 - P)(H_1 - \bar{\ell})^{-1}k^*\|^2$ is purely singular.

Theorem 2. $R(\ell) = \mathcal{G}P(H_1 - \ell)^{-1}\mathcal{G}^{-1}$, and thus $R(\ell)$ is a generalized resolvent—in the sense of M. A. Naimark—corresponding to the symmetric operator $L = \mathcal{G}PH_1\mathcal{G}^{-1}$.

We will see at the very end of the paper that L is the singular integral operator $\mathfrak{L}$ on an appropriately chosen domain.

Proof. By construction

$$\left(R(\ell) \frac{\bar{k}(t)}{t-w}, \frac{\bar{k}(t)}{t-\bar{z}}\right) = \pi i \frac{[1 - E(\ell, w)\bar{E}(\bar{\ell}, \bar{z})]}{z-w}.$$

But an elementary relation satisfied by determining functions is: (The proof uses only the basic commutator identity)

$$\begin{aligned} (z_1 - \bar{w}_1)(z_2 - \bar{w}_2)k(H_2 - z_2)^{-1}(H_1 - z_1)^{-1}(H_1 - \bar{w}_1)^{-1}(H_2 - \bar{w}_2)^{-1}k^* \\ = iE(\bar{w}_1, z_2)^{-1}E(\bar{w}_1, \bar{w}_2) - iE(z_1, z_2)^{-1}E(z_1, \bar{w}_2). \end{aligned}$$

Thus, since $E(z_1, z_2) \rightarrow 1$ as $|\operatorname{Im} z_2| \rightarrow \infty$ we have

$$\begin{aligned} k(H_2 - z)^{-1}(H_1 - \ell)^{-1}(H_2 - w)^{-1}k^* \\ = ((H_1 - \ell)^{-1}(H_2 - w)^{-1}k^*, (H_2 - \bar{z})^{-1}k^*) \\ = (P(H_1 - \ell)^{-1}(H_2 - w)^{-1}k^*, (H_2 - \bar{z})^{-1}k^*) \\ = \pi i \frac{[1 - E(\ell, w)\bar{E}(\bar{\ell}, \bar{z})]}{z-w} = \left(R(\ell) \frac{\bar{k}(t)}{t-w}, \frac{\bar{k}(t)}{t-\bar{z}}\right). \end{aligned}$$

Since sums of the form $\sum_i a_i(H_2 - w_i)^{-1}k^*$ are dense in H , the result follows.

Lemma 3. Let H_k be the smallest invariant subspace of H_2 that contains $(H_1 - i)^{-1}k^*$. Then $PH_k = H$.

Proof: Since $(H_2 - z)^{-1}(H_1 - i)^{-1}k^* = (H_1 - i)^{-1}(H_2 - z)^{-1}k^*E^{-1}(i, z)$, H_k is the image of H under $(H_1 - i)^{-1}$.

We will show that vectors of the form $P(H_1 - i)^{-1} \sum a_i(H_2 - z_i)^{-1}k^*$ are dense in H . As we have already seen, for $\lambda \notin N(k)$,

$$\mathcal{G}P(H_1 - i)^{-1}(H_2 - z)^{-1}k^* = \frac{1}{2} \frac{\tilde{H}(-i, \lambda)}{k(\lambda)} \frac{1}{\lambda - z} E(i, z).$$

Here $(1/2)\tilde{H}(-i, \lambda)/k(\lambda) = R(i)k^*$. Furthermore, we claim that the function $(1/2)\tilde{H}(-i, \lambda)/k(\lambda)$ does not vanish on the support of $k(\lambda)$. For [2]

$$\frac{E(i, \lambda + i0)}{E(i, \lambda - i0)} = \frac{A(\lambda) - i + |k(\lambda)|^2}{A(\lambda) - i - |k(\lambda)|^2}$$

and $E(i, \lambda + i0) = E(i, \lambda - i0)$ if and only if $|k(\lambda)| = 0$. Let us denote the support of a measurable function $G(\lambda)$ by $S(G)$.

Lemma 4. If $\int_{S(G)} |G(\lambda)|^2 d\lambda < \infty$, then sums of the form $G(\lambda) \sum_{i=1}^r A_k/(\lambda - z_k)$ are dense in $L_2(S(G))$.

Proof: Let $g(\lambda)$ be a continuous function defined for all λ . S. N. Bernstein has shown that any continuous function can be approximated by sums of the form $B_1 + \sum_{k=1}^N B_k/(\lambda - z_k)$, even uniformly.

Since $(\lambda - z_1)g(\lambda)$ is continuous, we must be able to find, given $\epsilon > 0$, constants $\{B_k\}$, $\{z_k\}$ such that

$$\left| (\lambda - z_1)g(\lambda) - B_1 - \sum \frac{B_k}{\lambda - z_k} \right| < \epsilon$$

Thus

$$\left| g(\lambda) - \sum \frac{C_k}{\lambda - z_k} \right| < \left| \frac{\epsilon}{\lambda - z_1} \right|$$

and

$$\int_{S(G)} \left| g(\lambda) - \sum \frac{C_k}{\lambda - z_k} \right|^2 |G(\lambda)|^2 d\lambda < \epsilon \int_{-\infty}^{\infty} \frac{|G(\lambda)|^2}{|\lambda - z_1|^2} d\lambda.$$

Thus sums of the form $\sum C_k/(\lambda - z_k)$ are dense in $L_2(S(G), |G(\lambda)|^2 d\lambda)$. But $f(\lambda)/G(\lambda) \in L_2(S(G), |G(\lambda)|^2 d\lambda)$ if and only if $f(\lambda) \in L_2(S(G), d\lambda)$. Thus

$$\int_N \left| \frac{f(\lambda)}{G(\lambda)} - \sum \frac{C_k}{\lambda - z_k} \right|^2 |G(\lambda)|^2 d\lambda = \int \left| f(\lambda) - G(\lambda) \sum \frac{C_k}{\lambda - z_k} \right|^2 d\lambda$$

can be made arbitrarily small whenever $f(\lambda)$ is in $L_2(S(G), d\lambda)$. Q.E.D.

We have therefore shown that sums of the form $(H(-i, \lambda)/k(\lambda)) \sum a_k/(\lambda - z_k)$ are dense in $L_2(S(k), d\lambda)$.

Thus, we have proved that $PH_k = H$. Let

$$h_\ell(t) = \begin{cases} \frac{\text{Im } \ell}{\pi} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x - \ell|^2} dx & \text{Im } \ell > 0 \\ \frac{|\text{Im } \ell|}{\pi} \int_{-\infty}^{\infty} \frac{1 - g(t, x)}{|x - \ell|^2} dx & \text{Im } \ell < 0 \end{cases}$$

Then $h_i(t) = (1/\pi) \tan^{-1} [A(t) + |k(t)|^2] - (1/\pi) \tan^{-1} [A(t) - |k(t)|^2]$ and

$$\exp \left(- \int_{-\infty}^{\infty} h_i(t) \frac{dt}{t - z} \right) = 1 - \int_{-\infty}^{\infty} \frac{d\mu_{-i}(t)}{t - z}.$$

Theorem 3. Let $M_\ell = \{g|(L - \ell)f = g, \text{ for some } f \in H\}$. Then, $d\mu_{-i}(\cdot)$ is absolutely continuous if and only if $M_i = H$.

This theorem asserts that L is maximal symmetric with deficiency indices of the form $(0, n)$, for some n if and only if $d\mu_{-i}(\cdot)$ is absolutely continuous.

Proof. All generalized resolvents of L coincide on M_ℓ . Indeed if $g \in M_\ell$, then $R(\ell)g = R(\ell)(L - \ell)f = P(H_1 - \ell)^{-1}(H_1 - \ell)f = Pf = f = (L - \ell)^{-1}g$ because H_1 is a self adjoint extension of L .

By definition L is maximal symmetric if and only if either $M_i = H$ or $M_{-i} = H$. Suppose that we consider the first case, $M_i = H$. Then $R(\ell)g = (L - \ell)^{-1}g = (H_1 - \ell)^{-1}g$ for $\text{Im } \ell > 0$. Thus, for $\text{Im } \ell > 0$

$$\begin{aligned} \left\| R(\ell) \frac{k(\lambda)}{\lambda - z} \right\|^2 &= ((H_1 - \ell)^{-1}(H_2 - z)^{-1}k^*, (H_1 - \ell)^{-1}(H_2 - z)^{-1}k^*) \\ &= \left(\frac{(H_1 - \ell)^{-1} - (H_1 - \bar{\ell})^{-1}}{\ell - \bar{\ell}} (H_2 - z)^{-1}k^*, (H_2 - z)^{-1}k^* \right) \\ &= \frac{\pi i}{(\ell - \bar{\ell})(z - \bar{z})} [E(\ell, z)\bar{E}(\bar{\ell}, z) - E(\bar{\ell}, z)\bar{E}(\ell, z)]. \end{aligned}$$

But

$$\left\| R(\ell) \frac{k(t)}{t - z} \right\|^2 = \frac{1}{4} E(\ell, z)\bar{E}(\bar{\ell}, z) \int_{-\infty}^{\infty} \left| \frac{H(\bar{\ell}, t)}{k(t)} \right|^2 \frac{dt}{|t - z|^2}.$$

Equality can hold only if

$$\frac{1}{2\pi} \int \left| \frac{H(\ell, t)}{k(t)} \right|^2 \frac{dt}{|t - z|^2} = \frac{2i}{(\ell - \bar{\ell})(z - \bar{z})} [E(\ell, z)\bar{E}(\bar{\ell}, z) - E(\bar{\ell}, z)\bar{E}(\ell, z)].$$

But

$$E(-i, z)\bar{E}(-i, \bar{z}) = 1 - \frac{1}{2\pi} \int \left| \frac{H(-i, t)}{k(t)} \right|^2 \frac{dt}{t - z}$$

if and only if $d\mu_{-i}$ is absolutely continuous since

$$E(\bar{\ell}, z)\bar{E}(\bar{\ell}, \bar{z}) = \exp \left(- \int_{-\infty}^{\infty} h_{\ell}(t) \frac{dt}{t - z} \right) = 1 - \int \frac{d\mu_{\bar{\ell}}(t)}{t - z}.$$

Q.E.D.

Consider the operator $\tilde{W} = (H_1 + i)(H_1 - i)^{-1}$. We claim that $W \equiv P\tilde{W}P = (L + i)(L - i)^{-1}$ if $f \in R(L - i)$. The proof is simple. If $y \in D(L)$ then $(L - i)y = x = (H_1 - i)y$ because H_1 extends L . Thus, $(H_1 - i)^{-1}x = (L - i)^{-1}x = y$. Hence, $P(H_1 + i)(H_1 - i)^{-1}x = P(H_1 + i)(L - i)^{-1}x = (L + i)(L - i)^{-1}x$.

Let us further observe that $Wf = [1 + 2i R(i)]f$ if $f \in R(L - i)$. This follows at once from the fact that if $f = (L - i)g$, then $R(i)f = P(H_1 - i)^{-1}(L - i)g = P(H_1 - i)^{-1}(H_1 - i)g = Pg = g$.

Thus, $1 + 2i R(i)$ coincides with the Cayley transform of L on the domain of the Cayley transform of L , $R(L - i)$.

Consider now the following perturbation problem: $H_2 \rightarrow H_2 - \mathfrak{D}$, where $\mathfrak{D} = (2/\pi)(H_1 - i)^{-1}k^*k(H_1 + i)^{-1}$. More precisely, consider the operator H_2 restricted to its smallest invariant subspace which contains the vector $(H_1 - i)^{-1}k^*, \mathfrak{H}_k$, and consider the perturbation problem $H_2 \rightarrow H_2 - \mathfrak{D}$ in $\mathfrak{H}_k$.

Let W_- denote the wave operator corresponding to this perturbation problem. That is, let $W_- = s\text{-}\lim_{t \rightarrow -\infty} \exp [it(H_2 - \mathfrak{D})] \exp (-itH_2) P$.

Note that the initial space of W_- is $\mathfrak{H}\text{Cac}(H_2) = H$ with corresponding projection P , and that the final space of W_- is $\mathfrak{H}\text{Cac}(H_2 - \mathfrak{D})$.

Theorem 4. *If $g \in R(L - i)$, then*

$$\mathcal{G}W_-^*Wg(\lambda) = \frac{A(\lambda) + i + |k(\lambda)|^2}{A(\lambda) - i + |k(\lambda)|^2} \mathcal{G}g(\lambda).$$

Proof: Any g in $R(L - i)$ can be expressed as an infinite sum $g = \sum a_i(H_2 - z_i)^{-1}k^*$ with some choice of constants $\{a_i\}$ and $\{z_i\}$.

We will now evaluate W_-^*Wg .

$$Wg = P[1 + 2i(H_1 - i)^{-1}]g = \sum a_i P[1 + 2i(H_1 - i)^{-1}](H_2 - z_i)^{-1}k^*.$$

But

$$(H_1 - \ell)^{-1}(H_2 - z)^{-1}k^* = (H_2 - z)^{-1}(H_1 - \ell)^{-1}k^*E(\ell, z).$$

Hence,

$$Wg = \sum a_i [(H_2 - z_i)^{-1}k^* + 2i(H_2 - z_i)^{-1}R(i)k^*E(i, z_i)].$$

Now, according to [11, section 3] if $f \in \mathfrak{H}\text{Cac}(H_2)$, then

$$\begin{aligned} W_-^*f(\lambda) &= \mathcal{G}W_-^*\mathcal{G}^{-1}f(\lambda) \\ &= f(\lambda) + \frac{1}{\pi} \frac{R(i)k^*(\lambda)}{\text{Det}(\lambda)} ((H_2 - (\lambda - i0))^{-1}f, (H_1 - i)^{-1}k^*), \end{aligned}$$

where

$$\text{Det}(\lambda) = \det(1 + \mathfrak{D}(H_2 - (\lambda - i0))^{-1}).$$

Here $f(\lambda) = \mathcal{G}f$ represents f in $L^2(-\infty, \infty)$.

In order to evaluate the scalar product which occurs above, we note the following:

If $g \in R(L - i)$, then $(H_1 - i)^{-1}g = (L - i)^{-1}g$. For, if $x \in R(L - i)$, then there exists y such that $Py = x$ and $(L - i)y = (H_1 - i)y = x$, so that $(H_1 - i)^{-1}x = (L - i)^{-1}x = y = R(i)x$. Thus, $[1 + 2i(H_1 - i)^{-1}]g = P[1 + 2i(H_1 - i)^{-1}]g$. Hence,

$$\begin{aligned} &((H_2 - q)^{-1}Wg, (H_1 - i)^{-1}k^*) \\ &= \sum_i a_i \{((H_2 - q)^{-1}(H_2 - z_i)^{-1}k^*, (H_1 - i)^{-1}k^*) \\ &\quad + 2i((H_2 - q)^{-1}R(i)g, (H_1 - i)^{-1}k^*)\} \\ &= \sum_i a_i \{((H_2 - q)^{-1}(H_2 - z_i)^{-1}k^*, (H_1 - i)^{-1}k^*) \\ &\quad + 2i((H_2 - q)^{-1}(H_2 - z_i)^{-1}(H_1 - i)^{-1}k^*, (H_1 - i)^{-1}k^*)E(i, z_i)\} \end{aligned}$$

Now we see immediately that

$$((H_2 - z)^{-1}k^*, (H_1 - \ell)^{-1}k^*) = k(H_1 - \ell)^{-1}(H_2 - z)^{-1}k^* = \pi i[E(\ell, z) - 1].$$

Hence,

$$\begin{aligned} & ((H_2 - (\lambda - i0))^{-1}(H_2 - z_i)^{-1}k^*, (H_1 - i)^{-1}k^*) \\ & = \pi i(\lambda - z_i)^{-1}[E(-i, \lambda - i0) - E(-i, z_i)]. \end{aligned}$$

But general identities for determining functions give us at once that

$$\begin{aligned} & \frac{1}{\pi} ((H_2 - (\lambda - i0))^{-1}(H_2 - z_i)^{-1}(H_1 - i)^{-1}k^*, (H_1 - i)^{-1}k^*) \\ & = \frac{E(-i, z_i)\bar{E}(-i, \bar{z}_i) - E(-i, \lambda - i0)\bar{E}(-i, \lambda + i0)}{\lambda - z} \end{aligned}$$

Thus,

$$\begin{aligned} W_-^* W g(\lambda) &= \sum a_i \left\{ W_-^* \frac{k^*(\lambda)}{\lambda - z_i} \right. \\ &+ \frac{i}{\text{Det}(\lambda)} \frac{\bar{H}(-i, \lambda)}{k(\lambda)} \left(\frac{E(-i, \lambda - i0) - E(-i, z_i)}{\lambda - z_i} \right) \\ &+ i \frac{E(i, z_i)}{\text{Det}(\lambda)} \frac{\bar{H}(-i, \lambda)}{k(\lambda)} \frac{E(-i, z_i)\bar{E}(-i, \bar{z}_i)}{\lambda - z_i} \Big\} \\ &= \sum a_i \left\{ \frac{k^*(\lambda)}{\lambda - z_i} + \frac{i}{\text{Det}(\lambda)} \frac{\bar{H}(-i, \lambda)}{k(\lambda)} \frac{E(-i, \lambda - i0)}{\lambda - z_i} \right\} \end{aligned}$$

But since $\bar{E}(\ell, z) = E^{-1}(\bar{\ell}, \bar{z})$, we can see that, because $E(\ell, z)\bar{E}(\ell, \bar{z}) = 1 + 1/\pi[(\ell - \bar{\ell})/i]k(H_1 - \ell)^{-1}(H_2 - z)^{-1}(H_1 - \bar{\ell})^{-1}k^*$, we have $1/\text{Det}(\lambda) = E(i, \lambda - i0)E^{-1}(-i, \lambda - i0)$. Thus using also that $H(\ell, \lambda) \equiv E(\ell, \lambda + i0) - E(\ell, \lambda - i0)$, we get

$$\begin{aligned} & \frac{k^*(\lambda)}{\lambda - z_i} + \frac{i}{\text{Det}(\lambda)} \frac{\bar{H}(-i, \lambda)}{k(\lambda)} \frac{E(-i, \lambda - i0)}{\lambda - z_i} \\ &= \frac{1}{\lambda - z_i} k^*(\lambda) + \frac{i}{\lambda - z_i} \frac{1}{k(\lambda)} (1 - E(i, \lambda - i0)E^{-1}(i, \lambda + i0)). \end{aligned}$$

But the fundamental property of the determining function is that it satisfies a Riemann–Hilbert barrier relation [2]. Thus,

$$E(i, \lambda - i0)E^{-1}(i, \lambda + i0) = \frac{A(\lambda) - i - |k(\lambda)|^2}{A(\lambda) - i + |k(\lambda)|^2},$$

and

$$\begin{aligned} & \frac{k^*(\lambda)}{\lambda - z_i} + \frac{i}{\text{Det}(\lambda)} \frac{\bar{H}(-i, \lambda)}{k(\lambda)} \frac{E(-i, \lambda - i0)}{\lambda - z_i} \\ &= \frac{k^*(\lambda)}{\lambda - z_i} + \frac{i}{\lambda - z_i} \frac{1}{k(\lambda)} \left(\frac{A(\lambda) - i + |k(\lambda)|^2 - A(\lambda) + i + |k(\lambda)|^2}{A(\lambda) - i + |k(\lambda)|^2} \right) \\ &= \frac{k^*(\lambda)}{\lambda - z_i} \frac{A(\lambda) + i + |k(\lambda)|^2}{A(\lambda) - i + |k(\lambda)|^2}. \end{aligned}$$

Hence, we have shown that if $g \in R(L - i)$, then $W_-^* W g(\lambda) = g(\lambda) f(\lambda)$ where $f(\lambda)$ is the unimodular function

$$f(\lambda) = \frac{A(\lambda) + i + |k(\lambda)|^2}{A(\lambda) - i + |k(\lambda)|^2}.$$

Lemma 5. *If $\overline{R(L - i)} = H$, then $R(W) \subset \text{Hac}(H_2 - \mathfrak{D})$.*

Proof. The hypotheses imply that $W_-^* W = f$. Hence, $P = ff^* P = W_-^* W W^* W_-$. Since $W_- W_-^* = \text{Pac}(H_2 - \mathfrak{D})$, we will also have $W_-^* = W_-^* W W^* \text{Pac}(H_2 - \mathfrak{D})$ or $W_- = \text{Pac}(H_2 - \mathfrak{D}) W W^* W_-$. Thus, $W_-^* W_- = P = W_-^* \text{Pac}(H_2 - \mathfrak{D}) W W^* W_-$. However, $W_-^* \text{Pac}(H_2 - \mathfrak{D}) = \text{Pac}(H_2) W_-^* = P W_-^* = W_-^*$. Thus, $P = W_-^* (W W^* W_-)$.

Suppose there exists $x \in H \neq 0$ with $y \equiv W W^* W_- x$ orthogonal to $\text{Hac}(H_2 - \mathfrak{D})$. Then $W_-^* y = 0$ and so $x = 0$. There is therefore no vector in the range of $W W^* W_-$ with a component orthogonal to $\text{Hac}(H_2 - \mathfrak{D})$. Hence, the range of $W W^* W_-$ is contained in $\text{Hac}(H_2 - \mathfrak{D})$.

But $W W^* W_- = W \bar{f}$, and since f is unitary on H , we can conclude that $R(W) \subset \text{Hac}(H_2 - \mathfrak{D})$.

Proposition. $W = W_- f$.

Proof. Since $W_-^* W_- = \text{Pac}(H_2 - \mathfrak{D})$, and thus $W_- f = \text{Pac}(H_2 - \mathfrak{D}) W$ the result follows from the preceding lemma.

For the moment let us drop the assumption of maximality.

Let us denote the projection onto $\text{Hac}(H_2 - \mathfrak{D})$ by P' , and let H_R denote the invariant subspace of H_2 generated by the (cyclic) vector $(1 - P) \cdot (H_1 - i)^{-1} k^*$. Then $H_k = H \oplus H_R$ is an orthogonal decomposition of the invariant subspace of H_2 generated by the cyclic vector $(H_1 - i)^{-1} k^*$ into reducing subspaces.

Let $f(\lambda) = (A(\lambda) + i + |k(\lambda)|^2) / (A(\lambda) - i + |k(\lambda)|^2)$, and define Q on $H_k = H \oplus H_R$ by setting

$$Qg = \begin{cases} P' W g & \text{if } g \in R(L - i) \\ W_- f g & \text{if } g \in [R(L - i)]^\perp \subseteq H \oplus H_R. \end{cases}$$

Theorem 5. $\tilde{W} = (H_1 + i)(H_1 - i)^{-1}$ is the minimal unitary dilation of Q , that is—the minimal unitary dilation of $W_- f$.

Proof. Let $H_L = H_s(H_2 - \mathfrak{D} | H_k)$, i.e., the singular part of the restriction of $H_2 - \mathfrak{D}$ to H_k .

Define the doubly infinite direct sum:

$$\hat{H} = \cdots \oplus H_L \oplus H_L \oplus H_k \oplus H_R \oplus H_R \oplus \cdots,$$

and let $\hat{W}$ on $\hat{H}$ be defined by setting for $\hat{x} \in \hat{H}$,

$$\hat{x} = [\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots],$$

$$\hat{W}\hat{x} = [\cdots, x_{-3}, x_{-2}, x_{-1} + Qx_0, P_R x_0, x_1, \cdots].$$

Define also $\hat{H}_2$ on $\hat{H}$ by setting

$$\hat{H}_2 \hat{x}_2 = [\cdots, (H_2 - \mathfrak{D})x_{-2}, (H_2 - \mathfrak{D})x_{-1}, H_2 x_0, H_2 x_1, H_2 x_2, \cdots].$$

Further, let $\hat{\mathfrak{D}}\hat{x} = [\cdots, 0, 0, \mathfrak{D}x_0, 0, 0, \cdots]$. We observe that $\hat{W}\hat{H}_2 - \hat{H}_2\hat{W} = -\hat{\mathfrak{D}}\hat{W}$. The proof that $\hat{W}$ is the minimal unitary dilation of Q is immediate. It is also true that $\text{Hac } (\hat{W})$ reduces both $\hat{W}$ and $\hat{H}_2$ and contains the range of $\hat{\mathfrak{D}}$. (see lemmas 1.2, 1.3, 1.4 of [11]).

Let h be the Hilbert space of doubly infinite sequences of complex numbers $\hat{\alpha} = [\cdots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \cdots]$ with $\sum_{i=-\infty}^{\infty} |\alpha_i|^2 < \infty$. Let $\hat{\mathfrak{H}}^*: h \rightarrow \hat{H}$ and $\hat{\mathfrak{H}}: \hat{H} \rightarrow h$ be chosen so that $\hat{\mathfrak{D}} = (1/\pi)\hat{\mathfrak{H}}^*\hat{\mathfrak{H}}$. The determining function of the triplet $[\hat{W}; \hat{H}_2, \hat{H}_2 - \hat{\mathfrak{D}}]$ is defined by setting

$$\hat{\varphi}(\ell, z) = 1_h - \frac{1}{\pi} \hat{\mathfrak{H}} \hat{W} (\hat{W} - z)^{-1} (\hat{H}_2 - \ell)^{-1} \hat{\mathfrak{H}}^*.$$

Since $\hat{\mathfrak{D}}$ has one dimensional range it follows from a result in Pincus [6] that $\hat{\varphi}(\ell, z)$ considered as an operator on h , has only one eigenvalue which is not identically equal to one.

In fact, a small modification of the cited results shows that this eigenvalue, which we henceforth denote by $\varphi(\ell, z)$ has the form

$$\varphi(\ell, z) = \exp \left(\frac{1}{2\pi i} \int_{|\tau|=1} \int_{-\infty}^{\infty} \tilde{g}(\nu, \tau) \frac{d\nu}{\nu - \ell} \frac{d\tau}{\tau - z} \right)$$

where $0 \leq \tilde{g}(\nu, \tau) \leq 1$.

We proceed to a determination of the function $\tilde{g}(\nu, \tau)$. To do this, we invoke the results of Carey-Pincus [11].

A minor modification of Theorem (4.3) of Carey-Pincus [11], asserts that $\tilde{g}(\lambda, \tau)$, for fixed real λ , is the characteristic function of the arc on the unit circle extending from $s(\lambda)f(\lambda)$ to $f(\lambda)$, measured in the clock-wise direction. Here $s(\lambda)$ is the scattering operator associated with the perturbation problem $H_2 \rightarrow H_2 - \mathfrak{D}$.

By general theory $s(\lambda) = e^{-2\pi i \delta(\lambda)}$ where $\delta(\lambda)$ is the phase shift associated with the perturbation problem, and

$$\exp \left(\int_{-\infty}^{\infty} \delta(\nu) \frac{d\nu}{\nu - \ell} \right) = \det (1 - \mathfrak{D}(H_2 - \ell)^{-1}).$$

Furthermore, $2\pi \delta(\lambda) = - \int_0^{2\pi} \tilde{g}(\lambda, e^{i\theta}) d\theta$, by theorem (2.5) of Carey-Pincus [11].

Let us recall now that

$$\exp \left(- \int_{-\infty}^{\infty} h_i(t) \frac{dt}{t - z} \right) = 1 - \int \frac{d\mu_{-i}(t)}{t - z},$$

and thus

$$\delta(t) = h_i(t) = \frac{1}{\pi} \tan^{-1} [A(t) + k(t)]^2 - \frac{1}{\pi} \tan^{-1} [A(t) - k(t)]^2.$$

Now $\arg (x + i)/(x - i) = -2 \tan^{-1} x$ with $-\pi/2 \leq \tan^{-1} x \leq \pi/2$. Thus, $\arg f(t) = -2 \tan^{-1} [A(t) + |k(t)|^2]$. Hence $f(t) = \exp (-2i \tan^{-1} (A(t) + |k(t)|^2))$, and $s(t)f(t) = \exp (-2i \tan^{-1} [A(t) - |k(t)|^2])$.

Thus $\tilde{g}(\lambda, \tau)$ is the characteristic function of the negative arc on the unit circle extending from $\exp (-2i \tan^{-1} (A(\lambda) - |k(\lambda)|^2))$ to $\exp (-2i \tan^{-1} (A(\lambda) + |k(\lambda)|^2))$ evaluated at τ . In fact, $\tilde{g}(\lambda, (\nu + i)/\nu - i) = g(\nu, \lambda)$. Set $\kappa^* = 2^{1/2}(H_1 - i)^{-1}\kappa^*$, and $k = 2^{1/2}k(H_1 + i)^{-1}$ and define, for non-real ℓ and $|z| \neq 1$

$$\tilde{\nu}(\ell, z) = 1 - \frac{1}{\pi} \kappa \tilde{W}(\tilde{W} - z)^{-1}(H_2 - \ell)^{-1}\kappa^*.$$

The following lemma is an elementary consequence of the basic commutator identities:

Lemma 6. $\tilde{\nu}(\ell, (w + i)/(w - i)) = E^{-1}(\ell, w)E(\ell, i)$.

The proof consists in setting $z = (w + i)/(w - i)$ and $\tilde{W} = (H_1 + i)(H_1 - i)^{-1}$ in $\tilde{\nu}(\ell, z)$ and proving that

$$\nu\left(\ell, \frac{w + i}{w - i}\right) = 1 + T(\ell, w) = 1 + \frac{1}{\pi i} [s(\ell, w) - s(\ell, i)] + \frac{1}{\pi i} T(\ell, w)s(\ell, i)$$

where

$$T(\ell, w) = \frac{1}{\pi i} k(w - i)(H_2 - w)^{-1}(H_1 - \ell)^{-1}(H_2 - i)^{-1}k^*$$

and

$$s(\ell, w) = k(H_2 - w)^{-1}(H_1 - \ell)^{-1}k^*.$$

If we solve for $T(\ell, w)$, we get $T(\ell, w) = (1/\pi i)[s(\ell, w) - s(\ell, i)][1 - (1/\pi i)s(\ell, i)]^{-1}$. But $(1 - (1/\pi i)s(\ell, i))^{-1} = E(\ell, i)$, and $(1/\pi i)[s(\ell, w) - s(\ell, i)] = E^{-1}(\ell, w) - E^{-1}(\ell, i)$, $\varphi(\ell, (w + i)/(w - i)) = 1 + T(\ell, w) = E^{-1}(\ell, w)E(\ell, i)$.

It follows at once that $\tilde{\nu}(\ell, z)$ has only one eigenvalue different from one; namely,

$$\nu(\ell, z) = \exp \left(\frac{1}{2\pi i} \int_{|\tau|=1} \int g(\lambda, \tau) \frac{d\lambda}{\lambda - \ell} \frac{d\tau}{\tau - z} \right).$$

Thus $\varphi(\ell, z) = \nu(\ell, z)$.

From this equality we can generate enough matrix elements to show that $\tilde{W} \cong \tilde{W}$, and $\hat{H}_2 \cong H_2$.

We note that this last observation is a special case of the general fact that the determining function is the complete unitary invariant of an intertwining pair.

We have proved above that the reproducing kernel Hilbert space of theorem one and the dilation space $\hat{H}$ coincide up to isomorphism.

Since, by the theorem, $Q = \mathcal{O}\tilde{W}\mathcal{O}$ where $\mathcal{O}$ denotes orthogonal projection onto $H \oplus H_R$, we have, for $g \in R(L - i)$, $P'Wg = \mathcal{O}\tilde{W}g = (P + P_R)\tilde{W}g = Wg + P_R\tilde{W}g$ where P_R denotes the projection onto H_R .

But, since $P'Wg \in H$ and $Wg \in H$ it must follow that $P_R\tilde{W}g = 0$. Hence $P'Wg = Wg$. Thus, $R(L + i) = R(W | R(L - i)) \subseteq \mathfrak{H}\text{ac}(H_2 - \mathfrak{D})$.

Now let us consider the consequences of the assumption that the operator L is maximal. We assume for this purpose that $\overline{R(L - i)} = H$. Then $(H_1 - i)^{-1}k^* = R(i)k^*$, and $(1 - P)(H_1 - i)^{-1}k^* = 0$. Thus, $H_R = \{0\}$ and $H_k = H$. In this case, $W = P\tilde{W}P = W_-f$. Thus, the range of W , which coincides with the range of $L + i$, is exactly $\mathfrak{H}\text{ac}(H_2 - \mathfrak{D})$.

Accordingly, the deficiency of range $(L + i)$ is exactly equal to the dimension of $H_s(H_2 - \mathfrak{D})$.

The previous results make it easy to see that this number, in turn, is equal to dimension $L_2(d\mu_i^{(s)})$ where $d\mu_i^{(s)}$ is the singular part of the measure $d\mu_i$ defined by setting

$$\exp\left(\int h_i(t) \frac{dt}{t - z}\right) = 1 + \frac{1}{\pi} \int \frac{d\mu_i(t)}{t - z}.$$

It is equally easy to see that if we had assumed that $R(L + i)$ is dense in H , rather than $\overline{R(L - i)} = H$, then the deficiency of $R(L - i)$ would be given by dimension $L_2(d\mu_{-i}^{(s)})$. Thus, we have

Theorem 6. *If $\overline{R(L - i)} = H$, then the non-zero deficiency index of L is equal to $\dim L_2(d\mu_i^{(s)})$; while, if $\overline{R(L + i)} = H$, then the non-zero deficiency index of L is equal to $\dim L_2(d\mu_{-i}^{(s)})$.*

Note that $H_R = \{0\}$ if $d\mu_{-i}$ is absolutely continuous, that is if L is maximal symmetric with deficiency $(0, n)$.

These results make it possible to compute the spectral multiplicity of the absolutely continuous part of H_1 .

Theorem 7. (See Pincus [1] or [6]). *Let $I_\nu = \{\lambda | g(\nu, \lambda) = 1\}$. If I_ν is the union of m disjoint intervals (an interval ending at $-\infty$ is amalgamated in this count with one ending at $+\infty$) then $\nu \in \sigma_{ac}(H_1)$ has spectral multiplicity m ; otherwise the spectral multiplicity of ν is infinite.*

As an elementary application, consider the singular integral operator discussed in Pincus-Rovnyak [3], which motivated the present study.

$$A(\lambda) = 0, \quad k(\lambda) = \begin{cases} \lambda^{-1/4} & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}.$$

It is easy to see, from Theorem (6), that the corresponding symmetric operator L has deficiency index $(0, 1)$. Since I_ν in theorem (7) above consists of a single interval whenever ν is positive, we can conclude that L is simple, i.e., that there is no reducing subspace of L on which it is self-adjoint.

To see this we simply note that, by the Wold decomposition, $(L + i)(L - i)^{-1}$ has an orthogonal decomposition into a unitary part and a completely non-unitary part. The completely non-unitary part is equivalent to a unilateral shift of multiplicity equal to the defect of L . The dilation of a unilateral shift is a bilateral shift of the same multiplicity with purely absolutely continuous spectrum.

Thus the spectral multiplicity of $\nu \in \sigma_{ac}(H_1)$ is equal to the defect, which is one in our example, plus the spectral multiplicity of ν coming from the unitary part of $(L + i)(L - i)^{-1}$, if any. Since the total spectral multiplicity by Theorem (7) above is one, there can be no unitary part.

It is not always true that H_1 has purely absolutely continuous spectral measure.

If $A(\lambda) - |k(\lambda)|^2 = M_-$ a.e. or $A(\lambda) + |k(\lambda)|^2 = M_+$ a.e. when $\sigma(H_2) = (-\infty, \infty)$ then M_- or M_+ will be point eigenvalues for L with infinite multiplicity, and thus also will have the same properties relative to H_1 . Otherwise, however, the spectrum is absolutely continuous.

We omit the verification.

Finally, we turn to a specification of the domain of the maximally symmetric operator L and the proof that the symmetric operator that we have constructed actually corresponds to the singular integral operator

$$\mathfrak{L}X(\lambda) = A(\lambda)X(\lambda) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{k}(\lambda)k(\mu)}{\mu - \lambda} x(\mu) d\mu.$$

Consider the case where the first defect index is zero.

Let $\mathfrak{D}(i)$ be the set of all finite sums of the form $R(i) \sum c_i \bar{k}(\lambda)/(\lambda - z_i)$. We have seen that vectors of this form are dense in H .

We will prove by direct calculation that $(\mathfrak{L} - \ell)R(\ell) \sum c_i \bar{k}(\lambda)/(\lambda - z_i) = \sum c_i \bar{k}(\lambda)/(\lambda - z_i)$ whenever $\text{Im } \ell \neq 0$.

Let $g(\lambda) = (\mathfrak{L} - \ell)R(\ell)k(\lambda)/(\lambda - z)$. We will show that $g(\lambda) = k(\lambda)/(\lambda - z)$.

By definition

$$\begin{aligned} f(\lambda) &= (\mathfrak{L} - \ell) \left[\frac{1}{2} \frac{1}{\lambda - z} \frac{E^{-1}(\ell, \lambda - i0) - E^{-1}(\ell, \lambda + i0)}{k(\lambda)} E(\bar{\ell}, z) \right] \\ &= \frac{1}{\pi i} \bar{k}(\lambda) \int_{-\infty}^{\infty} \frac{k(\mu)}{\mu - \lambda} \frac{1}{2} \frac{E(\bar{\ell}, z)}{\mu - z} \frac{E^{-1}(\ell, \mu - i0) - E^{-1}(\ell, \mu + i0)}{k(\mu)} d\mu \\ &\quad + (A(\lambda) - \ell) \frac{1}{2} \left(\frac{1}{\lambda - z} \frac{E^{-1}(\ell, \lambda - i0) - E^{-1}(\ell, \lambda + i0)}{k(\lambda)} E(\bar{\ell}, z) \right). \end{aligned}$$

Let

$$\theta(P) = \frac{1}{2\pi i} \int_c \frac{E^{-1}(\ell, w)}{(w - z)(w - P)} dw = -\frac{E^{-1}(\ell, z)}{z - P} + \frac{E^{-1}(\ell, P)}{z - P}$$

where c consists of two lines parallel to the real axis, one above the axis, and one below it.⁵

Then,

$$\theta(\lambda + i0) + \theta(\lambda - i0) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{E^{-1}(\ell, \mu - i0) - E^{-1}(\ell, \mu + i0)}{(\mu - \lambda)(\mu - z)} d\mu.$$

Thus,

$$\begin{aligned} \frac{1}{2\pi i} \bar{k}(\lambda) \int_{-\infty}^{\infty} \frac{k(\mu)}{\mu - \lambda} \frac{E(\bar{\ell}, z)}{\mu - z} \frac{E^{-1}(\ell, \mu - i0) - E^{-1}(\ell, \mu + i0)}{k(\mu)} d\mu \\ = \frac{E(\bar{\ell}, z)}{2} \frac{\bar{k}(\lambda)}{z - \lambda} [E^{-1}(\ell, \lambda + i0) + E^{-1}(\ell, \lambda - i0)] - \frac{\bar{k}(\lambda)}{z - \lambda}. \end{aligned}$$

We deduce from this that

$$\begin{aligned} 2k(\lambda)E^{-1}(\bar{\ell}, z)E(\ell, \lambda + i0)g(\lambda) &= (A(\lambda) - \ell)[E(\ell, \lambda + i0)E^{-1}(\ell, \lambda - i0) - 1] \\ &\quad - \frac{|k(\lambda)|^2}{\lambda - z} [1 + E(\ell, \lambda + i0)E^{-1}(\ell, \lambda - i0)] + \frac{2|k(\lambda)|^2}{\lambda - z} E(\ell, \lambda + i0)E^{-1}(\bar{\ell}, z) \\ &= \frac{(A(\lambda) - \ell)}{\lambda - z} \left(\frac{2|k(\lambda)|^2}{A(\lambda) - \ell - |k(\lambda)|^2} \right) - \frac{2|k(\lambda)|^2}{\lambda - z} \left(\frac{A(\lambda) - \ell}{A(\lambda) - \ell - |k(\lambda)|^2} \right) \\ &\quad + \frac{2|k(\lambda)|^2}{\lambda - z} E(\ell, \lambda + i0)E^{-1}(\bar{\ell}, z) = \frac{2|k(\lambda)|^2}{\lambda - z} E(\ell, \lambda + i0)E^{-1}(\bar{\ell}, z). \end{aligned}$$

Thus,

$$g(\lambda) = \frac{k(\lambda)}{\lambda - z}.$$

Thus, for $r(t) = \sum a_i(k(t)/(t - z_i))$, we have $(\mathfrak{L} - i)(L - i)^{-1}r(t) = r(t)$. Hence, if we set $q(t) = (L - i)^{-1}r(t)$, we will get $(\mathfrak{L} - i)(L - i)^{-1}(L - i)q(t) = (L - i)q(t)$. Thus $\mathfrak{L}q = Lq$, for q in $\mathfrak{D}(i)$. Accordingly, for x, y in $\mathfrak{D}(i)$, $(\mathfrak{L}x, y) = (Lx, y) = (x, Ly) = (x, \mathfrak{L}y)$ and $\mathfrak{L}$ on $\mathfrak{D}(i)$ is symmetric. But $\mathfrak{L}$ is also maximally symmetric since we have shown above that the range of $(\mathfrak{L} - i)$ is dense in H . Hence since L extends $\mathfrak{L}$ and L is symmetric (even maximal), we can conclude that $\mathfrak{L} = L$. When the second deficiency index is zero, we define $\mathfrak{D}(-i)$ to be the set of all finite sums of the form $R(-i) \sum c_i \bar{k}(\lambda)/(\lambda - z_i)$, and argue in exactly the same way.

Appendix (A)

Proof of Theorem 1. We will provide only a sketch of the rather long proof. If b is a vector in h , let $\bar{b}$ be the linear functional defined by $\bar{b}a = (a, b)$ for each vector $a \in h$. By (2) there exists a unique Hilbert space $\mathfrak{H}$ whose elements are h -valued analytic functions $F(z_1, z_2)$ defined for non-real z_1 and z_2 such that for every vector b and non-real w_1 and w_2 $k(w_1, w_2; z_1, z_2)\bar{b}$ belongs to $\mathfrak{H}$ as a function of z_1 and z_2 , and

$$\bar{b}F(w_1, w_2) = \langle F(t_1, t_2), k(w_1, w_2, t_1, t_2)\bar{b} \rangle$$

for every $F(z_1, z_2)$ in $\mathcal{H}$. We assert that the map:

$$F(z_1, z_2) \rightarrow \frac{[F(z_1, z_2) - F(z_1, w_2)]}{z_2 - w_2},$$

for $w_2 \neq \overline{w_2}$, is the resolvent of a self-adjoint transformation H_2 in $\mathcal{H}$.

The proof consists of studying the map $u(w_2) : F(z_1, z_2) \rightarrow \tilde{F}(z_1, z_2)$, defined for all $F(z_1, z_2) \in \mathcal{H}$, by

$$\tilde{F}(z_1, z_2) = F(z_1, z_2) + (w_2 - \bar{w}_2) \frac{F(z_1, z_2) - F(z_1, w_2)}{z_2 - w_2}$$

for fixed non-real w_2 and showing that $u(w_2)$ is unitary in $\mathcal{H}$. This follows by a tedious but elementary three step verification:

First we show that

$$\frac{k(\alpha_1, \alpha_2, z_1, z_2) - k(\alpha_1, \alpha_2; z_1, w_2)}{z_2 - w_2} = \frac{k(\alpha_1, \alpha_2, z_1, z_2) - k(\alpha_1, \bar{w}_2, z_1, z_2)}{\bar{\alpha}_2 - w_2}.$$

Then one takes

$$\begin{aligned} F(z_1, z_2) &= k(\alpha_1, \alpha_2; z_1, z_2)a \\ b(z_1, z_2) &= k(\beta_1, \beta_2; z_1, z_2)b \end{aligned}$$

for some vectors a, b in $\mathcal{H}$ and shows that

$$\begin{aligned} \langle \tilde{F}(t_1, t_2), \tilde{b}(t_1, t_2) \rangle &= \bar{b} \left\{ \frac{w_2 - \beta_2}{\bar{w}_2 - \beta_2} \frac{\bar{w}_2 - \bar{\alpha}_2}{w_2 - \bar{\alpha}_2} k(\alpha_1, \alpha_2; \beta_1, \beta_2) \right. \\ &\quad + \frac{w_2 - \beta_2}{\bar{w}_2 - \beta_2} \frac{w_2 - \bar{w}_2}{w_2 - \bar{\alpha}_2} k(\alpha_1, \bar{w}_2; \beta_1, \beta_2) \\ &\quad + \frac{\bar{w}_2 - w_2}{\bar{w}_2 - \beta_2} \frac{\bar{w}_2 - \bar{\alpha}_2}{w_2 - \bar{\alpha}_2} k(\alpha_1, \alpha_2; \beta_1, \bar{w}_2) \\ &\quad \left. + \frac{\bar{w}_2 - w_2}{\bar{w}_2 - \beta_2} \frac{w_2 - \bar{w}_2}{P_2 - \bar{\alpha}_2} k(\alpha_1, \bar{w}_2; \beta_1, \bar{w}_2) \right\} a \\ &= \bar{b} k(\alpha_1, \alpha_2; \beta_1, \beta_2) a \\ &= \langle F(t_1, t_2), b(t_1, t_2) \rangle. \end{aligned}$$

We then extend the map $F \rightarrow \tilde{F}$ by linearity and continuity to an everywhere defined isometry, $u(w_2)$ on $\mathcal{H}$, and note that $u^*(w_2) = u(\bar{w}_2)$.

Further, we note that 1 is not an eigenvalue for $u(w_2)$ because if $u(w_2)F(z_1, z_2) = F(z_1, z_2)$, then $(F(z_1, z_2) - F(z_1, w_2))/(z_2 - w_2) = 0$ identically and thus, $F(z_1, z_2) = F(z_1, w_2)$. But $\lim_{y_2 \rightarrow \infty} F(z_1, iy_2) = 0$, in fact,

$$\begin{aligned} |F(z_1, iy_2)|^2 &= \sup_{\|b\| \leq 1} |\langle F(t_1, t_2), k(z_1, iy_2; t_1, t_2)b \rangle|^2 \\ &\leq \sup_{\|b\| \leq 1} \|F\|^2 \bar{b} k(z_1, iy_2; z_1, iy_2)b \\ &= \|F\|^2 |k(z_1, iy_2; z_1, iy_2)| \end{aligned}$$

and (3) implies that this goes to zero as $y_2 \rightarrow \infty$. Thus, $F(z_1, z_2) = F(z_1, w_2) = 0$.

We introduce H_2 by taking $u(i) = (H_2 + i)(H_2 - i)^{-1}$ and it follows at once that

$$(H_2 - w_2)^{-1} \cdot F(z_1, z_2) \rightarrow \frac{F(z_1, z_2) - F(z_1, w_2)}{z_2 - w_2}.$$

In a similar way we show that there is a self-adjoint transformation H_1 in $\mathcal{H}$ such that

$$(H_1 - w_1)^{-1} : F(z_1, z_2) \rightarrow \frac{F(z_1, z_2) - Q(z_1, z_2)Q(w_1, z_2)^{-1}F(w_1, z_2)}{z_1 - w_1}.$$

Now define $k^* : b \rightarrow [1 - Q(z_1, z_2)]b$ and prove, using (3), that k^* is a continuous linear map from $\mathcal{h}$ into $\mathcal{H}$ and that the commutator of $(H_1 - z_1)^{-1}$ and $(H_2 - z_2)^{-1}$ has the indicated form.⁶

Although the present paper discusses explicitly only the case where $A(\lambda)$ and $k(\lambda)$ are complex valued functions, the theory which we are using in the present study actually extends also to the case where $A(\lambda)$ and $k(\lambda)$ are operator valued, and certain theorems, for example those specifying deficiency indices, can be obtained with only minor changes from those we prove in the present paper.

The full study of the operator valued coefficient case, however, requires an extensive development and certain pieces are still incomplete. We therefore have chosen to present the scalar results separately and to postpone the general discussion.

Added in proof: An alternate method for proving theorems three and six which makes no use of the representation furnished by theorem one has now been given in a Stony Brook thesis by R. N. Mukherjee. Theorem four of the present paper is however still employed.

FOOTNOTES

1. The fact that $E(z_1, z_2)$ satisfies hypothesis 3) requires an estimate—see [4].
2. $\mathcal{h}$ is one dimensional in our present application.
3. [To do this we use $(w - i)(H_2 - w)^{-1}(H_1 - \ell)^{-1}(H_2 - i)^{-1} = (w - i)(H_2 - w)^{-1}(H_2 - i)^{-1}(H_1 - \ell)^{-1} + (w - i)(H_2 - w)^{-1}(H_1 - \ell)^{-1}(H_2 - i)^{-1}(C/\pi i)(H_2 - i)^{-1}(H_1 - \ell)^{-1}$.]
4. This is simply a restatement in the present form of results from [2].
5. The full justification here rests on certain estimates analogous to those of pg. 697 in [11].
6. After determining functions were introduced in [1] [2] as the solutions of Riemann Hilbert problems associated with singular integral operators, extended discussions about their meaning were held with Louis deBranges. This theorem arose out of these discussions, and was found somewhat later by both deBranges and the present author.

REFERENCES

1. J. D. PINCUS, *Commutators, generalized function expansions, and singular integral operators*, Trans. Amer. Math. Soc. **121** (1966), 358–377.
2. J. D. PINCUS, *Commutators and systems of singular integral equations, I*, Acta Math. **121** (1968), 219–249.

3. J. D. PINCUS & J. ROVNYAK, *A spectral theory for some unbounded self-adjoint singular integral operators*, Amer. J. Math. **61**, (1969) 619–636.
4. J. D. PINCUS & J. ROVNYAK, *A representation theorem for determining functions*, Proc. Amer. Math. Soc.
5. J. D. PINCUS, *The determining function method in the treatment of commutator systems*, Colloq. Math. Soc., Janos Bolyai 5, Hilbert space operators, Tihany, Hungary, 1970.
6. J. D. PINCUS, *Some applications of operator valued analytic functions of two complex variables*, Linear operators and approximation, Edited by P. L. Butzer, J.-P. Kahane & B. Sz.-Nagy, Proceedings of the conference at Oberwolfach, August 14–22, 1971, ISNM **20** (1972), 68–79, Birkhäuser Verlag, Basel und Stuttgart.
7. J. D. PINCUS, *On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutator* (to appear).
8. R. CAREY & J. D. PINCUS, *On an invariant for non-commutative Banach algebras* (to appear).
9. J. D. PINCUS, *The spectrum of semi-normal operators*, Proc. National Acad. Sci. U.S.A., **68** (1971), 1684–1685.
10. T. KATO, *Smooth operators and commutators*, Studia Math. **31** (1968), 535–546.
11. R. W. CAREY & J. D. PINCUS, *The structure of intertwining isometries*, Indiana Univ. Math. J. **22** (1973), 679–703.

State University of New York
at Stony Brook

Date communicated: APRIL 23, 1973